Gentle Beukers's Proofs that $\zeta(2,3)$ are Irrational

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Abstract

Although Beukers's proof that $\zeta(2)$ and $\zeta(3)$ are irrational are at the level of advanced calculus, they are condensed. This article slows down the development and adds examples of the techniques used. In so doing it is hoped that more people might enjoy these mathematical results. We focus on the easier of the two $\zeta(2)$.

Introduction

Apery proved $\zeta(3)$ is irrational [1]. His proof was difficult [5, 12]. Beukers simplified it. Beukers in his four and a half page paper [3] proves both $\zeta(2)$ and $\zeta(3)$ are irrational. The former uses less complicated cases of a pattern than the latter, but still the math in both is highly condensed.

There is a mixture of number theory and calculus which is not typical in either number theory or calculus courses. In this article we slow down Beukers's presentation and provide tutorials, in effect, for the harder parts. Where possible we give examples and references. As the $\zeta(3)$ case is largely a repeat of $\zeta(2)$, we focus mainly only on it, but do cover $\zeta(3)$ as well, generally using Maple, a computer algebra system, wherever possible.

If the presentation works, teachers (and students) might find good examples of challenging high school and undergraduate topics and all within the context of a truly marvelous, yet serious mathematical results.

Preliminaries

Divisions can generate infinite series. Perhaps the easiest example of this is using long division to arrive at

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots.$$

When we take integrals, we've solved an integral with a series:

$$\int \frac{1}{1-x}\,\mathrm{dx} = \int 1 + x + x^2 + x^3 + \cdots = x + x^2/2 + x^3/3 + x^4/4 + \dots. \quad (1)$$

The definite integral with 0 and 1 limits of integration gives the harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

which doesn't converge. We suspected as much, as $1/(1-x)$, the integrand on the left of (1), gets large as x approaches 1.

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

squares the terms of the harmonic. So if we had a way to repeat the ideas just given, we might be able to arrive at an integral (or a double integral) that equals this series; its notated with $\zeta(2)$.

We find that

$$\frac{1}{1-xy} = 1 + xy + x^2y^2 + x^3y^3 + \dots \quad (2)$$

and

$$\int 1 + xy + x^2y^2 + x^3y^3 + \dots\ \mathrm{dx}\ = x + \frac{x^2}{2}y + \frac{x^3}{3}y^2 + \dots.$$

Taking the limits of integration to be 1 and 0, yields

$$1 + \frac{1}{2}y + \frac{1}{3}y^2 + \dots,$$

and then integrating with the same limits with respect to y, we arrive at $\zeta(2)$:

$$\begin{aligned}\int_0^1 1+\frac{1}{2}y+\frac{1}{3}y^2+\dots \text{ dx } &= \Big|_0^1 y+(1/2)y^2/2+(1/3)y^3/3+\dots \\ &= \zeta(2).\end{aligned}$$

This can all be summarized succinctly with

$$\int_0^1\int_0^1 \frac{1}{1-xy} \text{ dx dy } = \zeta(2). \tag{3}$$

Our mission is to prove this number is irrational. This is a result known prior to Apery and Beukers work.

Imagining a solution

Looking at proofs of irrationality that involve integrals [6, 7, 11], the structure is to generate bounds that are violated somehow when the integral is evaluated. An increasing n value needs to be incorporated into (3) so that as this n goes to infinity the violation of a bound is guaranteed. We note that in the articles just cited, we see the form $x^n(p-qx)^n$ used as an integrand with 0 and p/q as limits of integration. The form $x^n(1-x)^n$ is used in [2] to show e is irrational. We have two variables in our double integral with 1 as the upper limit of integration for both variables, so we will explore

$$\int_0^1\int_0^1 \frac{y(1-y)x(1-x)}{1-xy} \text{ dx dy }. \tag{4}$$

Now, if we have bounds for the nth power of the integrand in (4), say

$$0 < \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^n} \leq B_n,$$

then perhaps we can say

$$\int_0^1\int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^n}\frac{1}{1-xy} \text{ dx dy } \leq B_n\zeta(2). \tag{5}$$

	d/dx	$\int$
+	x^2	$\sin x$
-	$2x$	$-\cos x$
+	2	$-\sin x$
	0	$\cos x$
	$\int$	d/dx

Table 1: Example of tabular integration

Note: the lower bound is automatic as the function is positive on the intervals in question.[1] Then following the example in [7], we assume $\zeta(2)$ is rational and evaluate this integral and violate this limit as n goes to infinity.

The template for a lot of irrationality proofs is to force a positive integer to be less than one. Something like

$$0 < x_n < q(5/6)^n \tag{6}$$

where x_n and q are positive integers, shows the trick: as $\lim_{n\to\infty}(5/6)^n = 0$, for any fixed q (6) becomes impossible. There are four tasks: form an integral (done); evaluate that integral; form an upper bound for it; and, show the combination gives a contradiction.

Evaluation, part 1

Doing multiple integrations by parts can be a daunting challenge. Tabular integration by parts can help [9]. As the integral in (5) looks like a prime candidate for such multiple integrations by parts, we'll provide a tutorial on tabular integration by parts and then use this method.

Finding the integral

$$\int x^2 \sin x \ \mathrm{dx}$$

becomes mechanical using tabular integration. Looking at Table 1 multiplying the first row's, second column's x^2 with the third row's $-\cos x$ and

[1]Technically this is an improper integral, but this is easily resolvable, see Beukers article.

	d/dx	$\int$
+	$(1-y)^n(1-xy)^{-1}$	$(1/n!)\frac{d^n}{dx^n}x^n(1-x)^n$
-	$(-1)y(1-y)^n(1-xy)^{-2}$	$(1/n!)\frac{d^{n-1}}{dx^{n-1}}x^n(1-x)^n$
+	$(-1)(-2)y^2(1-y)^n(1-xy)^{-3}$	$(1/n!)\frac{d^{n-2}}{dx^{n-1}}x^n(1-x)^n$
$\pm$	$\vdots$	$\vdots$
$\pm$	$(-1)^n n! y^n(1-y)^n/(1-xy)^{n+1}$	$(1/n!)x^n(1-x)^n$
	$\int$	d/dx

Table 2: Bottom up tabular integration

incorporating the sign in the first column and then repeating and adding each such product gives the answer:

$$\int x^2 \sin x \text{ dx} = -x^2 \cos x + 2x \sin x + 2 \cos x$$

If, for some reason, we wanted to keep an integral in our solution, we can adjust our algorithmic assembly process. For example, we simply reset our table at row 2, observing the minus sign in the first column:

$$\int x^2 \sin x \text{ dx } = -(-x^2 \cos x + \int 2x(-\cos x) \text{ dx}).$$

One can, as usual, take derivatives and confirm these statements.

Table 1 repeats in reverse order the table header in the last row: $\int$ and d/dx. This is to indicate that one can reverse direction and construct the top row as one equivalent to the bottom. Usually the function that promises to end with a 0 after derivatives are taken is chosen for column one, as in the first $x^2 \sin x$ example. We use this bottom up integration by parts, see Table 2, to construct a easier form of (5):

$$\int_0^1 \int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}} \text{ dx dy } = \int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} \text{ dx dy} \quad (7)$$

where

$$P_n(x) = (-1)^n \frac{1}{n!} \frac{d^n}{dx^n}[x^n(1-x)^n].$$

$$> \ expand\left((1-y)^3 \cdot \frac{d^3}{dx^3}\left(x^3 \cdot (1-x)^3\right) \right)$$

$$120 x^3 y^3 - 360 x^3 y^2 - 180 x^2 y^3 + 360 x^3 y + 540 x^2 y^2 + 72 x y^3 - 120 x^3 - 540 x^2 y - 216 x y^2 - 6 y^3$$
$$+ 180 x^2 + 216 x y + 18 y^2 - 72 x - 18 y + 6$$

Figure 1: The numerator of the integrand using Maple's expand function.

Studying the bottom line of Table 2 one sees the integrand desired and then works up rather than down to arrive at an equivalent integrand.

Evaluation, part 2

Have we accomplished anything? The integrand in

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} \text{ dx dy} \tag{8}$$

has the same denominator as the one in (2) and we were able to give an infinite series form for it: the right hand side of (2). The definition of $P_n(x)$ reveals that it is a integer polynomial in x and its a multiplicand with an integer polynomial in y. Using Maple we can look at an example computation, see Figure 1. Imagining this result multiplied by (2), we see an integer polynomial in various powers of x and y. So if we can evolve a way to evaluate integrals of the form

$$\int_0^1 \int_0^1 \frac{c_{r,s} x^r y^s}{1-xy} \text{ dx dy} \ , \tag{9}$$

where $c_{r,s}$ is an integer, we have made progress. The terms of interest in the integrand are

$$\sum_{r \geq 0, s \geq 0} c_{r,s} x^r y^s. \tag{10}$$

Find a case equation for (9) without the $c_{r,s}$ coefficient – they are integers, which have the right form for our template, (6). This is relatively easy. Just multiply (2) by $x^s x^r$ and do the integrations; the integrand is

$$x^r y^s = x^r y^s (1 + xy + x^2 y^2 + x^3 y^3 + \ldots). \tag{11}$$

We find

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} \text{ dx dy } = \begin{cases} \zeta(2) & \text{if } r = s = 0 \\ \zeta(2) - \sum_{k=1}^{r} \frac{1}{k^2} & \text{if } r = s > 0 \\ \frac{m}{d_r^2} & \text{if } r > s. \end{cases}$$

We will show each of the cases.

If $r = s = 0$, then (11) is just one times the infinite series, which we've seen is just $\zeta(2)$. Now is about the time we should start imagining that we assume $\zeta(2)$ is rational and what that might mean. It will have then a denominator – hold that thought.

If $r = s > 0$, this just moves the terms up by r terms and we can express this as the whole series minus the early terms:

$$\zeta(2) - \sum_{k=1}^{r} \frac{1}{k^2}.$$

The sum is finite and is equal to a fraction. Under the assumption that $\zeta(2)$ is rational we can blast out its denominator with dispatch. What about this sum? Well multiplying it by the least common multiple of all its denominators will make it too a positive integer. When we calculate an upper bound, multiplication by the least common multiple of integers 1 through n will be necessary.

The next case, $r > s$, requires more work. Simply integrating, we have

$$\int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^{r+k} y^{s+k} \text{ dx dy} = \sum_{k=0}^{\infty} \frac{1}{(k+r+1)(k+s+1)}$$

When adding or subtracting fractions using cross multiplication, the product of denominators appears; we use this in

$$\frac{1}{k+s+1} - \frac{1}{k+r+1} = \frac{(k+r+1)-(k+s+1)}{(k+r+1)(k+s+1)} = \frac{r-s}{(k+r+1)(k+s+1)}.$$

So, we now have

$$\begin{aligned} \sum \frac{1}{(k+r+1)(k+s+1)} &= \sum \frac{1}{r-s} \left(\frac{1}{k+s+1} - \frac{1}{k+r+1} \right) \\ &= \frac{1}{r-s} \sum \left(\frac{1}{k+s+1} - \frac{1}{k+r+1} \right). \end{aligned}$$

It's best to consider an example to see what's going on. Letting $r = 5$ and $s = 2$ we see a cascading phenomenon:

$$(1/3 - 1/6) + (1/4 - 1/7) + (1/5 - 1/8) + (1/6 - 1/9) + (1/7 - 1/10) + \ldots = (1/3) + (1/4) + (1/5),$$

that is, the terms from $s+1$ up to r remain. The net is

$$\frac{1}{r-s}\left(\frac{1}{s+1} + \cdots + \frac{1}{r}\right).$$

Which, for $s = 2$, $r = 5$, is

$$\frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right). \tag{12}$$

For such sums we can use a fraction that is not necessarily reduced, such as

$$\frac{m}{d_r^2},$$

where d_r is the least common multiple of $\{1, \ldots, r\}$. So (12) gives

$$\frac{20}{60}\left(\frac{20}{60} + \frac{15}{60} + \frac{12}{60}\right)$$

or

$$\frac{20}{60}\left(\frac{47}{60}\right) = \frac{940}{60^2} = \frac{m}{d_r^2}$$

as the least common multiple of $\{1, \ldots, 5\}$ is 60.

One more time; it is as simple as

$$\begin{aligned} &\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots \\ &\qquad\qquad\quad - \frac{1}{6} - \frac{1}{7} - \ldots \end{aligned} \tag{13}$$

is (14)

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5}.$$

Evaluation, part 3

As a result of part 2, we have three sums to consider in our integrand:

$$\sum_{r=s=0} c_{r,s}x^r y^r + \sum_{r=s>0} c_{r,s}x^r y^r + \sum_{r>s\geq 0} c_{r,s}x^r y^r.$$

As the coefficients, the $c_{r,s}$s, are all integers, this becomes

$$\zeta(2)\sum_{r=s=0} c_{r,s} + \sum_{r=s>0} c_{r,s}[\zeta(2) - \sum_{k=1}^{r}\frac{1}{k^2}] + \sum_{r>s\geq 0} c_{r,s}\sum_{k=1}^{r-s}\frac{1}{k^2}$$

or

$$\zeta(2)(\sum_{r=s=0} c_{r,s} + \sum_{r=s>0} c_{r,s}) - \sum_{r=s>0} c_{r,s}\sum_{k=1}^{r}\frac{1}{k^2} + \sum_{r>s\geq 0} c_{r,s}\sum_{k=1}^{r-s}\frac{1}{k^2}$$

or

$$\zeta(2)a_n - \sum_{r=s>0} c_{r,s}\frac{m_1}{d_n^2} + \sum_{r>s\geq 0} c_{r,s}\frac{m_2}{d_n^2}$$

or

$$\zeta(2)a_n + b_n\frac{m_2}{d_n^2}$$

or

$$(\zeta(2)A_n + B_n)/d_n^2, \tag{15}$$

where d_n^2 is the square of the least common multiple of the set $\{1, 2, \ldots, n\}$, a_n, b_n, A_n and B_n are integers. It should be noted that when a fraction is multiplied by an integer, cancellations can occur, but no additional factors can be created in the denominator: factors go out, not in. Also, as Figure 1 implies, the coefficients are all functions of n per force of the numerator of the integrand in (8) consisting consisting of the multiplication of two polynomials of degree n. In [4] these integer coefficients are calculated, but this is not really necessary.

The upper bound, part 1

Summing up, the integral evaluates to (15). Next up in our program is to create an upper bound. Calculus books, like [9], use the area under the curve feature of integrals to determine upper bounds: if the definite integral gives the area under the curve, the maximum of the integrand times the length of the limits of integration should give the area of a rectangle that encloses the area under the curve, an upper (and a lower) bound:

$$m(b-a) \leq \int_a^b f(x)\ \mathrm{dx} \leq M(b-a).$$

An upper bound for

$$F_n(x,y) = \left(\frac{x(1-x)y(1-y)}{1-xy}\right)^n$$

will be the nth power of the upper bound of

$$F_1(x,y) = \left(\frac{x(1-x)y(1-y)}{1-xy}\right).$$

Given the limits of integration, we seek then a relative extrema for $(x, y) \in ([0,1] \times [0,1])$. Such a maxima will occur when $x = y$ because otherwise the lesser of the two will diminish the value of the product. This gives a function in one variable:

$$f(x) = \frac{x^2(1-x)^2}{1-x^2}.$$

A theorem in calculus says that the maximum of the integrand times the length of the interval of integration gives the maximum of the integral [9]. We need then to find the maximum of the integrand.

Using elementary calculus, a maximum of a function occurs when its derivative is 0 and the second derivative is negative at this point. We will determine the appropriate 0 for f'. First, lets reduce f:

$$f(x) = -\frac{x^2(x-1)(x-1)}{(x+1)(x-1)} = -\frac{x^3-x^2}{(x+1)}.$$

The derivative of f is then

$$f'(x) = -\frac{(3x^2-2x)(x+1)-(1)(x^3-x^2)}{(x+1)^2} = \frac{(-2x)(x^2+x-1)}{(x+1)^2}.$$

After using the quadratic formula, we find the appropriate 0 of the numerator is

$$\Phi = \frac{\sqrt{5}-1}{2}.$$

One doesn't have to bother with the second derivative test because the other roots of the numerator, 0 and $(-\sqrt{5}-1)/2$, can't work: too small and out of domain, respectively.

Next we need to evaluate $f(\Phi)$. This seems like a complicated evaluation, but this number is the reciprocal of ϕ, Greek phi, a famous number with lots of properties [10]. We'll use some of them. So, we need to evaluate

$$\frac{x^2(1-x)^2}{1-x^2}$$

at $\Phi = \phi^{-1}$. Well as Φ is a root of x^2+x-1, we have $\Phi^2 = 1-\Phi$ and $(1-\Phi)^2 = \Phi^4$, so

$$\frac{x^2(1-x)^2}{1-x^2} = \frac{\Phi^6}{1-\Phi^2},$$

when $x = \Phi$. For the denominator, using $x^2+x-1 = 0$ once again, $1-x^2 = x$ and so $1-\Phi^2 = \Phi$. We can conclude that for $(x,y) \in [0,1) \times [0,1)$,

$$\frac{x(1-x)y(1-y)}{1-xy} \leq \Phi^5 = \left(\frac{\sqrt{5}-1}{2}\right)^5 \approx .0902.$$

Deriving a contradiction, part 1

So on the one hand we have the evaluation of an integral yielding something of the form

$$I_n = b_n\zeta(2) - \frac{p_n}{d_n^2} \tag{16}$$

and on the other hand we have a bound for the integral in this form

$$|I_n| \leq \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2). \tag{17}$$

With the assumption that $\zeta(2)$ is rational, we can get (16) is of the form

$$\frac{q_n\zeta(2) - p_n}{d_n^2} = \frac{q_n p - q p_n}{q d_n^2}.$$

Putting this with (17), we have

$$\frac{q_n p - q p_n}{q d_n^2} \leq \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2)$$

or

$$q_n p - q p_n \le q d_n^2 \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2) = p d_n^2 \left(\frac{\sqrt{5}-1}{2}\right)^{5n}.$$

Now if we can get an upper bound for d_n^2 such that when multiplied by an upper bound for Φ^{5n} gives a fraction less than one to the nth power, we could get a contradiction, as the far left side is an integer.

Deriving a contradiction, part 2

Now d_n is the least common denominator of the integers 1 through n. We can get an interesting way to calculate this using the fundamental theorem of arithmetic:

$$d_n = \prod_{p \le n} p^{Int(\log n / \log p)}.$$

So for $n = 10$

$$\begin{aligned} d_n &= \prod_{p \le 10} p^{Int(\log n / \log p)} = 2^{Int(\log_2 10)} 3^{Int(\log_3 10)} 5^{Int(\log_5 10)} 7^{Int(\log_7 10)} \\ &= 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \end{aligned}$$

and this is correct; the least common multiple of the set of natural numbers $\{1, \ldots, 10\}$ is $2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1$. Now if we remove the Int, the integer floor function, we can form inequalities:

$$d_n = \prod_{p \le n} p^{Int(\log n / \log p)} \le \prod_{p \le n} p^{\log_p n} = \prod_{p \le n} n = n^{\pi(n)} = e^{\ln n^{\pi(n)}} = e^{\pi(n) \ln n}$$

where $\pi(n)$ is the number of primes less than or equal to n. We'd like a bound on this. According to the prime number theorem $\pi(n) \sim \frac{\ln n}{n}$, so $\pi(n) \ln n \sim n$. This means for a given epsilon we can find N_ϵ such that

$$\pi(n) \ln n \le (1 + \epsilon) n,$$

for all $n > N_\epsilon$, so we have an upper bound for d_n:

$$d_n \le e^{(1+\epsilon)n}.$$

Now $e = 2.71828 \cdots < 3$, so we can safely say $d_n < 3^n$ and $d_n^2 < 9^n$.

Deriving a contradiction, part 3

We need to find an upper bound for

$$\left(\frac{\sqrt{5}-1}{2}\right)^{5n} 9^n = \left(\sqrt{5}-1\right)^{5n} \left(\frac{9}{32}\right)^n.$$

Using $\sqrt{5} < 2.24$, we have $\sqrt{5}-1 < 1.24 = 31/25$. Some calculations show

$$\left(\frac{31^5}{25^5}\right)\left(\frac{9}{32}\right) < \frac{5}{6},$$

so

$$\left(\frac{\sqrt{5}-1}{2}\right)^{5n} 9^n < \left(\frac{5}{6}\right)^n$$

Conclusion

A follow up document will cover the irrationality of $\zeta(3)$.

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