Relativistic Analysis of Sagnac Effect in the Reference Frame of the Accelerating Observer

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Abstract

We present a simple, clean relativistic derivation of the Sagnac effect from the perspective of the rotating observer/detector. The well-known Sagnac formula just emerged right on the first run of the calculation, without any tweaking of arguments to get the desired result. The key, novel idea introduced in this paper is that two observers moving with the same velocity at the same point of space simultaneously at a given instant of time will observe the same, identical phenomenon (for example, interference fringe position) at that point, regardless of their motion history before that instant of time. Therefore, we introduce an imaginary inertial observer who is moving with the same instantaneous velocity at the point of light detection as the accelerating observer. The two observers (the real accelerating observer and the imaginary inertial observer) will observe identical phenomena at that point of space and time. Thus, acceleration of the detector/observer is irrelevant and the known non-relativistic Sagnac effect can be understood within the framework of special relativity theory. However, for relativistic speeds, Lorentz transformation / special relativity predicts different fringe shifts for the stationary and the rotating observers. This is a theoretical disproof of special relativity, along with experimental evidences such as the Silvertooth and the Marinov experiments.

Introduction

The Sagnac effect is a light interference effect induced by rotation of a ring interferometer. A beam of light is split and sent in opposite directions in a closed light path. The counter-propagating light beams return to their starting point and form interference fringes. When the whole device is set into rotation, a shift in the interference fringes is observed that is proportional to the angular velocity of the apparatus. The classical explanation based on the ether is that rotation of the device causes change in path length of each light beam, increasing one and decreasing the other and creating a change in phase relationship at the detector.

However, the Sagnac effect is still one of the controversial topics in physics because it appears to prove the ether and disprove relativity theory. But on the other hand, the ether has been disproved by the Michelson-Morley experiment. Although the mainstream view is that the Sagnac effect doesn’t disprove relativity theory, the relativistic explanation of the effect is usually criticized for its apparent inconsistency with the constancy of the speed of light. While it is true that relativity theory can explain the effect in the laboratory reference frame, to this date there is no satisfactory explanation of the effect in the reference frame of the rotating/ accelerating observer. There is not even a consensus on whether the Sagnac effect should be treated within the framework of special relativity or general relativity. In this paper, we resolve this longstanding problem by introducing a novel approach to the problem.

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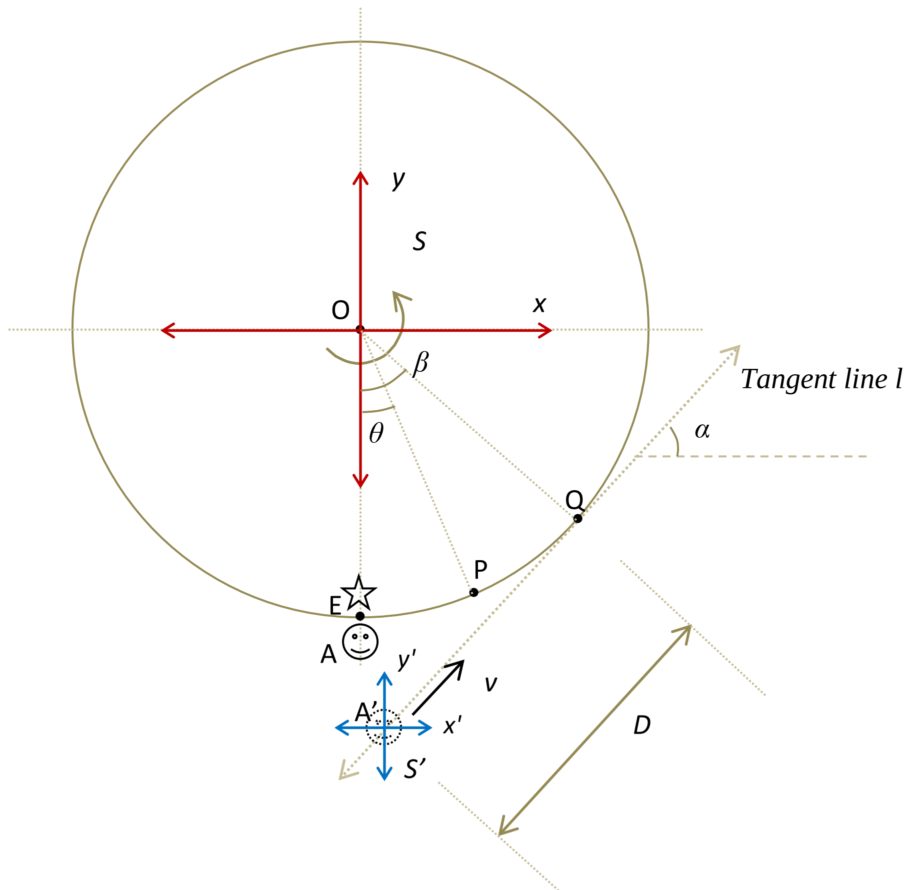
Rotational Sagnac effect

We will show that the Sagnac effect can be treated fully within the framework of special relativity. For simplicity let us consider a hypothetical circular Sagnac apparatus.



Let the light source emit a short light pulse at point E in the lab frame S. Let the observer A detect the clockwise propagating light beam at point P and the counterclockwise light beam at point Q. The procedure of analysis is first to find points P and Q and the times of detection of the light beams at points P and Q, in the lab inertial reference frame S.

The Sagnac effect is easy to understand in the lab frame S. But it has been difficult to understand the Sagnac effect in relativity theory from the perspective of the accelerating observer/detector A who is moving in the circular path towards one of the light beams and away from the other, in frame S. To this date there is no simple, consistent, satisfactory treatment of the problem within the framework of relativity theory, both special relativity and general relativity. There isn't even a consensus on whether special relativity or general relativity applies to it. The problem is that the observer is accelerating, so special relativity cannot be applied. On the other hand, no real solution has appeared based on general relativity either.

In this paper we show that the Sagnac effect can be analyzed fully within the framework of special relativity theory and show that, in fact, acceleration of the observer/detector is irrelevant in the case of Sagnac effect.

A key idea is that the accelerating observer A is in continuously changing reference frame. But the time difference in arrival of the two light beams should be evaluated in a single *inertial* reference frame.

Another novel idea introduced in this paper is that ***two observers moving with the same instantaneous velocity at the same point in space at a given instant of time will observer the same, identical phenomenon (for example, the same fringe positions), and the motion histories of the two observers before that time instant are irrelevant.*** This idea is adopted from my paper on Apparent Source Theory [2].

Therefore, since observer A is not inertial we need to *find* an ***imaginary inertial observer*** A' who reaches point Q simultaneously with accelerating observer A and who is moving with the same velocity as the instantaneous velocity of observer A at the point of light detection, point Q. Since we can easily determine points P and Q in frame S, we can *find* the position of this imaginary inertial observer A' at the instant of light emission, in frame S. The velocity of this imaginary observer is the same as the tangential velocity (ωR) of observer A at point Q, and the velocity direction is the same as the velocity direction of observer A at point Q, which can be found by drawing a tangent line at point Q. We find the position of this imaginary inertial observer at the instant of light emission from the fact that both the real accelerating observer A and the imaginary inertial observer A' reach point Q simultaneously. We can also determine the time instants, in frame S, of detection of the light pulses at point P and at point Q, from which we can find the time difference between these events. Using the time and space separation of these two events in frame S, we can calculate the time difference of these events in frame S', from which the fringe shift in frame S' can be determined and compared with the fringe shift in frame S.

Note that the velocity of frame S' relative to S is not in the usual $+x$ direction, but along a line making an angle α with the $+x$ axis, and therefore the generalized Lorentz transformation equations should be used. We simply use Lorentz transformation from S to S' to find the difference in time of detection of the two pulses in frame S'.

Now we analyze the Sagnac effect in frame S. This means that we determine the x, y, t – coordinates of three events in frame S: emission of light, detection of clockwise propagating beam at point P, detection of counterclockwise propagating beam at point Q.

We have three events, Event 1, Event 2, Event 3, with coordinates:

(x_1,y_1,t_1), (x_2,y_2,t_2), (x_3,y_3,t_3), respectively, in frame S, and
(x_1',y_1',t_1'), (x_2',y_2',t_2'), (x_3',y_3',t_3'), respectively, in frame S'

Event 1 is the emission of light from point E
Event 2 is the detection of the clockwise propagating light beam at point P
Event 3 is the detection of the counterclockwise propagating light beam at point Q

(actually we need to consider only Event 2 and Event 3 , and Event 1 could be skipped)

Sagnac effect in frame S

Event 1: Emission of light pulse at point E.

$$x_1 = 0 \quad , \quad y_1 = -R \quad , \quad t_1 = 0$$

Event 2: Detection of clockwise propagating light beam at point P

We find the angular position θ (in radians) of point P as follows.

Since the clockwise beam and the observer A take equal times to arrive at point P,

$$\frac{2\pi R - R\theta}{c} = \frac{R\theta}{v}$$

$$\Rightarrow \frac{2\pi R - R\theta}{c} = \frac{R\theta}{\omega R}$$

$$\Rightarrow \theta = \frac{2\pi\omega R^2}{Rc + \omega R^2} \quad . \; . \; . \quad [1]$$

Therefore,

$$x_2 = R\sin\theta \quad , \quad y_2 = -R\cos\theta \quad , \quad t_2 = \frac{\theta}{\omega} \quad . \; . \; . \quad [2]$$

Event 3: Detection of counterclockwise propagating light beam at point Q

We find the angular position β (in radians) of point Q as follows.

Since the counterclockwise beam and the observer A take equal times to arrive at point Q,

$$\frac{2\pi R + R\beta}{c} = \frac{R\beta}{v}$$

$$\Longrightarrow \frac{2\pi R + R\beta}{c} = \frac{R\beta}{\omega R}$$

$$\Longrightarrow \beta = \frac{2\pi\omega R^2}{Rc - \omega R^2} \qquad . . . \quad [3]$$

Therefore,

$$x_3 = R\sin\beta \quad , \quad y_3 = -R\cos\beta \quad , \quad t_3 = \frac{\beta}{\omega} \qquad . . . \quad [4]$$

<u>Time interval between detection at point P and detection at point Q</u>

This is the time difference between Event 2 and Event 3 in frame S:

$$\Delta t = t_3 - t_2 = \frac{\beta}{\omega} - \frac{\theta}{\omega} = \frac{\beta - \theta}{\omega} \qquad . . . \quad [5]$$

We will use this time difference to determine the time difference in the reference frame (S') of the imaginary inertial observer A'. For this, however, we need not only the time difference of the two events in S, but also their spatial difference in S.

$$\Delta x = x_3 - x_2 = R\sin\beta - R\sin\theta = R\,(\sin\beta - \sin\theta) \qquad . . . \quad [6]$$

$$\Delta y = y_3 - y_2 = -R\cos\beta - (-R\cos\theta) = R\,(\cos\theta - \cos\beta) \qquad . . . \quad [7]$$

Therefore, we will use both Δx and Δt determined above to determine the time difference $\Delta t'$ in frame S' , from which the fringe shift in S' can be obtained.

Determining the position and velocity of the inertial reference frame S' of the imaginary inertial observer

We determine the position and velocity of the imaginary inertial observer A' in frame S , at the time of light emission ($t = 0$) , from the fact that the real (accelerating) observer A and the imaginary inertial observer A' arrive at point Q simultaneously. Therefore, the position of the imaginary inertial observer at $t = 0$, along the tangent line l , is determined as follows.

(actually we need only the velocity in this case, we calculate the position just for the sake of completeness)

$$\frac{R\beta}{v} = \frac{D}{v}$$

$$D = R\beta$$

The velocity (hence the path) of imaginary inertial observer A' makes an angle of α with the horizontal (with the x - axis), where, it can be shown that:

$$\alpha = \beta \quad . . . \quad [8]$$

Now that the relative positions and velocities of inertial reference frames S and S' are determined, we can use Lorentz transformations from S to S' to determine the time interval between Event 3 and Event 2 in frame S'.

We use the generalized Lorentz transformation equation:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = B(v) \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

where

$$\mathbf{B}(\boldsymbol{v}) = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{{\beta_x}^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{{\beta_y}^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{{\beta_z}^2}{\beta^2} \end{bmatrix}$$

Since there is no relative motion in the *z*- direction, the Lorentz transformation matrix will be:

$$\begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & 0 \\ -\gamma\beta_x & 1+(\gamma-1)\frac{{\beta_x}^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 0 \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{{\beta_y}^2}{\beta^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad . \; . \; . \; [9]$$

We first need to determine β_x , β_y and γ .

$$\beta_x = \frac{v_x}{c} = \frac{v\cos\alpha}{c} = \frac{v\cos\beta}{c}$$

$$\beta_y = \frac{v_y}{c} = \frac{v\sin\alpha}{c} = \frac{v\sin\beta}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where

$$v = \omega R$$

From the generalized Lorentz transformation matrix [9], the time difference between Event 3 (detection ccw beam) and Event 2 (detection of the cw beam) will be:

$$\Delta (ct') = \gamma \Delta(ct) - \gamma \beta_x (\Delta x) - \gamma \beta_y (\Delta y)$$

$$\Rightarrow \Delta t' = \gamma \Delta t - \gamma \frac{\beta_x}{c} (\Delta x) - \gamma \frac{\beta_y}{c} (\Delta y)$$

$$\Rightarrow \Delta t' = \gamma \left(\Delta t - \frac{\beta_x}{c} \Delta x - \frac{\beta_y}{c} \Delta y \right) \quad . . . \quad [10]$$

From equations [5], [6], [7] , we have already obtained:

$$\Delta t = t_3 - t_2 = \frac{\beta - \theta}{\omega}$$

$$\Delta x = x_3 - x_2 = R (\sin\beta - \sin\theta)$$

$$\Delta y = y_3 - y_2 = R (\cos\theta - \cos\beta)$$

From equation [10]:

$$\Rightarrow \Delta t' = \gamma \left(\Delta t - \frac{\beta_x}{c} \Delta x - \frac{\beta_y}{c} \Delta y \right)$$

$$\Rightarrow \Delta t' = \gamma \left(\frac{\beta - \theta}{\omega} - \frac{\frac{v \cos\beta}{c}}{c} R (\sin\beta - \sin\theta) - \frac{\frac{v \sin\beta}{c}}{c} R (\cos\theta - \cos\beta) \right) \quad . . . \quad [10]$$

The values of θ and β from equations [1] and [3] can be substituted in the above equation for an exact solution.

$$\theta = \frac{2\pi\omega R^2}{Rc + \omega R^2}$$

$$\beta = \frac{2\pi\omega R^2}{Rc - \omega R^2}$$

Let us first try to simplify the above equation [10]:

$$\Delta t' = \gamma \left(\frac{\beta - \theta}{\omega} - \frac{\frac{v\cos\beta}{c}}{c} R(\sin\beta - \sin\theta) - \frac{\frac{v\sin\beta}{c}}{c} R(\cos\theta - \cos\beta) \right)$$

$$\Delta t' = \gamma \left(\frac{\beta - \theta}{\omega} - \frac{\mathrm{R}\, v}{c^2} \left(\cos\beta \sin\beta - \cos\beta \sin\theta - \sin\beta \cos\theta + \sin\beta\cos\beta \right) \right)$$

$$\Delta t' = \gamma \left(\frac{\beta - \theta}{\omega} - \frac{\mathrm{R}\, v}{c^2} \left(\sin 2\beta - \sin(\theta + \beta) \right) \right) \quad . \; . \; . \quad (11)$$

Since angles β and θ are very small:

$$\sin 2\beta \approx 2\beta \quad \text{and} \quad \sin(\theta + \beta) \approx \theta + \beta$$

Therefore:

$$\Delta t' = \gamma \left(\frac{\beta - \theta}{\omega} - \frac{\mathrm{R}\, v}{c^2} \left(2\beta - (\theta + \beta) \right) \right)$$

$$\Rightarrow \Delta t' = \gamma \left(\frac{\beta - \theta}{\omega} - \frac{\mathrm{R}\, v}{c^2} (\beta - \theta) \right)$$

$$\Rightarrow \Delta t' = \gamma (\beta - \theta) \left(\frac{1}{\omega} - \frac{\mathrm{R}\, v}{c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma (\beta - \theta) \left(\frac{1}{\omega} - \frac{\omega R^2}{c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma (\beta - \theta) \left(\frac{c^2 - \omega^2 R^2}{\omega c^2} \right)$$

Substituting the values of β and θ ,

$$\Rightarrow \Delta t' = \gamma \left(\frac{2\pi\omega R^2}{Rc - \omega R^2} - \frac{2\pi\omega R^2}{Rc + \omega R^2} \right) \left(\frac{c^2 - \omega^2 R^2}{\omega c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma \, 2\pi\omega R^2 \left(\frac{1}{Rc - \omega R^2} - \frac{1}{Rc + \omega R^2} \right) \left(\frac{c^2 - \omega^2 R^2}{\omega c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma \, 2\pi\omega R^2 \left(\frac{2\,\omega R^2}{\left(Rc - \omega R^2\right)\left(Rc + \omega R^2\right)} \right) \left(\frac{c^2 - \omega^2 R^2}{\omega c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma \, 2\pi\omega R \left(\frac{2\,\omega R}{(c - \omega R)(c + \omega R)} \right) \left(\frac{c^2 - \omega^2 R^2}{\omega c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma\, 2\pi\omega R \left(\frac{2\,\omega R}{\omega c^2} \right)$$

$$\Rightarrow \Delta t' = \gamma \left(\frac{4\pi\ (\omega R)^2}{\omega c^2} \right) = \gamma\, \frac{4\pi R^2\, \omega}{c^2} = \gamma\, \frac{4\, A\, \omega}{c^2} \qquad . \ . \ . \quad [12]$$

The fringe shift observed by the rotating observer will be:

$$\Delta N = \frac{c\,\Delta t'}{\lambda'} = \gamma\, \frac{4\, A\, \omega}{c\, \lambda'}$$

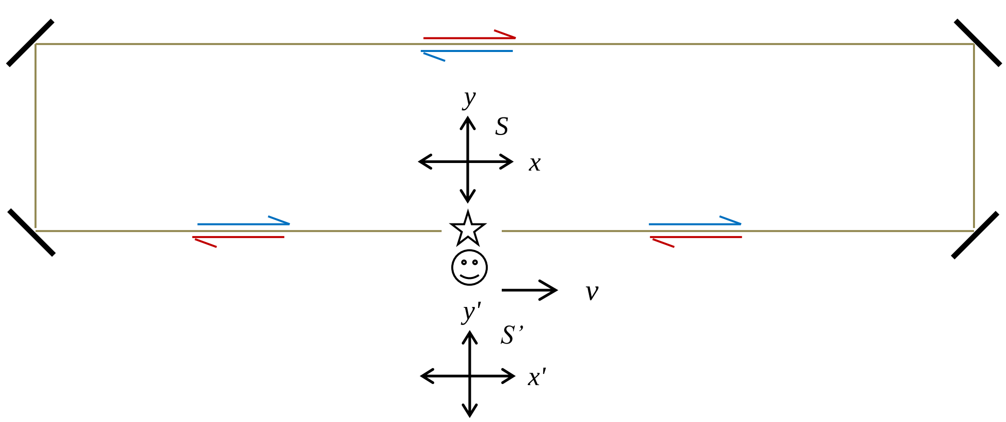
For this fringe shift to be equal to the fringe shift observed by the stationary observer,

$$\lambda' = \gamma\, \lambda$$

because of time dilation[5]. Both the stationary and the rotating observers will see equal fringe shifts. Therefore, the non-relativistic Sagnac effect is consistent with SRT.

The linear Sagnac effect

The Wang linear Sagnac experiment [3] is conceptualized and simplified as follows. The mirrors are stationary in the lab frame and only the observer/detector and the light source are co-moving to the right with velocity *v* in the lab frame S. It is found that the observer detects a fringe shift that is proportional to the linear velocity.



The light source and the observer are co-moving to the right with velocity *v*. Consider two inertial frames S and S'. S is the lab frame. S' is the reference frame of the observer. S' is moving with velocity *v* in the +*x* direction relative to S. The observer and the source are both at the origin of S'.

We have three events:

Event 1 : emission of light from the source
Event 2 : detection of the ccw light beam
Event 3: detection of the cw light beam

The phase difference of the counter-propagating light beams in frame S is [4]:

$$\Delta\phi = \frac{4\pi}{\lambda c} \oint v \, . \, dx$$

The difference in time *Δt* between Event 3 and Event 2 in frame S will be:

$$\Delta t = \frac{\Delta\phi}{\frac{c}{\lambda}} = \frac{\frac{4\pi}{\lambda c} \oint v \, . \, dx}{\frac{c}{\lambda}} = \frac{4\pi}{c^2} \oint v \, . \, dx \quad . \; . \; . \quad [\,13\,]$$

To get the time difference *Δt'* of Event 3 and Event 2 in frame S' , in addition to the time separation *Δt*, we also need the spatial separation of the two events in frame S:

$$\Delta x = v \, \Delta t$$

where *Δt* is from equation [13].

Using the Lorentz transformation equation, the difference in time between Event 3 and Event 2 in frame S' will be:

$$\Delta t' = \gamma \left(\Delta t - \frac{v \Delta x}{c^2} \right)$$

$$\Longrightarrow \Delta t' = \gamma \left(\Delta t - \frac{v * v \, \Delta t}{c^2} \right)$$

$$\Longrightarrow \Delta t' = \gamma \, \Delta t \left(1 - \frac{v^2}{c^2} \right)$$

$$\Longrightarrow \Delta t' = \gamma \, (\frac{4\pi}{c^2} \oint v \, . \, dx \,) \left(1 - \frac{v^2}{c^2} \right)$$

$$\Longrightarrow \Delta t' = \frac{\frac{4\pi}{c^2} \oint v \, . \, dx}{\gamma}$$

A hole in the argument

Unfortunately for relativity, our analysis so far has a hole in it. This concerns the approximation we used to simplify equation (11):

For very small angles β and θ:

$$\sin 2\beta \;\approx\; 2\beta \quad \text{and} \quad \sin(\theta + \beta\,) \;\approx\; \theta + \beta$$

For large values of Θ and β , that is for relativistic rotation of the Sagnac device, this approximation will not hold and the exact equation (11) should be used. Let us see whether or not the stationary and the rotating observers will see the same fringe shift in this case.

Using Excel I found that, for ω = 10^7 Π rad/sec , R = 1m

v/c ≈ 1.04719755119660000000E-01 , γ = 1.00552862646959

Δt = 4.38649084492860000000E-09 , Δt'= 4.45135189041167000000E-09

The fringe shift observed by the stationary observer, i.e. fringe shift relative to the lab frame will be, for λ = 600 nm

$$\Delta N = \frac{c\;\Delta t}{\lambda} = \frac{300000000 * \;\; 4.38649084492860000000\text{E} - 09}{0.0000006} =$$

$$\Delta N = 2.19324542246430000000E + 06$$

The fringe shift observed by the rotating observer will be:

$$\Delta N' = \frac{c\;\Delta t'}{\lambda'}$$

where

$$\lambda' = \gamma\lambda$$

which is due to time dilation[5].

Therefore,

$$\Delta N' = \frac{c\;\Delta t'}{\lambda'} = \frac{c\;\Delta t'}{\gamma\;\lambda} = \frac{300000000 * \;4.45135189041167000000\text{E} - 09}{1.00552862646959 * \;0.0000006} =$$

$$\Delta N' \;=\; 2.21343866958834000000E + 06$$

Thus, the principle that all observers should agree on the observable quantity (fringe shift) is violated because ΔN' ≠ ΔN . This disproves special relativity.

Conclusion

In this paper we have successfully resolved the longstanding enigmatic problem in the special relativity theory of the non-relativistic Sagnac effect in the reference frame of the accelerating observer. Since the reference frame of the accelerating observer is continuously changing, we cannot apply special relativity directly. A key idea is that the time difference of the two light beams should be evaluated within a single reference frame. We can see that the time difference can be evaluated only at the instant it occurs, at point Q, i.e. at the instance of detection of the late arriving light beam, which is the ccw propagating light beam. Obviously the time difference cannot be evaluated at the instant of detection of the early arriving light beam, i.e. the cw propagating beam, because the ccw light beam has not arrived yet. A novel idea introduced in this paper is *that two observers that happen to be at the same point in space simultaneously at a given instant of time and moving with the same velocity will observe the same, identical phenomenon at that point of space and time (for example, position of interference fringes), irrespective of their motion histories before that time instant*. Therefore, we created an imaginary inertial observer who will be at the point of detection of the counterclockwise (ccw) propagating light beam simultaneously with the real accelerating observer. The problem is solved in the reference frame of this imaginary inertial observer using the Lorentz transformation equations.

We have shown that both the non-relativistic rotational and linear Sagnac effects can be explained within the framework of special relativity theory using Lorentz transformation equations. In this case, we have shown that both the stationary observer and the rotating observer will see equal fringe shifts. Therefore, the known non-relativistic Sagnac effect cannot be used as an experimental evidence against the theory of relativity. However, we have seen that special relativity / Lorentz Transformation predicts different fringe shifts for the stationary and the rotating observer, at relativistic speeds of the Sagnac device. Therefore, we have shown that although the experimental Sagnac effect we know may not contradict special relativity, we have shown a theoretical disproof of Lorentz transformations and special relativity by a rigorous application of Lorentz transformations to the Sagnac effect. This is a theoretical evidence against special relativity theory, along with the Marinov, the Silvertooth and the Roland Dewitte experiments.

Thanks to Almighty God Jesus Christ and His Mother Our Lady Saint Virgin Mary

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