# Evaluating the Alignment of the Polarized Starlight from 99 Stars in a Region off the Disk of the Milky Way

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Abstract

Detecting polarized starlight projects an intriguing pattern of polarization directions on the Galaxy. Polarized starlight is a well known tracer of Galactic Magnetic fields and acts as a tool for understanding the electrodynamics of the dust that contaminates the view of more distant objects. The polarization data is taken from the Heiles 2000 agglomeration catalog appended with data from the Berdyugin 2014 catalog. Here, the alignment of the polarization directions of a sample of stars well off the Galactic Disk is investigated with a recently devised test. The sample of 99 stars, located from Galactic longitude 15° to 35° and latitude 23° to +35°, is one of the more highly aligned regions among the many significantly aligned regions in the Galaxy. As determined by the Hub Test, the alignment occurs at the 20$\sigma$ level, so chance alignment can be ruled out. The Hub Tests's alignment function neatly separates the Celestial Sphere into four parts, two of alignment and two of avoidance, providing a full-sphere depiction of the collective alignment. This article is a Mathematica notebook which can be accessed via a currently viable link in the References.

Keywords: Polarized Starlight; Alignment; Computer Program; Uncertainties; Hub Test; Galactic Structure; Galactic Magnetic Field

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## 0. Preface

The pdf version of this notebook is available online from the viXra archive.
To find the ready-to-run notebook follow the link in Ref. 1. The notebooks in this series were created using Wolfram Mathematica, Version Number: 12.1, Ref. 2.

Note(s):

(1) The numerical quantities in the pdf version can be replaced with user-generated numerics by processing the Mathematica notebook in Ref. 1. These different sets of random runs, and uncertainty runs, should yield results that, for a reasonable number of runs, differ only slightly.

(2) This notebook is based on Galactic coordinates with most figures, all but Fig. 4, drawn as the sky is seen from the ground, North is upward and East is to the left. Polarization position angles are measured counterclockwise from North with East to the left.

(3) A similarly organized notebook in Equatorial Coordinates, with views from outside the Celestial Sphere, as one views a weather map, with North upward and East to the right, can be found in Ref. 11.

CONTENTS

Part 1: the Article

1. Introduction

For those interested in the structure of the Milky Way, polarized starlight infers the direction of the Galactic magnetic field, see for example, Refs. 3 & 4. For those interested in deep space objects on the far side of the Galaxy, polarized starlight helps uncover the physics of the contaminating dust that obscures the more distant objects of interest, see, for example, Refs. 5 & 6.

The polarization of starlight is a well-known phenomenon that has been important in understanding the structure of the magnetic field of the Milky Way Galaxy. The scale of the magnetic fields is large enough that starlight from regions containing large numbers of stars should confront similar environments on the way to being detected. So, it is not surprising to find that the polarization directions for stars in many good-sized regions of the Galaxy are parallel.

The stellar data is taken from the catalogs Heiles 2000 and Berdyugin 2014, Ref. 7 - 10. To keep this notebook largely self-contained, the needed catalog data is included in Part II Appendix Sec. 3.

The sample is chosen because it is off the Galactic Disk and has extreme alignment behavior. An irrelevant, inconsequential feature distinguishes this particular sample: an accident of nature has put the alignment direction closely coincident with the Celestial Equator. Compare the Celestial Equator in Galactic Coordinates with one of the gray Great Circles in Fig. 5. A glance at Fig. 2 shows that there are many other significantly aligned off-disk regions that could have been chosen to be explored here.

This work looks at a very significantly aligned sample of 99 stars occupying a region about 30° off the Galactic Disk. The stars' polarization directions are aligned at the $20\sigma$ level, with the chance that the alignment is random being nil. The alignment is evident from simply plotting the polarization directions, Figs. 6 and 8. The stars in the sample and the surrounding region are well-known to be polarized in the general direction from southeast to northwest, compare Fig. 4 in Ref. 4 and Fig. 3 in Ref. 9, for example, with Figs. 6 and 8 here.

Applying an alignment test provides a numerical basis to judge the evident alignment as well as supplying other data. For example, the Hub Test locates where on the Celestial Sphere these polarization directions converge. Also, analyzing this sample with the Hub Test provides another demonstration of that test's numerical metrics. A previous application to a complementary situation, one with hubs close to the sample, can be found in Ref. 11.

The sample selection process and the Hub Test is described in Sec. 2 of this article, not to be confused with Sec. 2 of the Appendix. In Sec. 3 the alignment function is discussed and depicted. Sec. 4 describes the propagation of experimental uncertainties through the calculations. In Sec. 5, the random runs are generated and analyzed. These are needed to assess the significance of the alignment. Some concluding remarks in Sec. 6 mark the end of Part I the Article.

After Part I the Article, we attach Part II the Appendix which displays a Mathematica notebook that produces, and in part verifies, the results discussed in this study.

2. Sample selection and the Hub Test

The Selection Process:

The data that underlies the present study is available online in two catalogs, Heiles 2000 and Berdyugin 2014, Refs. 7 - 10. Without access to their work, this study would not be possible. Fig. 1 gives an idea of the data set by color-coding the average polarization directions of a set of 5° radius regions in a survey described below.

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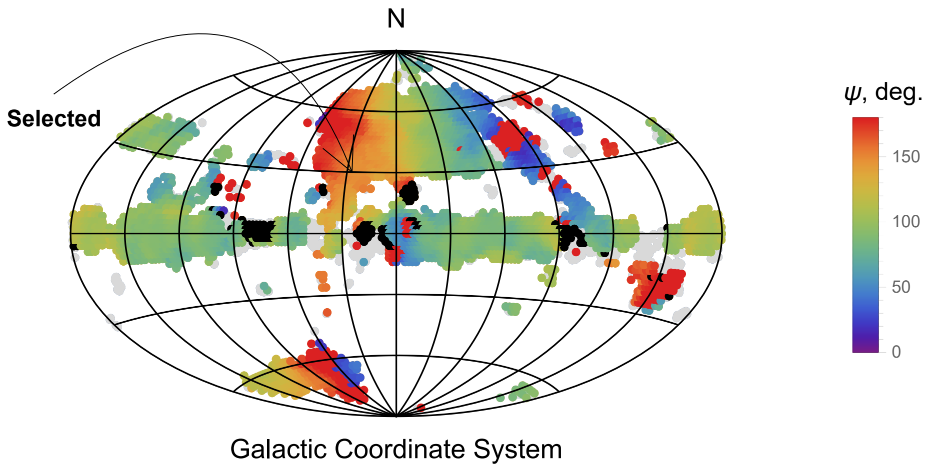


Figure 1. The catalog data represented as color-coded polarization directions. The average polarization direction $\psi$ for 5° radius regions is shown, each region is centered on the grid point of a 2° mesh. Regions with wide-ranging $\psi$, $\Delta\psi > 40°$, are shaded Black, ■ . Since polarization is not oriented, the red 180° direction is equivalent to the purple 0° direction, so red wraps around into purple, ■ = ■ or South = North. Note the alternating Blue, Red, Blue, ..., stripe along the northern -90° meridian. Where the regions on the Disk favor green, ■ or $\psi$ = 90°, it means that $\psi$ points parallel to the Disk.

To find suitable regions to investigate, we conduct a survey. We create 5° radius regions on an evenly-spaced, whole-sphere, 10,518-grid-point mesh, see Fig. 12. For valid statistics, we require at least 7 stars in a region. It turns out that there are 3632 populated regions with from seven to 314 stars each. Of these, 2983 had very significant alignment. Clearly, Galactic stars form an abundant resource for polarization studies. By "very significant", we mean that the significance is less than one percent, sig $\leq$ 1% = $1 \times 10^{-2}$, and at most one in a hundred samples with randomly directed polarization directions would be equally well aligned. See Figs. 2 and 3 for topographical maps of the significance of the 2983 very significantly aligned 5° radius regions.

To aid in visualizing the significance, let us define a quantity "$h$" that increases as the polarization directions become better aligned. The significance, "Sig", of a quantity is the chance that random data would produce a better result. It follows that the better the result, the less likely random data would produce it and the significance is less. To map significance conveniently we invert it by defining the quantity "h", height, for a given significance "Sig", by the formula

$$h = -\log_{10}(\text{Sig}) \text{ or } \text{Sig} = 10^{-h}. \qquad (1)$$

The function $h$ is mapped in Figs. 2 and 3.

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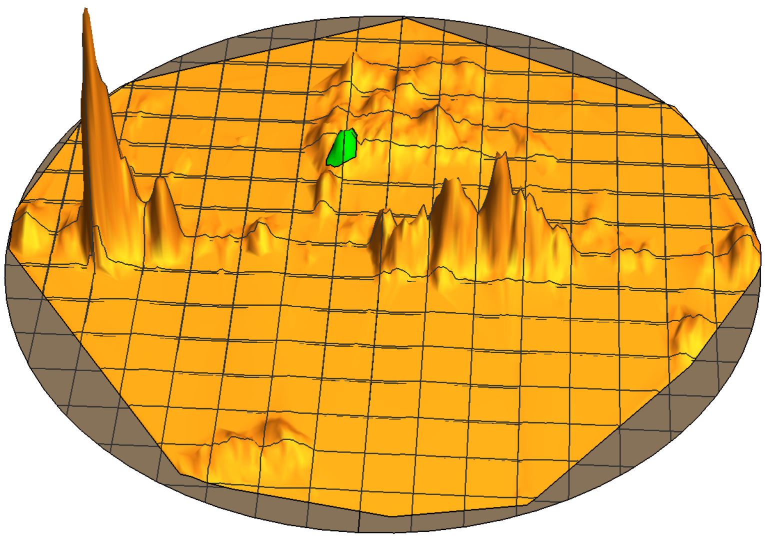

Figure 2. *A whole-sky plot of the significance survey, Galactic Coordinates centered on (gLON,gLAT) = (0,0), East to the Left, Aitoff Plot.* The heights $h$ are the negative logarithm of the significance, by Eq. 1, so that the more significant 5° radius regions are higher. The largest peaks occur along the Galactic Disk. The top-most peak on the left, East, rises to a value of about $h = 250$, meaning that fewer than one in $10^{250}$ randomly directed regions would have better aligned polarization directions. The hill shaded Green is composed of 35 of the 5° radius regions, located off-Disk and Northeast of the Galactic Center. The significances of the alignments of the 5° radius regions are determined by the Hub Test.

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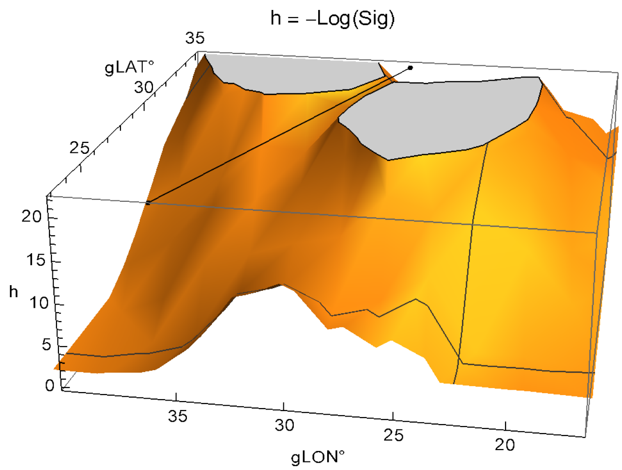


Figure 3. The map of $h$ near the sample. The hilltops are flat because they are cut-off at $h = 22$, *i.e.* Sig = $10^{-22}$. The line follows the valley between the two hills. The sample selected for study, shaded green in Fig. 2, is to the right of the line. The hill consists of 35 regions. The polarization directions of the 99 stars in the 35 regions, when collected together, are much more significantly aligned than any one of the 35 regions.

The 99 stars selected for the sample studied are all the stars in 35 regions. Each such region satisfies the following conditions: (*i*) 7 or more stars, (*ii*) longitude $17° \leq$ gLON $\leq 34°$, (*iii*) latitude $23° \leq$ gLAT $\leq 35°$, (*iv*) gLAT $< 23° - 1.5$(gLON - 36°), and (*v*) whose stars align with a significance less than a billionth, sig $\leq 10^{-9}$ and $h > +9$. Requirement (*iv*) separated the region of interest from an adjacent peak, as shown in Fig. 3.

The Hub Test

The Hub Test is discussed more fully in Ref. 12. The basic idea is analogous to a well-known guide to find Polaris, the North Star. Assume one can find the stars Merak and Dubhe which are two stars in the constellation Ursa Major. Then the direction from Merak to Dubhe aligns with the direction from Merak to Polaris. While Fig. 4 is not drawn for this case, with the labelling of Fig. 4, let the source *S* be the star Merak, take the direction from Merak to Dubhe to be the direction of polarization $\hat{v}_\psi$, and let Polaris be the point *H*. Then the alignment of the Merak-to-Dubhe direction $\hat{v}_\psi$ with the direction toward Polaris, the point *H*, illustrates the concept of alignment in the Hub Test. The alignment angle $\eta$ for Merak-Dubhe and Merak- Polaris would be about $\eta = 3.47°$ and the blue great circle would almost coincide with the purple great circle .

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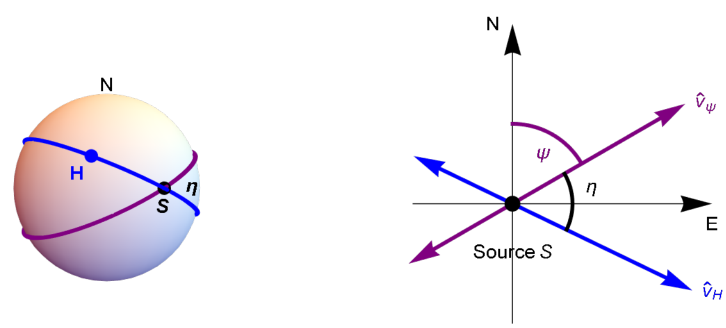


Figure 4: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source *S*. The linear polarization direction $\hat{v}_\psi$ lies in the tangent plane and determines the purple great circle on the sphere. A point *H* on the sphere together with the point *S* determine a second great circle, the blue circle drawn on the sphere. Clearly, *H* and *S* must be distinct in order to determine a great circle. The angle $\eta$ measures the alignment of the polarization direction $\psi$ with the point *H*.

In Fig. 4, the “alignment angle” $\eta$ is the acute angle $\eta$ between two great circles at *S*, $0° \leq \eta \leq 90°$ . The alignment angle $\eta$ measures how well the polarization direction $\hat{v}_\psi$ matches the direction $\hat{v}_H$ toward the point *H*. Perfect alignment occurs when $\eta = 0°$ and the two great circles overlap. Perpendicular great circles, $\eta = 90°$, indicates maximum “avoidance” of the polarization direction $\hat{v}_\psi$ from the point *H* on the sphere. The halfway value, $\eta = 45°$, favors neither alignment nor avoidance of $\psi$ with *H*.

With *N* sources $S_i$, $i = 1, ..., N$, there are *N* alignment angles $\eta_{\mathrm{iH}}$ at each point *H* . One can calculate an average alignment angle $\overline{\eta}$ at *H*,

$$\overline{\eta}(\mathrm{H}) = \frac{1}{N} \sum_{i=1}^{N} \eta_{\mathrm{iH}} , \qquad (2)$$

where

$$\cos(\eta_{\mathrm{iH}}) = | \hat{v}_\psi . \hat{v}_H | . \qquad (3)$$

Each angle $\eta_{\mathrm{iH}}$ is taken to be the acute angle from Eq. 3. The average of acute angles is acute, so, the average alignment angle $\overline{\eta}(\mathrm{H})$ at the point H must be acute.

The alignment angle $\overline{\eta}(\mathrm{H})$ is a function of position *H* on the sphere. It is symmetric across diameters, $\overline{\eta}(\mathrm{H}) = \overline{\eta}(\mathrm{-H})$, because great circles are symmetric across diameters. The function $\overline{\eta}$(H) measures convergence and divergence of the great circles determined by the polarization directions. For random polarization directions, the average $\overline{\eta}$(H) should be near 45°, since each alignment angle $\eta_{iH}$ is acute, $0° \leq \eta_{iH} \leq 90°$, and random polarization directions should not favor any one value. Points $H$ where the alignment angle $\overline{\eta}$(H) is smaller than 45°, the great circles of the polarization directions tend to converge, where $\overline{\eta}$(H) is larger than 45°, the great circles of the polarization directions can be said to diverge.

In this article and notebook, the terms "min" and "max" are often given special meanings. We often use "min" to label the smallest alignment angle $\overline{\eta}_{\min}$ and the associated points on the sphere, the "hubs" $H_{\min}$ and $-H_{\min}$. Thus "min" is associated with the best convergence of the polarization directions. For divergence, the hubs $H_{\max}$ and $-H_{\max}$ locate places that the polarization directions avoid, the places with the largest alignment angle $\overline{\eta}_{\max}$. Thus, we very often label an avoidance related quantity with "max".

3. The alignment of the polarization directions

The polarization directions of the 99 stars have the same position angle within a few degrees and with very few exceptions. The data is displayed in Figs. 6 and 8 as short lines at the sources. Even without any calculations, it is clear that the polarization directions are well-aligned. The alignment angle function $\overline{\eta}$(H) in Eq. 2 is calculated in Part II Appendix Sec. 5b,c, and makes the following contour maps. The polarization directions converge in the Blue regions and the directions are least dense in the Red regions of the map.

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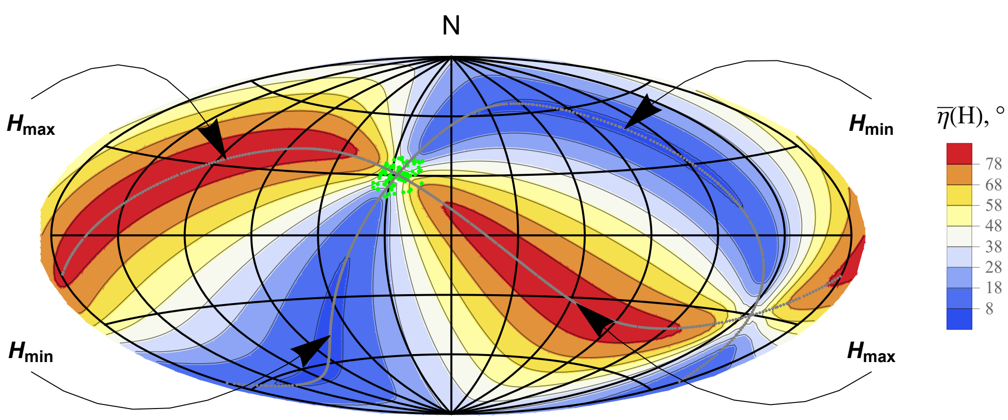


Figure 5: The alignment angle function $\overline{\eta}$(H) mapped on the Celestial Sphere (Aitoff plot, Galactic Coordinates centered on (gLON,gLAT) = (0,0) , East to the left). The sources are shaded green ■. To guide the eye, two Great Circles are plotted in gray, one through the sources' center point and the avoidance hubs $H_{\max}$ and $-H_{\max}$ while the other Great Circle runs through the sources' enter and the alignment hubs $H_{\min}$ and $-H_{\min}$. The smallest alignment angle, $\overline{\eta}_{\min}$ = 7.01°, is located at the hubs $H_{\min}$ and $-H_{\min}$, where the polarization directions converge best. The hubs $H_{\max}$ and $-H_{\max}$ pinpoint the largest avoidance angle $\overline{\eta}_{\max}$ = 83.12° where the polarization directions diverge.

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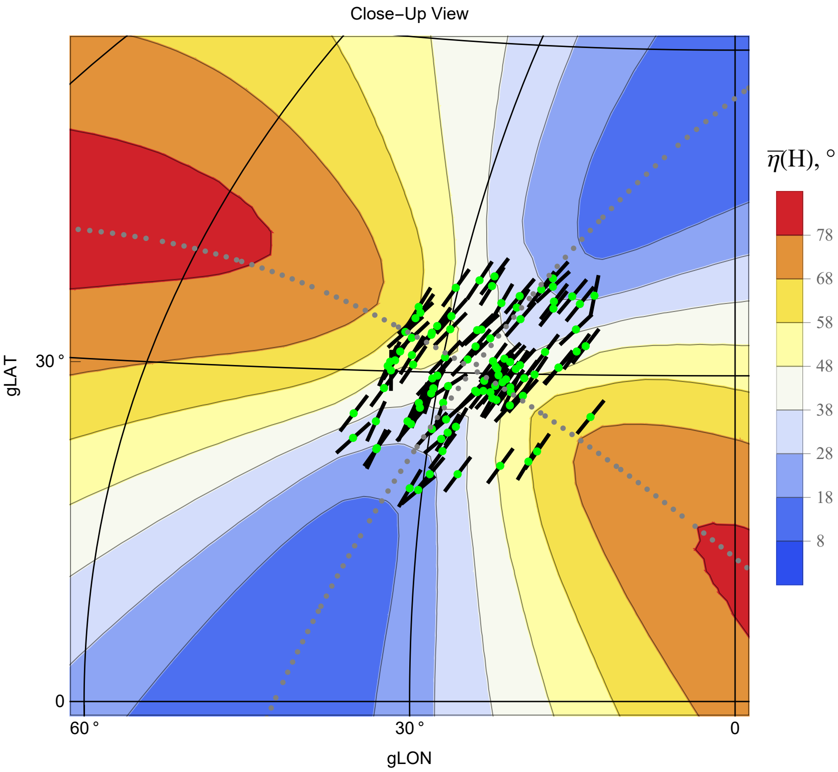


Figure 6: The region near the sources. The sources are located at the green dots. The short black lines indicate the polarization directions.  Measuring polarization directions $\psi$ counterclockwise from North, with East to the Left, one sees that the angles $\psi$ point in the direction of $\psi = 150°$ , in agreement with Fig. 10. The alignment of these polarization directions with one another is apparent.

By Fig. 6, one sees that the polarization directions are parallel with one another. This direct comparison is the basis for a different type of tests than the Hub Test, the 'S' and 'Z' tests, Refs. 13,14,15.  With the Hub Test, one determines alignment of the polarization directions with points on the Celestial Sphere, thereby making the Hub Test an indirect measure of alignment. The Hub Test supplements these other tests by finding alignments in cases where the polarization directions focus on a nearby point and have considerable parallax, unlike the case here. See, for example Ref. 11. Whether direct or indirect, any alignment test should find these 99 stars' polarization directions well-aligned.

In sections 4 and 5 we will quantify the alignment by calculating various numerical values. In the next section, Sec. 4, we find how accurately the polarization directions determine the smallest alignment angle $\overline{\eta}_{\text{min}}$ and other quantities, based on the experimental uncertainties for the polarization directions $\psi$ listed in the catalogs. In the following section, Sec. 5, the experimental data for the $\psi$ is set aside, replaced by random values inserted for $\psi$. The process in Sec. 5 determines the significance of the alignment of the observed polarization directions $\psi$.

## 4. Experimental uncertainty

The data reported in the Heiles 2000 and Berdyugin 2014 catalogs include the experimental measurements of the polarization directions together with their experimental uncertainties recorded alongside. In Part II the Appendix Sec. 6, below, the uncertainties are carried through the calculations yielding the uncertainties in the results. We discuss those results in this section.

An observed polarization direction and its uncertainty makes a probability distribution, a normal distribution, *i.e.* a Gaussian that integrates to unity. For example, one of the stars, the sixth one, has a measured polarization position angle of $\psi_{\text{obs}} \pm \sigma = 152.70° \pm 0.90°$.  We take this to mean that the probability that the actual value of $\psi$ was some value $\psi_1$, and possibly not $\psi_{\text{obs}} = 152.7°$, is given by the Gaussian

$$P(\psi_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{\psi_1-\psi_{obs}}{\sigma}\right)^2\right]. \qquad (4)$$

Random values of $\psi_1$ consistent with the probability distribution in Eq. (4) are determined with the special command "RandomVariate", which is part of the Mathematica software.

An "uncertainty run" begins by selecting a set of 99 polarization directions $\psi$ for the 99 stars, each $\psi$ conforming to the star's uncertainty distribution, Eq. (4). The alignment angle function $\overline{\eta}$(H) in Eq. (2) is evaluated to find the smallest alignment angle $\overline{\eta}_{min}$ for the uncertainty run. Small changes to the observed polarization directions should make small changes to the resulting angle $\overline{\eta}_{min}$. Repeating the process makes many uncertainty runs and one obtains a distribution of values for the smallest alignment angle $\overline{\eta}_{min}$, as well as other quantities of interest.

With the many uncertainty runs constructed for this article, the resulting distribution of the smallest alignment angle $\overline{\eta}_{min}$ is displayed in Fig. 7. Other quantities, such as the locations of alignment hubs, also spread out in their own distributions. Each of these distributions has a mean value and a distribution width which can be combined to summarize the distribution. For the smallest alignment angle $\overline{\eta}_{min}$ in Fig. 7 we write $\overline{\eta}_{min} = 7.25° \pm 0.29°$. As noted previously, the recorded polarization directions $\psi_{obs}$, the "best" values of $\psi$, give the observed value, $\overline{\eta}_{min} = 7.01°$, and that value makes the low end of the range determined by experimental uncertainty. Thus, the best recorded polarization directions $\psi_{obs}$ listed in the catalogs produce a value, $\overline{\eta}_{min} = 7.01°$, that is about one sigma lower than the most likely value, $\overline{\eta}_{min} = 7.25°$, obtained by invoking experimental uncertainty.

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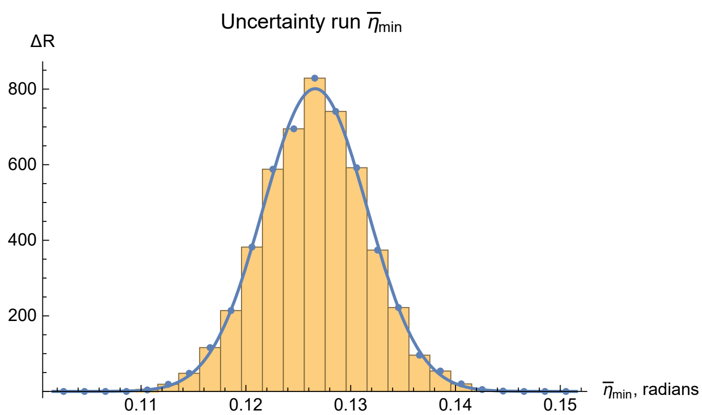


Figure 7: Histogram of the smallest alignment angle $\overline{\eta}_{min}$ for R = 10,000 uncertainty runs. The height ΔR is the number of uncertainty runs with a value of $\overline{\eta}_{min}$ in the 'bin', the range covered by each bar. This Gaussian distribution peaks at a mean value of $\overline{\eta}_{min}$ of 0.1266 radians = 7.25° and has a half-width of $\sigma = 0.0050 = 0.29°$, measured at points on the distribution that are down from the peak by a fraction $e^{-1/2} = 0.607 = 60.7\%$. One writes the result as $\overline{\eta}_{min} = 0.1266 \pm 0.0050$ radians $= 7.25° \pm 0.29°$.

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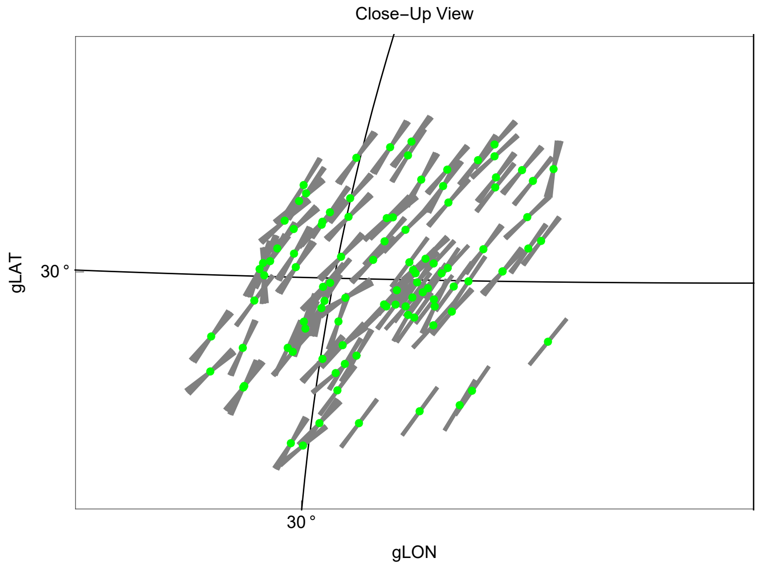


Figure 8: The stars as green dots plotted with the experimental uncertainties in polarization directions. Each polarization direction is plotted three times: once for the best direction $\psi_{obs}$, once for the angle $\psi_{obs} + \sigma\psi$, once for the angle $\psi_{obs} - \sigma\psi$; these are the one-sigma plus/minus values.

Besides the uncertainty in the smallest alignment angle $\overline{\eta}_{min}$, the uncertainty runs yield uncertainty ranges for other quantities such as the largest avoidance angle $\overline{\eta}_{max}$. Each uncertainty run has its own set of alignment and avoidance hubs, $H_{min}$ and $H_{max}$, respectively. See Part II Appendix Sec. 6, for more detail. A plot of the polarization directions with their uncertainties is displayed in Fig. 8.

5. Significance

We need to determine the significance of the alignment found for the polarization directions of these 99 stars. ‘Significance’ means how likely it is that randomly directed polarization vectors would give the same or better alignments than the observed polarization directions give.

This section involves a process that is much like that in the previous section. Finding the uncertainties required repeating the underlying calculations over-and-over many times to obtain sufficiently accurate statistical results. In this section, similarly, the underlying calculations are carried through repeatedly, but we discard the experimentally observed $\psi$ values. Instead of experimental values of $\psi_{obs}$, one substitutes $\psi$ that are chosen at random from the allowed range of values of $\psi$, $0 \leq \psi \leq 180°$. The goal is to see what fraction of random runs yields a value with a lower $\overline{\eta}_{min}$ than the experimentally observed value $\overline{\eta}_{min} = 7.01°$ obtained with the measured data.

For this article we completed 10,000 random runs. By sorting those 10,000 runs by the value of $\overline{\eta}_{min}$, smaller $\overline{\eta}_{min}$ before larger $\overline{\eta}_{min}$, one can find how many of those 10,000 runs gives a smaller alignment angle $\overline{\eta}_{min}$ than the observed value of $\overline{\eta}_{min}$. However, the smallest $\overline{\eta}_{min}$ of these 10,000 random runs is $\overline{\eta}_{min} = 33.2°$, which is far larger than the observed $\overline{\eta}_{min} = 7.01°$. Clearly, we would need many more random runs for such simple considerations to produce a value of significance.

Rather than expending a large amount of computer time to follow the simple straightforward method, we make assumptions. We manipulate the 10,000 random runs. We start by finding a function that fits the distribution of the 10,000 values of $\overline{\eta}_{min}$; there is one smallest alignment angle $\overline{\eta}_{min}$ per random run. Having found a function that fits the distribution, we make the assumption that the fit accurately describes the distribution far down the “tail” of the function where our well-aligned stars have their $\overline{\eta}_{min}$ .

A histogram of the resulting smallest alignment angles $\overline{\eta}_{\mathrm{min}}$ from 10,000 runs is displayed in Fig. 9. To see the observed result, we must change the scale on the $\overline{\eta}_{\mathrm{min}}$-axis so that the blue arrow is visible in the figure on the right.

There is a technical problem. Look closely at the distribution on the left in Fig. 9. The side toward $\overline{\eta}_{\mathrm{min}} \to \pi/4 \sim 0.79$ , has a steeper slope than the side toward $\overline{\eta}_{\mathrm{min}} \to 0$. Thus, the low $\overline{\eta}_{\mathrm{min}}$ side is favored and probability is pushed from the right side to the left side. A simple, symmetrical Gaussian would not fit the data well. The fitting curve shown combines a Gaussian with a unit step-function, that is unity to the left, and zero to the right, of the peak. Since the sample has an alignment angle $\overline{\eta}_{\mathrm{min}}$ that is about 0.13 radians, it occurs far down the tail of the curve on the side where the step-function is unity and the curve is a Gaussian. In fact, it occurs so far down the tail that any nuances concerning asymmetric distributions are inconsequential.

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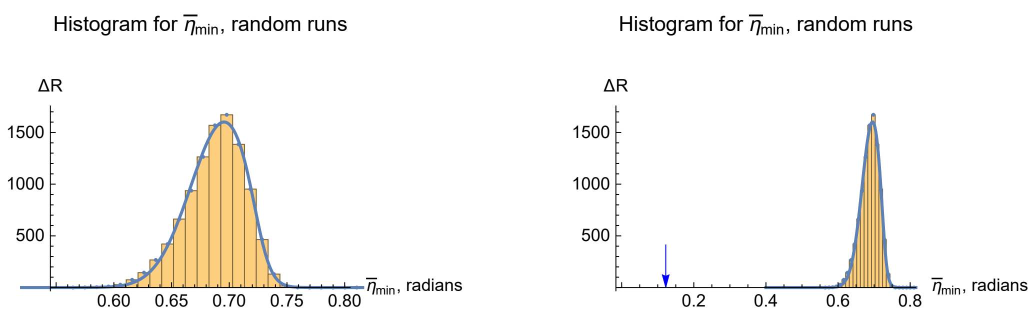


Figure 9. *Left*: The distribution of the smallest alignment angle $\overline{\eta}_{\mathrm{min}}$ for R = 10,000 random runs. Each run assigns a random polarization direction to each of the 99 stars. The height ΔR is the number of runs with $\overline{\eta}_{\mathrm{min}}$ in the designated range of each bin. The fraction ΔR/R represents the likelihood that a random run result $\overline{\eta}_{\mathrm{min}}$ is in the bin. Thus the histogram approximates the shape of the probability distribution, aside from a normalizing scale factor. *Right:* The observed polarization directions determine a value of $\overline{\eta}_{\mathrm{min}}$ at the blue arrow far down the tail, so far down the tail that the $\overline{\eta}_{\mathrm{min}}$-axis scale is reset so you can see it.

To find the significance of the observed smallest alignment angle $\overline{\eta}_{\mathrm{min}} = 7.01°$, we integrate the probability distribution to find the likelihood that a random run would produce a smaller value. The significance is essentially zero with a calculated value of Sig. = $10^{-80}$ ( $\approx$ 0), and about one in $10^{+80}$ ( think $\infty$) random runs with a smaller alignment angle $\overline{\eta}_{\mathrm{min}}$. It is, therefore, an understatement to conclude that the alignment of the polarization directions with the hub $H_{\mathrm{min}}$ is very significant.

Alternatively, since the peak $\eta_0$ of the random-run $\overline{\eta}_{\mathrm{min}}$ distribution in Fig. 9 and the half-width $\sigma$ are well-defined values, we can describe the significance in terms of these two quantities. Since $\eta_0 = 39.94°$ and $\sigma = 1.74°$ , which are the values found below in Part II the Appendix Sec. 7c, we see that the observed smallest alignment angle $\overline{\eta}_{\mathrm{min}} = 7.01°$ is 18.9$\sigma$s below the most likely random run value.

Avoidance is also significant, with the observed largest avoidance angle $\overline{\eta}_{\mathrm{max}} = 83.12°$ occurring 18.7$\sigma$s above the most likely random run value. The generally accepted standard is 5$\sigma$s, so both alignment and avoidance are well-confirmed to be highly significant for the polarization directions of the 99 stars in this sample.

6. Concluding Remarks

If the 99 stars in the sample had randomly directed polarization directions, they would have more widely scattered polarization directions with an alignment angle near 40° and an avoidance angle near 50°. The observed polarization directions converge to a value of $\overline{\eta}_{\min}$ of about 7° and an avoidance angle $\overline{\eta}_{\max}$ of about 83°. Both for alignment and for avoidance, *i.e.* for the observed $\overline{\eta}_{\min}$ and $\overline{\eta}_{\max}$ , the results occur about 20$\sigma$s from the results with random polarization values. The significance is infinitesimal, about $10^{-83}$ or less. One concludes that the alignment is not explained by chance. Note that the Hub Test supplies numerical values to help make that determination.

The 99 stars in this region have polarization correlations that illustrate an extreme case. With samples of Galactic stellar polarization, the extreme case may not be unusual. However, when, as here, the polarization directions are all nearly equal, the smallest alignment angles $\overline{\eta}(H)$ arrange themselves along an "equator" , a Great Circle moving away from the sources in the direction of the best convergence point at the alignment hubs $H_{\min}$ and $-H_{\min}$ . The sample is an extreme case because the directions perpendicular to the polarization directions are also well-correlated, making a second Great Circle through the avoidance hubs $H_{\max}$ and $-H_{\max}$. See the grey Great Circles in Fig. 4. The two Great Circles are perpendicular within experimental uncertainty at the points of intersection. The Hub Test treats alignment and avoidance equally, equally uncovering significant correlations of a sample's alignment as well as its avoidance. In the extreme case here, both alignment and avoidance are remarkably strong.

Just 45 of the 99 stars have distances listed in the catalogs. The closest of these is 63 parsecs, while the furthest is 725 pc. All but 4 are within about 400 pc, about 5% of the distance to the Galactic Center. On the scale of the Galaxy, these 45 of the 99 stars with known distances are fairly close to the Sun.

One hopes the results are of interest and potentially useful.

## 7. References


1. R. Shurtleff, the ready-to-run Mathematica version of this notebook is available at the following URL:
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# Part II: the Appendix, a Computer Program

## 1. Introduction

The following computer program, a Mathematica notebook, performs the calculations made to evaluate the alignment of the sources in the sample under consideration.

Since Mathematica encodes the instructions, it is inconvenient to try to run the computer program from the pdf version of this work. A viable .nb version that runs on Mathematica is available by following the link in Ref. 1.

## 2. Coordinates, utility functions, derivation of basic formula

### 2a. Coordinates

Consider the “Celestial Sphere”, a sphere in 3 dimensional Euclidean space. See Figs. 1,2,4,5,13. The sphere is also called the “sphere” or sometimes “the sky”. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates ($x$, $y$, $z$). The direction of the positive $z$ -axis is due “North”. Galactic longitude, gLON and latitude, gLAT, are measured as in the Heiles 2000 catalog with the direction of the positive $x$-axis along (gLON,gLAT) = (0°, 0°). The similar appearance of the letter “*l*” and the number “1” when typed keep us from using the ($l$,$b$) notation.

The view of the Galaxy is generally from inside the sphere, let us say from the origin to be specific. Then the direction of increasing gLON, *i.e.* local East, is to the left with up toward North. Latitude gLAT = 90° indicates the North Galactic Pole, the direction from the origin (0,0,0) to (0,0,1) . We do not use the conventional *UVW* notation. When Galactic Coordinates are used in this article, the viewpoint is from inside the sphere, looking up at the sky, so, with North upward, East is to the Left.

Somewhat contrarily, from a point-of-view located outside the sphere, as in the sketch in Fig. 4, one pictures a source $S$ plotted on the sphere and, in the 2D tangent plane at $S$, local North is upward and local East is to the right. A “position angle” at the point $S$ on the sphere, such as the angle $\psi$ in Fig. 4, is measured in the 2D plane tangent to the sphere at $S$. In the tangent plane as drawn in Fig.4, the position angle $\psi$ is measured clockwise from local North with East to the right.

It is important to note that from a point of view inside the sphere, position angles are measured counterclockwise from North, since increasing gLON, *i.e.* East, is to the left when viewed from inside the sphere. But, for some purposes, it is much easier to draw a sphere from the outside looking inward, as with Fig. 4.

Definitions:

er, eN, eE are unit vectors in a 3D Cartesian coordinate system
(gLON,gLAT) = galactic longitude and latitude
er(gLON,gLAT) = radial unit vectors from Origin
eN(gLON,gLAT) = local North at a point on the Celestial Sphere
eE(gLON,gLAT) = local East at a point on the Celestial Sphere
gLONFROMr(er) = gLON determined by radial unit vector er
gLATFROMr(er) = gLAT determined by radial unit vector er

The following Aitoff Plot formulas can be found in Wikipedia, Ref. 16.
$\alpha$H(gLON,gLAT) , xH(gLON,gLAT) , yH(gLON,gLAT), where xH is centered on gLON = 0 and gLON increases from left-to-right.
xH180(gLON,gLAT) , yH180(gLON,gLAT), where xH is centered on gLON = 180° and gLON increases from left-to-right.
xHGal(gLON,gLAT) , yHGal(gLON,gLAT), where xH is centered on gLON = 0 and gLON increases from right-to-left, so gLON = +180° is on the left and gLON = -180° is to the right.

The mean and standard deviation are convenient functions. And we identify directories for getting and putting data.

mean — the arithmetic average of a set of numbers, $\frac{1}{N}\sum_{i=1}^{N} n_i$

stanDev — the standard deviation. Given a set of *N* numbers $n_i$ with mean value *m*, the standard deviation is $\left(\frac{1}{N}\sum_{i=1}^{N}(n_i - m)^2\right)^{1/2}$, the square root of the average of the squares of the differences of the numbers with the mean. Note that we divide by *N* to get the average of the deviations squared.

```
In[433]:= Print["The computer time expended so far is ", TimeUsed[], " seconds."]
```

```
The computer time expended so far is 91.655 seconds.
```

```
In[434]:= homeDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\
   20210221StellarPolarization\\20210221Notebooks\\20210228GalacticCoordsNotebooks\\
   20210320Lon30Lat30offDisk";
 (*The notebook file and data files for this notebook are put in this directory. *)
```

```
In[435]:= (* For a Source at (gLON,gLAT) = (gLON,gLAT): er, eN,
eE are unit vectors from Origin to Source, local North, local East, resp.  *)
er[gLON_, gLAT_] := er[gLON, gLAT] = {Cos[gLON] Cos[gLAT], Sin[gLON] Cos[gLAT], Sin[gLAT]}
eN[gLON_, gLAT_] := eN[gLON, gLAT] = {-Cos[gLON] Sin[gLAT], -Sin[gLON] Sin[gLAT], Cos[gLAT]}
eE[gLON_, gLAT_] := eE[gLON, gLAT] = {-Sin[gLON], Cos[gLON], 0}
{"Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN
   = 1, eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: ",
 {0} == Union[Flatten[Simplify[{er[gLON, gLAT].er[gLON, gLAT] - 1, er[gLON, gLAT].eN[gLON, gLAT],
     er[gLON, gLAT].eE[gLON, gLAT], eN[gLON, gLAT].eN[gLON, gLAT] - 1, eN[gLON, gLAT].
      eE[gLON, gLAT], eE[gLON, gLAT].eE[gLON, gLAT] - 1, Cross[er[gLON, gLAT], eE[gLON, gLAT]] -
      eN[gLON, gLAT], Cross[eE[gLON, gLAT], eN[gLON, gLAT]] - er[gLON, gLAT],
     Cross[eN[gLON, gLAT], er[gLON, gLAT]] - eE[gLON, gLAT]}]]]}
```

```
Out[438]= {Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN = 1,
   eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: , True}
```

Get (gLON,gLAT) in radians from a radial vector r:

```
In[439]:= gLONFROMr[r_] := N[ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] > 0)
gLONFROMr[r_] := N[π - ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] < 0)
gLONFROMr[r_] := N[-π + ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] < 0)
gLONFROMr[r_] := N[-ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] > 0)
gLONFROMr[r_] := π / 2. /; (r[[2]] ≥ 0 && r[[1]] == 0)
gLONFROMr[r_] := -π / 2. /; (r[[2]] < 0 && r[[1]] == 0)
```

In[445]:=
```
gLATFROMr[r_] := N[ArcTan[r[[3]] / (√(r[[1]]^2 + r[[2]]^2))]] /; (√(r[[1]]^2 + r[[2]]^2) > 0)
gLATFROMr[r_] := Sign[r[[3]]] (π/2.) /; (√(r[[1]]^2 + r[[2]]^2) == 0)
```

For these formulas the angles gLON and gLAT should be in degrees.
They give an Aitoff Plot that is centered on (0°,0°)

In[447]:=
```
αH[gLON_, gLAT_] := αH[gLON, gLAT] = ArcCos[Cos[((2. π) / 360.) gLAT] Cos[((2. π) / 360.) gLON / 2.]]
xH[gLON_, gLAT_] :=
 xH[gLON, gLAT] = (2. Cos[((2. π) / 360.) gLAT] Sin[((2. π) / 360.) gLON / 2.]) / Sinc[αH[gLON, gLAT]]
yH[gLON_, gLAT_] := yH[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[gLON, gLAT]]
```

Using the following functions produces an Aitoff Plot that is centered on (180°,0°)

In[450]:=
```
xH180[gLON_, gLAT_] := xH180[gLON, gLAT] =
  (2. Cos[((2. π) / 360.) gLAT] Sin[((2. π) / 360.) (gLON - 180.) / 2.]) / Sinc[αH[(gLON - 180.), gLAT]]
yH180[gLON_, gLAT_] := yH180[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[(gLON - 180.), gLAT]]
```

For Galactic Coordinates, the following functions produces an Aitoff Plot that is centered on gLON = 0° and the gLON axis runs from +180° on the left to 0° at the center to -180° on the right. The viewpoint is inside the Celestial Sphere, looking out.

In[452]:=
```
(*The plots of the sky in Galactic coordinates have the gLON axis running from +
 180° on the left to -180° on the right. Angles gLON and gLAT are in degrees*)
xHGal[gLON_, gLAT_] := xHGal[gLON, gLAT] =
  (2. Cos[((2. π) / 360.) gLAT] Sin[-(2. π / 360.) gLON / 2.]) / Sinc[αH[-gLON, gLAT]]
yHGal[gLON_, gLAT_] := yHGal[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[-gLON, gLAT]]
```

In[454]:=
```
mean[data_] := (1 / Length[data]) Sum[data[[i4]], {i4, Length[data]}];
(* arithmetic average *)
stanDev[data_] :=
 ((1 / Length[data]) Sum[(data[[i5]] - mean[data])^2, {i5, Length[data]}])^(1/2)
 (*standard deviation*)
```

2b. Derivation of a formula for the alignment angle $\eta_{\mathrm{iH}}$ given the position $r_S$ of the $i$th source , the location $r_H$ of point $H$ , and the polarization direction $\psi$ for the $i$th source

From Fig 4b, we see that $\cos\eta = \mathrm{v}\psi.\mathrm{vH}$, which is Eq. 3.

By Fig. 4, the vector vH is in the plane of rH and rS. It is perpendicular to rS, so we can get vH by subtracting: rH - (the part of rH parallel to rS) and then normalizing. It is the Gram-Schmitt process.

$\mathrm{vH} = \frac{\mathrm{rH} - (\mathrm{rH.rS})\,\mathrm{rS}}{[(\mathrm{rH} - (\mathrm{rH.rS})\,\mathrm{rS}).(\mathrm{rH} - (\mathrm{rH.rS})\,\mathrm{rS})]^{1/2}}$ : unit vector in the 2D tangent plane at S, in the direction of H from S, $\mathrm{vH.rS} = 0$, where the vectors rH and rS are unit vectors from the origin to point H and source S

Since $\mathrm{v}\psi$ is also perpendicular to rS, it follows that $\mathrm{v}\psi.\mathrm{rS} = 0$, and we have $\frac{\mathrm{rH}}{[(\mathrm{rH} - (\mathrm{rH.rS})\,\mathrm{rS}).(\mathrm{rH} - (\mathrm{rH.rS})\,\mathrm{rS})]^{1/2}}$ as the part of vH that contributes to the dot product $\cos\eta = \mathrm{v}\psi.\mathrm{vH}$ . Therefore, define

$$\text{vHperpS} = \frac{\text{rH}}{[(\text{rH} - (\text{rH.rS})\,\text{rS}).(\text{rH} - (\text{rH.rS})\,\text{rS})]^{1/2}}$$

Simplify the denominator,

In[456]:=
```
denoSquared1 =
  FullSimplify[(er[gLONH, gLATH] - (er[αH, δH].er[gLONS, gLATS]) er[gLONS, gLATS]).
    (er[gLONH, gLATH] - (er[gLONH, gLATH].er[gLONS, gLATS]) er[gLONS, gLATS])];
(* denoSquared = [rH - (rH.rS) rS].[rH - (rH.rS) rS] =
    rH.rH - 2(rH.rS)^2 + (rH.rS)^2 rS.rS =
     1 - 2(rH.rS)^2 + (rH.rS)^2 = 1 - (rH.rS)^2*)
```

In[457]:=
```
FullSimplify[denoSquared1 - (1 - (er[gLONH, gLATH].er[gLONS, gLATS])^2)] (*check that*)
```

Out[457]= 0

Write the formula for the vector vHperpS, with a denominator of $(1 - (\text{rH.rS})^2)^{1/2}$ :

In[458]:=
```
vHperpS[gLONS_, gLATS_, gLONH_, gLATH_] :=
 er[gLONH, gLATH] / (1 - (er[gLONH, gLATH].er[gLONS, gLATS])^2)^(1/2)
```

In[459]:=
```
(*Make H = S*)
Simplify[vHperpS[gLONH, gLATH, gLONH, gLATH]];(* BANG,
BOOM!! See Fig. 4 for why η is undefined when H = S.*)
```

... Simplify: Expression $\frac{\text{Cos[gLATH] Cos[gLONH]}}{\sqrt{1 - (\text{Power[≪2≫] Power[≪2≫] + Sin[≪1≫]}^2 + \text{Power[≪2≫] Power[≪2≫]})^2}}$ simplified to ComplexInfinity.

... Simplify: Expression $\frac{\text{Cos[gLATH] Sin[gLONH]}}{\sqrt{1 - (\text{Power[≪2≫] Power[≪2≫] + Sin[≪1≫]}^2 + \text{Power[≪2≫] Power[≪2≫]})^2}}$ simplified to ComplexInfinity.

... Simplify: Expression $\frac{\text{Sin[gLATH]}}{\sqrt{1 - (\text{Power[≪2≫] Power[≪2≫] + Sin[≪1≫]}^2 + \text{Power[≪2≫] Power[≪2≫]})^2}}$ simplified to Indeterminate.

... General: Further output of Simplify::infd will be suppressed during this calculation.

Back to the derivation:

The other vector we need is v$\psi$, the unit vector in the 2D tangent plane at S pointing in the direction of the polarization position angle $\psi$. By Fig. 4b, one sees that

$$\text{v}\psi = \cos(\psi)\,\text{N} + \sin(\psi)\,\text{E}\,,$$

where N and E are local north and east unit vectors in the 2D tangent plane at S.

In[460]:=
```
vψ[gLONS_, gLATS_, gLONH_, gLATH_, ψ_] := Cos[ψ] eN[gLONS, gLATS] + Sin[ψ] eE[gLONS, gLATS]
(*vψ[gLONS,gLATS,gLONH,gLATH,ψ]*)
```

The alignment angle $\eta$ is the acute angle between v$\psi$ and vH in the 2D tangent plane at S. By Eq. 3,

In[461]:=
```
ηiH0[gLONS_, gLATS_, gLONH_, gLATH_, ψ_] :=
 ArcCos[ Abs[vψ[gLONS, gLATS, gLONH, gLATH, ψ].vHperpS[gLONS, gLATS, gLONH, gLATH] ] ]
(*ηiH0[gLONS,gLATS,gLONH,gLATH,ψ]*)
FullSimplify[ηiH0[gLONS, gLATS, gLONH, gLATH, ψ]]
```

Out[462]=
```
ArcCos[Abs[(Cos[gLATS] Cos[ψ] Sin[gLATH] +
     Cos[gLATH] (-Cos[gLONH - gLONS] Cos[ψ] Sin[gLATS] + Sin[gLONH - gLONS] Sin[ψ])) /
   (Sqrt[1 - (Cos[gLATH] Cos[gLATS] Cos[gLONH - gLONS] + Sin[gLATH] Sin[gLATS])^2])]]
```

In[463]:=
```
(*The following function is well-
 behaved everywhere except where ±H coincides with ±S.*)
ηiHwithIndeterminate[gLONS_, gLATS_, gLONH_, gLATH_, ψ_] :=
 ArcCos[Abs[(Cos[gLATS] Cos[ψ] Sin[gLATH] +
      Cos[gLATH] (-Cos[gLONH - gLONS] Cos[ψ] Sin[gLATS] + Sin[gLONH - gLONS] Sin[ψ])) /
    (Sqrt[(1 - (Cos[gLONH - gLONS] Cos[gLATH] Cos[gLATS] + Sin[gLATH] Sin[gLATS])^2)])] ]
```

In[464]:=
```
(*Since η is an acute angle, let us take the halfway value,
η = π/4 in the neighborhood where H ≈ S.*)
ηiH[gLONS_, gLATS_, gLONH_, gLATH_, ψ_] :=
 ηiHwithIndeterminate[gLONS, gLATS, gLONH, gLATH, ψ] /;
  ((1 - (er[gLONH, gLATH].er[gLONS, gLATS])^2) ≥ 0.000001)
ηiH[gLONS_, gLATS_, gLONH_, gLATH_, ψ_] :=
 π / 4. /; ((1 - (er[gLONH, gLATH].er[gLONS, gLATS])^2) < 0.000001)
Print["Thus η_iH = π/4 wherever ±H is 'close'
   to ±S, with 'close' meaning within an angle of ",
 ArcSin[0.000001^(1/2)], " radians, or ", ArcSin[0.000001^(1/2)] (360./(2. π)), "°."]
```

Thus $\eta_{iH} = \pi/4$ wherever ±H is 'close' to ±S, with 'close' meaning within an angle of 0.001 radians, or 0.0572958°.

## 3. Polarization and Position Data

The stars in the combined Heiles 2000 and Berdyugin 2014 catalog, Refs. 7 - 10 , are filtered for the % polarization,% p≥0.1%,the fractional uncertainty in % p, $\sigma p/p \le 0.25$ , and the uncertainty in PPA $\psi$, $|\sigma\psi| \le 7°$. See Part I the Article Sec. 2, and Figs. 2 and 3, for the selection process.

Definitions:

| | |
|---|---|
| cat | the catalog data, Heiles 2000 Refs. 7,8 combined with Berdyugin 2014 Refs. 9,10 |
| allClumpsofStarsIDsInCatalog | record numbers of the stars in the catalog for all clumps |
| clumpOfStarsIDinCatalog | record numbers of the sample’s stars in the catalog (we treat this clump) |
| nSrc | number of stars |
| gLONSrc | galactic longitude (radians ) |
| gLATSrc | galactic latitude (radians) |

| | |
|---|---|
| $\psi$Src | PPA, polarization position angle: counterclockwise from North with East to the left, as seen from inside the Celestial Sphere. |
| $\sigma\psi$Src | uncertainty in PPA |
| percentPol | percentage of linear polarization |
| rSrc | unit vector from Origin to Sources on Celestial Sphere |
| eNSrc | Local North at the ith Source |
| eESrc | Local East at the ith Source |
| sourceCenter | unit radial vector to the arithmetic center of the sources |
| angleSourceToCenter | arc from Source to Center |
| showClump1 | map of significance for alignments in the catalog, needed to discuss sample selection |

Catalog data

The HD or BD numbers for the stars in the sample are given in the following cell.
Most records can be found by searching the Heiles 2000 or the Berdyugin 2014 catalogs for the HD or BD number.
Just one star had neither an HD nor a BD number. The exceptional star is record # 5361 in the Heiles 2000 catalog with (gLON,gLAT) = (16.726632 hours, 6.0061°). To find record # 5361, one can search the Heiles 2000 catalog for the dec.RA entry which is “60061.16726632” .

In[467]:=
```
starIDnumbers = {{"HD", 151061.`}, {"HD", 154445.`}, {"HD", 155195.`}, {"HD", 156247.`},
   {"HD", 152126.`}, {"HD", 150764.`}, {"HD", 145085.`}, {"HD", 152974.`}, {"HD", 151219.`},
   {"HD", 157999.`}, {"HD", 152310.`}, {"HD", 152087.`}, {"HD", 153115.`}, {"HD", 145892.`},
   {"HD", 150752.`}, {"HD", 151026.`}, {"HD", 151812.`}, {"HD", 153147.`}, {"HD", 152067.`},
   {"HD", 152466.`}, {"HD", 152897.`}, {"Heiles 2000", "Record # 5361"}, {"HD", 146815.`},
   {"HD", 151828.`}, {"HD", 158836.`}, {"HD", 160140.`}, {"HD", 153033.`}, {"HD", 157278.`},
   {"HD", 151556.`}, {"HD", 150873.`}, {"HD", 155593.`}, {"HD", 151494.`},
   {"HD", 153303.`}, {"HD", 152532.`}, {"HD", 156130.`}, {"HD", 153272.`},
   {"HD", 160311.`}, {"HD", 156655.`}, {"HD", 151291.`}, {"HD", 155500.`},
   {"HD", 156404.`}, {"HD", 154762.`}, {"HD", 153797.`}, {"HD", 156732.`},
   {"HD", 154619.`}, {"HD", 154302.`}, {"HD", 155644.`}, {"HD", 156681.`},
   {"HD", 153540.`}, {"HD", 155422.`}, {"HD", 153835.`}, {"HD", 152447.`},
   {"HD", 159082.`}, {"HD", 159005.`}, {"HD", 157606.`}, {"BD", 13.328`}, {"HD", 151627.`},
   {"HD", 151072.`}, {"HD", 151545.`}, {"HD", 159119.`}, {"HD", 155581.`},
   {"HD", 154512.`}, {"HD", 152308.`}, {"HD", 153898.`}, {"BD", 15.3104`},
   {"BD", 15.3101`}, {"HD", 157741.`}, {"HD", 151203.`}, {"HD", 148035}, {"HD", 148512},
   {"HD", 146047}, {"HD", 147510}, {"HD", 149755}, {"HD", 149755}, {"HD", 147189},
   {"HD", 149413}, {"HD", 146026}, {"HD", 145568}, {"HD", 146561}, {"HD", 148622},
   {"HD", 147548}, {"HD", 148229}, {"BD", {"BD+10", 3004}}, {"HD", 150305},
   {"HD", 147252}, {"HD", 147836}, {"HD", 150123}, {"HD", 151059}, {"HD", 150268},
   {"HD", 151879}, {"HD", 147868}, {"HD", 150905}, {"HD", 150257}, {"HD", 148765},
   {"HD", 153225}, {"HD", 150830}, {"HD", 150568}, {"HD", 152155}, {"HD", 153301}};
```

For example, the Heiles 2000 catalog listing for the first star in the sample, Record # 4698, HD151061. :
{-30849.16753181 151061.0 -2.424200-999.900000-999.900000 2.390 0.035 87.4 0.4 144.8 14.3393 26.1434 0.60 -0.1 1 7.2 199.5 M6III 0000000000001000000000 10}

The combined Heiles 2000 and Berdyugin 2014 data file that we use has the Heiles data first followed by Berdyugin 2014 data. The Heiles 2000 part of the file contains the original unaltered catalog entries, except that the declination and Right Ascension have been separated and the object’s record number is appended to each record.

The Berdyugin 2014 catalog data requires some work to get it into the same form as the Heiles 2000 catalog. Some 39 stars appear in both catalogs and are deleted from the Berdyugin 2014 catalog.
We kept the 399 stars in the Berdyugin catalog that do not have polarization directions.
Also, the polarization direction in the Berdyugin 2014 catalog need to be converted from Equatorial to Galactic coordinates.
Once determined, the data was rearranged to conform to the Heiles 2000 catalog format. Any unknown quantities were flagged as "-999", as in the Heiles 2000 catalog. The Bredyugin 2014 data is appended to the Heiles 2000 catalog, increasing the star count from 9286 to 11647 stars.

--------------------------------------------------------------------------------

1. Declination (deg) 2 RA (hr) 3. HD number 4. Bonner DM number 5. Cordoba DM number 6. Cape DM number 7. Percentage polarization (%) 8. rms uncertainty on Pol (%) 9. Position angle, equatorial (deg.) 10. rms uncertainty on PA (deg.) 11. Position angle, Galactic (deg.) 12. Galactic longitude (deg.) 13. Galactic latitude (deg.) 14. Reddening (mag.) 15. Discrepancy between PA and PAgal (deg.) 16. Primary stellar database 17. Visual magnitude (mag.) 18. Distance (pc) 19. Spectral type 20. Polarization catalog numbers 21. Distance catalog 22. Object # in the catalog

--------------------------------------------------------------------------------

See the ReadMe files in Refs. 8 & 10 for details.

In[468]:=
```
(*galactic longitude in radians, rounded to six places*)
gLONSrc = 10^-6. {250268, 336777, 349877, 396797, 371553, 355747, 264845, 392120, 373212,
    467158, 393484, 395614, 416793, 306326, 383883, 390549, 404983, 424726, 410660,
    422532, 429238, 402380, 339266, 419410, 513891, 530055, 442034, 497609, 423139,
    413503, 480505, 426869, 452972, 443167, 493094, 456224, 544635, 502568, 433076,
    499134, 520949, 508835, 505271, 558219, 532561, 530339, 547761, 565138, 527496,
    551098, 535399, 520166, 610464, 609671, 619517, 574278, 553250, 546288, 553193,
    653334, 616906, 611382, 587635, 608045, 618923, 616325, 660516, 589513, 270352,
    285536, 290074, 291470, 314683, 314683, 338245, 342259, 345575, 348193, 366694,
    396015, 405091, 406662, 436856, 445059, 460243, 460941, 464083, 468970, 471762,
    479093, 487121, 521155, 524297, 528660, 580147, 582940, 588874, 601790, 602662};
```

In[469]:=
```
nSrc = Length[gLONSrc]
```

Out[469]=
```
99
```

In[470]:=
```
(*galactic latitude in radians, rounded to six places*)
gLATSrc =
  10^-6. {456288, 400235, 383845, 376501, 488809, 523040, 650730, 473078, 516980, 362236,
    494295, 502109, 481468, 648983, 537656, 531592, 514558, 484264, 509903, 503908,
    493726, 542429, 640304, 520890, 361623, 336287, 495586, 398525, 531022, 547522,
    437666, 534664, 493298, 511650, 428199, 495468, 338716, 417793, 543225, 449026,
    433278, 475457, 502053, 440561, 489956, 498113, 466692, 444870, 518293, 474419,
    513689, 547677, 399743, 401410, 443712, 534818, 582830, 596885, 586129, 416671,
    496506, 524756, 577118, 540722, 531568, 538403, 455449, 608338, 569675, 560949,
    637045, 596204, 534769, 534769, 629017, 559378, 663923, 677188, 659211, 611040,
    647866, 629366, 635998, 579449, 678235, 662702, 593063, 566010, 592016, 545066,
    671254, 592365, 613134, 658338, 549779, 617148, 626224, 586082, 555015};
```

In[471]:=
```
(* galactic position angle in radians, rounded to six places*)
ψSrc = 10^-6. {2 527 237, 2 614 503, 2 677 335, 2 590 069, 2 513 274, 2 665 118, 3 071 779, 2 492 330,
    2 617 994, 2 584 833, 2 830 924, 2 602 286, 2 638 938, 2 569 125, 2 513 274, 2 672 099, 2 635 447,
    2 752 384, 2 604 031, 2 631 957, 2 522 001, 2 523 746, 2 624 975, 2 644 174, 2 528 982, 2 328 269,
    2 614 503, 2 616 249, 2 445 206, 2 457 424, 2 745 403, 2 460 914, 2 406 809, 2 588 323,
    2 708 751, 2 501 057, 2 705 260, 2 352 704, 2 642 428, 2 389 356, 2 513 274, 2 972 296,
    2 227 040, 2 560 398, 2 993 240, 2 792 527, 2 912 955, 2 858 849, 2 595 305, 2 918 191,
    2 537 709, 2 523 746, 2 834 415, 2 647 664, 2 888 520, 2 724 459, 2 499 311, 2 703 515,
    2 560 398, 2 412 045, 2 679 080, 205 949, 2 483 604, 2 771 583, 2 935 644, 2 911 209, 2 645 919,
    2 471 386, 2 585 376, 2 610 617, 2 581 367, 2 422 949, 2 485 061, 2 554 874, 2 607 604,
    2 604 617, 2 440 555, 2 548 246, 2 661 401, 2 566 889, 2 593 929, 2 731 882, 2 801 263,
    2 482 942, 2 681 701, 2 783 884, 2 424 506, 2 579 781, 2 480 488, 2 494 087, 2 764 966,
    2 522 077, 2 597 706, 2 732 176, 2 784 442, 2 749 822, 2 824 960, 2 454 031, 2 725 736};
```

In[472]:=
```
hGramψ = Histogram[ψSrc (360./(2. π)), {5}, PlotLabel → "PPA ψ, number ΔR per bin",
   AxesLabel → {"ψ", "ΔR"}, PlotRange → {{0, 200}, Automatic}];
hGramψ
```

Out[473]=

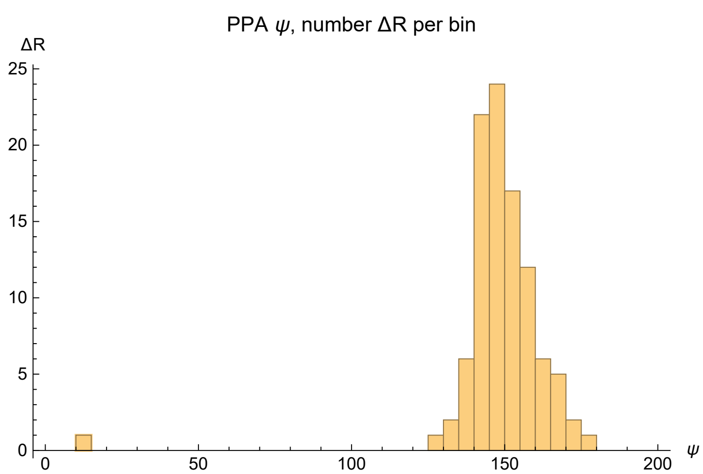



Figure 10: Distribution of position angles for the polarization directions in the sample. That one star at $\psi = 12°$ is not as special as it appears in the figure. Since the electric polarization vector oscillates, $\psi = 12°$ is equivalent to $\psi = 180° + 12° = 192°$, which is just 16° above the next-closest star at $\psi = 176°$. Note the narrowness of the distribution.

In[475]:=
```
Sort[Table[{ψSrc[[i]] (360./(2. π)), i}, {i, Length[ψSrc]}]];
i = 62;(*The star with ψ = 12°, actually ψ = 11.8° or 191.8°.*)
{ψSrc[[i]] (360./(2. π)), i, starIDnumbers[[i]]}
Clear[i]
```

Out[477]= {11.8, 62, {HD, 154 512.}}

```
In[479]:= (*uncertainty in ψ in radians, rounded to six places*)
σψSrc =
  10^-6· {6981, 3491, 8727, 8727, 43633, 15708, 83776, 15708, 10472, 17453, 34907, 3491, 8727,
    50615, 5236, 6981, 10472, 10472, 6981, 6981, 3491, 20944, 50615, 41888, 104720,
    38397, 26180, 19199, 41888, 61087, 41888, 12217, 40143, 47124, 24435, 3491, 69813,
    29671, 15708, 116937, 59341, 50615, 106465, 36652, 116937, 66323, 38397,
    26180, 26180, 54105, 43633, 71558, 45379, 80285, 52360, 83776, 33161, 41888,
    54105, 97738, 24435, 90757, 92502, 102974, 111701, 55851, 97738, 62832,
    34907, 34907, 27925, 34907, 34907, 34907, 26180, 52360, 34907, 52360, 52360,
    34907, 52360, 104720, 69813, 34907, 61087, 34907, 52360, 34907, 34907,
    52360, 69813, 69813, 52360, 69813, 69813, 34907, 34907, 69813, 52360};

In[480]:= (* % polarization, rounded to six places*)
percentPol =
  10^-6· {2390000, 3420000, 2300000, 2002000, 718000, 154000, 210000, 493000, 660000,
    1010000, 199000, 621000, 1706000, 340000, 728000, 645000, 931000, 1193000, 563000,
    1009000, 896000, 623000, 360000, 585000, 330000, 590000, 660000, 1150000, 545000,
    583000, 700000, 548000, 698000, 602000, 1460000, 811000, 410000, 1490000, 554000,
    150000, 610000, 420000, 340000, 920000, 150000, 550000, 460000, 660000, 680000,
    560000, 490000, 290000, 380000, 260000, 330000, 820000, 540000, 510000, 460000,
    180000, 710000, 460000, 190000, 290000, 390000, 790000, 180000, 280000, 443000,
    336000, 484000, 368000, 489000, 551000, 575000, 540000, 469000, 305000, 272000,
    507000, 306000, 219000, 375000, 285000, 211000, 305000, 481000, 408000, 390000,
    346000, 200000, 400000, 212000, 198000, 359000, 327000, 346000, 299000, 382000};

In[481]:= σpercentPol =
  10^-6· {35000, 24000, 42000, 34000, 63000, 5000, 35000, 16000, 13000, 35000, 14000, 5000,
    29000, 35000, 8000, 10000, 20000, 26000, 7000, 15000, 5000, 27000, 36000, 50000,
    69000, 46000, 35000, 46000, 45000, 72000, 58000, 14000, 57000, 57000, 73000,
    5000, 58000, 87000, 17000, 35000, 73000, 42000, 73000, 69000, 35000, 73000,
    35000, 35000, 35000, 60000, 42000, 42000, 35000, 42000, 35000, 138000,
    35000, 42000, 50000, 35000, 35000, 83000, 35000, 60000, 87000, 87000, 35000,
    35000, 32000, 23000, 27000, 25000, 35000, 40000, 30000, 49000, 30000, 26000,
    32000, 36000, 29000, 43000, 48000, 24000, 26000, 26000, 50000, 31000, 29000,
    34000, 26000, 61000, 24000, 27000, 51000, 25000, 25000, 38000, 37000};

In[482]:= (*distance in pc *)
distance = 10^-6 {199500000, 406000000, 109600000, 203000000, -999900000, -999900000,
    158500000, -999900000, -999900000, 151400000, -999900000, -999900000, -999900000,
    125900000, -999900000, -999900000, -999900000, -999900000, -999900000, -999900000,
    -999900000, -999900000, 724400000, -999900000, 363100000, 549500000, 251200000,
    182000000, -999900000, -999900000, 263000000, -999900000, -999900000, -999900000,
    416900000, -999900000, 288400000, 302000000, -999900000, 95500000, 346700000,
    331100000, 275400000, 288400000, 63100000, 263000000, 131800000, 131800000,
    208900000, 288400000, 199500000, 218800000, 158500000, 218800000, 251200000,
    602600000, 75900000, 199500000, 251200000, 288400000, 239900000, 660700000,
    151400000, 288400000, 83200000, 66100000, 151400000, 166000000, -999000000,
    -999000000, -999000000, -999000000, -999000000, -999000000, -999000000,
    -999000000, -999000000, -999000000, -999000000, -999000000, -999000000,
    -999000000, -999000000, -999000000, -999000000, -999000000, -999000000,
    -999000000, -999000000, -999000000, -999000000, -999000000, -999000000,
    -999000000, -999000000, -999000000, -999000000, -999000000, -999000000};
```

In[483]:=
```
rSrc = Table[er[ gLONSrc[[i]], gLATSrc[[i]] ],  {i, nSrc}];(*calculated from Input.*)
eNSrc = Table[eN[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
eESrc = Table[eE[  gLONSrc[[i]], gLATSrc[[i]]  ], {i, nSrc}];(*calculated from Input.*)
```

In[486]:=
```
ηBarAtHwithAnyψ[gLONH_, gLATH_, ψ_] :=
 1/nSrc Sum[ηiH[gLONSrc[[i]], gLATSrc[[i]], gLONH, gLATH, ψ[[i]] ], {i, nSrc}]
(*ηBarAtHwithAnyψ[1.45,0.6,ψSrc]*)(* An example with a selected
 gLONH and gLATH and with the observed polarization directions for ψ*)
```

In[487]:=
```
sourceCenter0 = 1/nSrc Sum[rSrc[[i]], {i, nSrc}];
sourceCenter = sourceCenter0/(sourceCenter0.sourceCenter0)^(1/2);
(*unit radial vector to the arithmetic center of the sources.*)
angleSourceToCenter = Table[ArcCos[rSrc[[i]].sourceCenter], {i, nSrc}];
ρRgnRadius = Sort[angleSourceToCenter][[-1]] (*Furthest source from center*)

ρRMS = (1/nSrc Sum[angleSourceToCenter[[i]]^2, {i, nSrc}])^(1/2)
```

Out[490]= 0.206647

Out[491]= 0.119346

3b. Section Summary

In[492]:=
```
Print["There are ", nSrc, " stars in the sample."]
Print["Check that the Sample obeys the data cuts:"]
Print[
 "Check that the smallest % polarization p in the sample is 0.1% or more. Smallest: ",
 Sort[percentPol][[1]], "% ."]
Print["Check that the largest fractional uncertainty in % polarization, σp/p,
   is less than 0.25. Largest: ", Sort[σpercentPol / percentPol][[-1]], " ."]
Print["Check that the largest PPA ψ uncertainty σψ is less than 7°. Largest: ",
 Sort[σψSrc][[-1]] (360./(2. π)), "° ."]
```

There are 99 stars in the sample.

Check that the Sample obeys the data cuts:

Check that the smallest % polarization p in the sample is 0.1% or more. Smallest: 0.15% .

Check that the largest fractional uncertainty
in % polarization, σp/p, is less than 0.25. Largest: 0.233333 .

Check that the largest PPA ψ uncertainty σψ is less than 7°. Largest: 6.7° .

In[497]:=
```
lpgLONgLATSrc = ListPlot[Table[{-gLONSrc[[j]], gLATSrc[[j]]} (360./(2. π)), {j, nSrc}],
   PlotRange → {{-180, 180}, {-90, 90}},
   Ticks → {Table[{i, -i}, {i, -180, 180, 60}], Table[{j, j}, {j, -90, 90, 30}]},
   PlotLabel → "Sources", AxesLabel → {"°gLON", "°gLAT"}, PlotStyle → Green];
lpgLONgLATSrc
```

Out[498]=

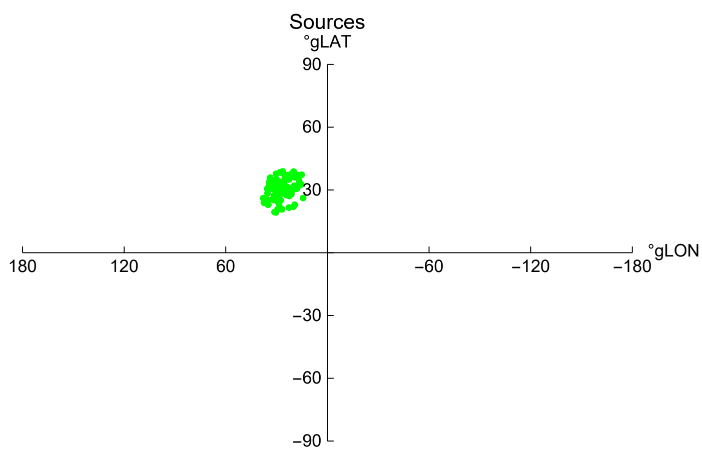



Figure 11. The locations of the 99 stars in the sample. The center of the sample has (gLON,gLAT) = {26.7517, 30.3533}, in degrees. Sample Size: The angular separation of the furthest star from the sample center is 11.84°. The RMS radius is 6.83803°.

## 4. Grid

### 4a. Construct the grid

We avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle d$\theta$.

We grid one hemisphere at a time, then the grids are combined.

Definitions:

| | |
|---|---|
| gridSpacing | separation in degrees between grid points on and between constant latitude circles |
| d$\theta$1 | grid spacing in radians |
| idN, ai, ji | dummy indices, ID #s for grid points, longitude, latitude |

| | |
|---|---|
| gLONpointH,gLATpointH | gLON and gLAT of the grid points $H_j$ |
| grid, gridN, gridS | tables data associated with grid points, listings are below |
| nGrid | number of grid points |
| gLONGrid | longitudes at the grid points ( $-\pi \le$ gLON $\le +\pi$ ) |
| gLATGrid | latitudes at the grid points ( $-\pi/2 \le$ gLON $\le \pi/2$ ) |
| rGrid | radial unit vectors from origin to grid points, in 3D Cartesian coordinates |

Tables: ***grid, gridN* and *gridS***

1. sequential point # 2. gLON index 3. gLAT index 4. gLON (rad) 5. gLAT (rad) 6. Cartesian coordinates of the grid point

In[500]:=
```
gridSpacing = 2.(*, in degrees.*);
```

In[501]:=
```
(*KEEP this cell - DO NOT DELETE*)
(*The Northern Grid "gridN". *)
dθ1 = ((2. π) / 360.) gridSpacing;
(*Convert gridSpacing to radians*) gridN = {};
idN = 1;
For[gLATj = 0., gLATj < π / (2. dθ1), gLATj++, gLATpointH = gLATj dθ1;
 For[ ai = 0., ai < Ceiling[((2. π) / dθ1) (Cos[gLATpointH] + 0.01)],
  ai++, gLONpointH = ai dθ1 / (Cos[gLATpointH] + 0.01);
  AppendTo[gridN, {idN, ai, gLATj, gLONpointH, gLATpointH, er[gLONpointH, gLATpointH]}];
  idN = idN + 1
 ]]
```

In[503]:=
```
(*KEEP this cell - DO NOT DELETE*)
(*The Southern Grid "gridS". *)
dθ1 = ((2. π) / 360.) gridSpacing; (*Convert gridSpacing to radians*)
gridS = {}; idN = 1;
For[gLATj = 1., gLATj < π / (2. dθ1), gLATj++, gLATpointH = -gLATj dθ1;
 For[ ai = 0., ai < Ceiling[((2. π) / dθ1) (Cos[gLATpointH] + 0.01)],
  ai++, gLONpointH = ai dθ1 / (Cos[gLATpointH] + 0.01);
  AppendTo[gridS, {idN, ai, gLATj, gLONpointH, gLATpointH, er[gLONpointH, gLATpointH]}];
  idN = idN + 1
 ]]
```

In[506]:=
```
(*KEEP this cell - DO NOT DELETE*)
grid = {}; j = 1;
For[jN = 1, jN ≤ Length[gridN], jN++, AppendTo[grid, {j, gridN[[jN, 2]], gridN[[jN, 3]],
    gLONFROMr[gridN[[jN, 6]] ], gLATFROMr[gridN[[jN, 6]] ], gridN[[jN, 6]]}];
 j = j + 1]
For[jS = 1, jS ≤ Length[gridS], jS++, AppendTo[grid, {j, gridS[[jS, 2]], gridS[[jS, 3]],
    gLONFROMr[gridS[[jS, 6]] ], gLATFROMr[gridS[[jS, 6]] ], gridS[[jS, 6]]}];
 j = j + 1]
```

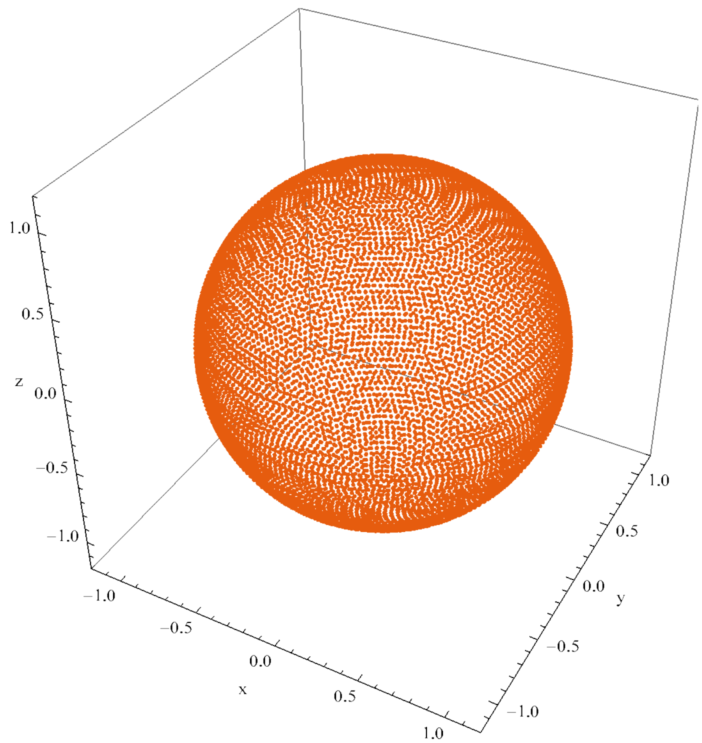
In[509]:=
```
nGrid = Length[grid];
gLONGrid = Table[grid[[j, 4]] , {j, nGrid}];
gLATGrid = Table[grid[[j, 5]] , {j, nGrid}];
rGrid = Table[grid[[j, 6]] , {j, nGrid}];
```

4b. Section Summary

In[513]:=
```
lpgrid = ListPointPlot3D[Table[grid[[i, 6]], {i, 1, Length[grid]}],
   PlotRange → {{-1.2, 1.2}, {-1.2, 1.2}, {-1.2, 1.2}},
   AxesLabel → {"x", "y", "z"}, BoxRatios → {1, 1, 1}, PlotTheme → "Scientific"];
```

In[514]:=
```
lpgrid
```

Out[514]=




```
Figure 12: The grid. There are 10518
  grid points on the sphere. The sphere is transparent, so the image may be somewhat deceptive.
   The grid points are separated by gridSpacing = 2.° arcs along latitude and longitude.
```

5. The alignment function $\overline{\eta}$(H) for the sample of sources

In this section, we use the “best” values of $\psi$Src, the values listed in the catalog to calculate the alignment function $\overline{\eta}$(H). Maps are drawn and alignment and avoidance hubs $\pm H_{\min}$ and $\pm H_{\max}$ are found.

5a. Determine the alignment angle function $\overline{\eta}$(H)

First find $\overline{\eta}$(Hj) at the grid points $H_j$ and find the smallest and largest values of the alignment function on the grid. Then use the function “$\eta$BarAtHwithAny$\psi$” derived in Secs. 2 and 3 to go between grid points $H_j$ and locate the smallest and largest angles, $\overline{\eta}_{\min}$

and $\overline{\eta}_{max}$, and their locations, the hubs $H_{min}$ and $H_{max}$ .

Here “min” indicates the extreme for convergence and “max” indicates the extreme for divergence of the 99 great circles from the sources along their polarization directions.

Definitions:

| | |
|---|---|
| v$\psi$Src | unit vectors along the polarization directions in the tangent planes of the sources |
| eN | local unit vectors along local North |
| eE | local unit vectors along local East |
| j$\eta$BarHj | {$j$,$\overline{\eta}$(H)}, where $j$ is the index for grid point $H_j$ and $\overline{\eta}$(H) is the average alignment angle at $H_j$. See Eq. (2) in Part I Sec. 2. |
| sortj$\eta$BarHj | {$j$,$\overline{\eta}$(H)}, sorted, with smallest angles $\overline{\eta}$(H) first. |
| j$\eta$BarMin | {$j$,$\overline{\eta}$(H)}, the $j$ and $\overline{\eta}$ for the smallest value of $\overline{\eta}$(H) , best alignment |
| $\eta$BarMin | the smallest value of $\overline{\eta}$(H) , measures alignment of the polarization directions |
| j$\eta$BarMax | {$j$,$\overline{\eta}$(H)}, the $j$ and $\overline{\eta}$ for the largest value of $\overline{\eta}$(H) , most avoided |
| $\eta$BarMax | the largest value of $\overline{\eta}$(H) , measures avoidance |

| | |
|---|---|
| nSx$\psi$Src | unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization |
| nSxHnj | unit vector, $S_i \times H_j$, cross product of the radial vector to the source with the radial vector to the grid point $H_j$ |
| $\eta$nHj | alignment angle between source and grid point $H_j$, see Fig. 4 |
| $\eta$BarHj | alignment angle $\overline{\eta}(H_j)$ between source and grid point $H_j$, avegLONged over all sources |
| j$\eta$BarHj | {$j$, $\overline{\eta}(H_j)$ }, the $j$ and $\overline{\eta}$ for grid point $H_j$ |

| | |
|---|---|
| sig$\eta$BarMin | significance of the smallest alignment angle |
| sigrange$\eta$BarMin | get the range of sigs using the plus/minus values on the parameters $c_i$, $a_i$ |
| sigSmall$\eta$BarMin | the smallest of the values in sigrange$\eta$BarMin |
| sigBig$\eta$BarMin | the largest of the values in sigrange$\eta$BarMin |
| sig$\eta$BarMax | significance of the largest alignment angle (i.e. avoidance) |
| sigrange$\eta$BarMax | get the range if sigs using the plus/minus values on the parameters $c_i$, $a_i$ |
| sigSmall$\eta$BarMax | the smallest of the values in sigrange$\eta$BarMax |
| sigBig$\eta$BarMax | the largest of the values in sigrange$\eta$BarMax |
| gLONHminDegrees | gLON of the point $H_{min}$where $\overline{\eta}$(H) is the smallest |
| gLATHminDegrees | gLAT of the point $H_{min}$where $\overline{\eta}$(H) is the smallest |
| gLONHmaxDegrees | gLON of the point $H_{max}$where $\overline{\eta}$(H) is the largest |
| gLATHmaxDegrees | gLAT of the point $H_{max}$where $\overline{\eta}$(H) is the largest |

In[516]:=

```
(* vψ, eN, eE unit vectors in the tangent plane of each source Si,
pointing along the polarization direction, local North,
and local East, respectively. See Fig. 4.*)
vψSrc = Table[Cos[ ψSrc[[i]] ] eN[ gLONSrc[[i]], gLATSrc[[i]] ] +
    Sin[ ψSrc[[i]] ] eE[  gLONSrc[[i]], gLATSrc[[i]]  ], {i, nSrc}];
```

In[517]:=
```
(* Analysis using Eq (5) in Ref. 12 to get η̄(Hj). First ηiH, cos(ηiH) = |v̂H.v̂ψi|,
where "v̂H" was called "vHperpS" in a previous discussion. Thus,
we can get η̄(Hj), by Eq. (3): *)
gridjηBarHj =
  Table[{j, (1/nSrc) Sum[ArcCos[ Abs[ rGrid[[j]].vψSrc[[i]] / ((rGrid[[j]] - (rGrid[[j]].
        rSrc[[i]]) rSrc[[i]]).(rGrid[[j]] - (rGrid[[j]].rSrc[[i]])
       rSrc[[i]]))^(1/2) ] - 0.000001 ] , {i, nSrc}]}, {j, nGrid}];
 sortgridjηBarHj = Sort[gridjηBarHj, #1[[2]] < #2[[2]] &];
gridjηBarMin = sortgridjηBarHj[[1]]; (*  {j,η̄(Hj)} for smallest η̄(Hj) *)
gridηBarMin = gridjηBarMin[[2]];
gridjηBarMax = sortgridjηBarHj[[-1]]; (*  {j,η̄(Hj)} for largest η̄(Hj) *)
gridηBarMax = gridjηBarMax[[2]] ;
```

The results just found on the grid should be close to even better results off-grid. Use FindMinimum and FindMaximum to go off-grid.

In[523]:=
```
ηmingLONgLATHObs = FindMinimum[
   ηBarAtHwithAnyψ[gLONH, gLATH, ψSrc], {{gLONH, gLONGrid[[ gridjηBarMin[[1]] ]]},
    {gLATH, gLATGrid[[ gridjηBarMin[[1]] ]]}}];
 ηmaxgLONgLATHObs =
  FindMaximum[ηBarAtHwithAnyψ[gLONH, gLATH, ψSrc],
   {{gLONH, gLONGrid[[ gridjηBarMax[[1]] ]]},
    {gLATH, gLATGrid[[ gridjηBarMax[[1]] ]]}}];
funcDataObs = {1, { ηmingLONgLATHObs[[1]], {gLONH, gLATH} /. ηmingLONgLATHObs[[2]]},
   { ηmaxgLONgLATHObs[[1]], {gLONH, gLATH} /. ηmaxgLONgLATHObs[[2]]}} ;
```

FindMinimum: The function value $0.122293 + 2.12863\times10^{-10}\, i$ is not a real number at {gLONH, gLATH} = {-1.86208, 0.835923}.

FindMaximum: The line search decreased the step size to within the tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient increase in the function. You may need more than MachinePrecision digits of working precision to meet these tolerances.

In[526]:=
```
ηBarMinfunDataObs = funcDataObs[[2, 1]];
ηBarMaxfunDataObs = funcDataObs[[3, 1]];
HmingLONfunDataObs = funcDataObs[[2, 2, 1]];
HmingLATfunDataObs = funcDataObs[[2, 2, 2]];
HmaxgLONfunDataObs =  funcDataObs[[3, 2, 1]];
HmaxgLATfunDataObs = funcDataObs[[3, 2, 2]];
```

In[532]:=
```
Print["When moving off-grid, check that  the
   hubs Hmin and Hmax did not move more than a grid spacing:"]
Print["When we found a local minimum, the hub Hmin moved off-grid by ",
 ArcCos[er[HmingLONfunDataObs, HmingLATfunDataObs].er[
     gLONGrid[[ gridjηBarMin[[1]] ]], gLATGrid[[ gridjηBarMin[[1]] ]]]   ] (360./(2. π)), "°."]
Print["When we found a local maximum, the hub Hmax moved off-grid by ",
 ArcCos[er[HmaxgLONfunDataObs, HmaxgLATfunDataObs].er[
     gLONGrid[[ gridjηBarMax[[1]] ]], gLATGrid[[ gridjηBarMax[[1]] ]]]   ] (360./(2. π)), "°."]
Print["Now compare that with the grid: The grid spacing is ", gridSpacing, "°. "]
```

```
When moving off-grid, check that  the hubs Hmin and Hmax did not move more than a grid spacing:
When we found a local minimum, the hub H_min moved off-grid by 0.109472°.
When we found a local maximum, the hub H_max moved off-grid by 1.90338°.
Now compare that with the grid: The grid spacing is 2.°.
```

5b. Plot the alignment angle function $\overline{\eta}$(H)

Definitions

| | |
|---|---|
| gLONHminDegrees | $H_{\rm min}$ location RA gLON in degrees |
| gLONHminHours | $H_{\rm min}$ location RA gLON in hours |
| gLATHminDegrees | $H_{\rm min}$ location Dec gLAT in degrees |
| gLONHmaxDegrees | $H_{\rm max}$ location RA gLON in degrees |
| gLONHmaxHours | $H_{\rm max}$ location RA gLON in hours |
| gLATHmaxDegrees | $H_{\rm max}$ location Dec gLAT in degrees |
| rHmin, rHmax | radial unit vectors to the alignment and avoidance hubs $H_{\rm min}$ and $H_{\rm max}$ |
| rPerpHmin (max) | a unit vector in the plane of the great circle combining sourceCenter and rHmin (max) |
| rGreatMinCircle($\theta$) (Max) | radial unit vector to a point on the great circle |
| gLONGreatMin (Max) | longitude at the point for $\theta$ |
| gLATGreatMin (Max) | latitude at the point for $\theta$ |
| xyAitoffGreatMin (Max) | Aitoff plot coordinates for the great circles |
| crossMin (Max) | unit vector perpendicular, normal to the plane of the great circle |
| $\theta$minMAXgreatcircles | angle between the vectors normal to the planes of the two great circles |

| | |
|---|---|
| gLONjgLATj$\eta$BarHjTable | {gLON$_j$, gLAT$_j$, $\overline{\eta}$(H)} at each grid point $H = H_j$, in degrees |
| xy$\eta$BarAitoffTable | {x, y, $\overline{\eta}$(x,y)} , where x,y are Aitoff coordinates and $\overline{\eta}$(x,y) is the alignment angle on grid |
| xyAitoffSources | {x,y} Aitoff coordinates for the sources' locations on the sphere |
| d$\eta$ContourPlot | separation of successive contour lines, in degrees |
| listCP | list contour plot of $\overline{\eta}$(H) from xy$\eta$BarAitoffTable |
| rPlus$\psi$ | unit vector in the polarization directions $\psi$ |
| polarLines | lines from each source along its polarization direction $\psi$ |
| mapOf$\eta$Bar | contour plot of the alignment angle $\overline{\eta}$(H) , adorned with source locations and labels |
| mapOf$\eta$BarLocal | magnified, local view of the map |

In[536]:=
```
(* Galactic coordinates (gLON,gLAT) for the hubs H_min and H_max .*)
gLONHminDegrees = HmingLONfunDataObs (360 / (2 π)); (*H_min*)
gLATHminDegrees = HmingLATfunDataObs (360 / (2 π));

gLONHmaxDegrees = HmaxgLONfunDataObs (360 / (2 π)); (*H_max*)
gLATHmaxDegrees = HmaxgLATfunDataObs (360 / (2 π));
```

In[540]:=
```
rHmin = er[ gLONHminDegrees (2. π/360.) + π , -gLATHminDegrees (2. π/360.) ];
rPerpHmin0 = rHmin - (rHmin.sourceCenter) sourceCenter;
rPerpHmin = rPerpHmin0/(rPerpHmin0.rPerpHmin0)^1/2.;
rGreatMinCircle[θ_] := Cos[θ] sourceCenter + Sin[θ] rPerpHmin
gLONGreatMin[θ_] := gLONFROMr[rGreatMinCircle[θ]]
gLATGreatMin[θ_] := gLATFROMr[rGreatMinCircle[θ]]
xyAitoffGreatMin =
  Table[{xHGal[ gLONGreatMin[θ] (360 / (2 π)), gLATGreatMin[θ] (360 / (2 π)) ],
    yHGal[ gLONGreatMin[θ] (360 / (2 π)), gLATGreatMin[θ] (360 / (2 π)) ]}, {θ, 1, 360}];
```

In[547]:=
```
rHmax = er[ gLONHmaxDegrees (2. π/360.) + π , -gLATHmaxDegrees (2. π/360.) ];
rPerpHmax0 = rHmax - (rHmax.sourceCenter) sourceCenter;
rPerpHmax = rPerpHmax0/(rPerpHmax0.rPerpHmax0)^1/2.;
rGreatMaxCircle[θ_] := Cos[θ] sourceCenter + Sin[θ] rPerpHmax
gLONGreatMax[θ_] := gLONFROMr[rGreatMaxCircle[θ]]
gLATGreatMax[θ_] := gLATFROMr[rGreatMaxCircle[θ]]
xyAitoffGreatMax =
  Table[{xHGal[ gLONGreatMax[θ] (360 / (2 π)), gLATGreatMax[θ] (360 / (2 π)) ],
    yHGal[ gLONGreatMax[θ] (360 / (2 π)), gLATGreatMax[θ] (360 / (2 π)) ]}, {θ, 1, 360}];
```

In[554]:=
```
crossMin0 = Cross[rHmin, sourceCenter];
crossMin = crossMin0/(crossMin0.crossMin0)^1/2.;
crossMax0 = Cross[rHmax, sourceCenter];
crossMax = crossMax0/(crossMax0.crossMax0)^1/2.;
θminMAXgreatcircles = ArcCos[crossMax.crossMin] (360./2. π);
```

In[559]:=
```
(*The following table gLONjgLATjηBarHjTable is created
 to generate a map of the alignment angle η̅(H) over the sphere.*)
(* Table gLONjgLATjηBarHjTable
 entries: 1.  gLON  2. gLAT  3. alignment angle ηBarRgnkj at grid point (all in degrees)*)
gLONjgLATjηBarHjTable = ( gLONjgLATjηBarHjTable0 = {};
   For[j = 1, j ≤ Length[gridjηBarHj], j++, AppendTo[ gLONjgLATjηBarHjTable0,
     {gLONGrid[[j]] * (360. / (2. π)), gLATGrid[[j]] * (360. / (2. π)),
      gridjηBarHj[[j, 2]] * (360. / (2. π))}]   ; If[ 180 ≥ gLONGrid[[j]] * (360. / (2. π)) > 174.,
     AppendTo[ gLONjgLATjηBarHjTable0,  {gLONGrid[[j]] * (360. / (2. π)) - 360.,
       gLATGrid[[j]] * (360. / (2. π)), gridjηBarHj[[j, 2]] * (360. / (2. π))}]   ] ;
    If[ -174. > gLONGrid[[j]] * (360. / (2. π)) ≥ -180., AppendTo[ gLONjgLATjηBarHjTable0,
      {gLONGrid[[j]] * (360. / (2. π)) + 360, gLATGrid[[j]] * (360. / (2. π)),
       gridjηBarHj[[j, 2]] * (360. / (2. π))}]   ]         ];
   gLONjgLATjηBarHjTable0);
```

In[560]:=
```
(*Transcribe the alignment function  η̅(H), the location of the sources,
and the Celestial Equator onto an Aitoff plot.*)
xyηBarAitoffTable = Table[{xHGal[gLONjgLATjηBarHjTable[[i, 1]], gLONjgLATjηBarHjTable[[i, 2]]],
    yHGal[gLONjgLATjηBarHjTable[[i, 1]], gLONjgLATjηBarHjTable[[i, 2]]],
    gLONjgLATjηBarHjTable[[i, 3]]}, {i, Length[gLONjgLATjηBarHjTable]}];

xyAitoffSources = Table[{xHGal[ gLONSrc[[n]] (360 / (2 π)), gLATSrc[[n]] (360 / (2 π)) ],
    yHGal[ gLONSrc[[n]] (360 / (2 π)), gLATSrc[[n]] (360 / (2 π)) ]}, {n, nSrc}];
 (*The Aitoff coordinates for the sources' locations.*)
```

In[562]:=
```
(* Contour plot of the alignment function ηBarHjSmooth. *)
dηContourPlot = 10 ;
(*, in degrees. *) listCP = ListContourPlot[Union[xyηBarAitoffTable (*,{{xHGal[gLONHminDegrees,
       gLATHminDegrees],yHGal[gLONHminDegrees,gLATHminDegrees],ηBarMin*(360./(2.π))-1.0}},
   {{xHGal[gLONHmaxDegrees,gLATHmaxDegrees],yHGal[gLONHmaxDegrees,gLATHmaxDegrees],
     ηBarMax*(360./(2.π))+1.0}}*)], AspectRatio → 1 / 2, Contours → Table[η, {η,
     Floor[gridjηBarMin[[2]] * (360. / (2. π))] + 1, Ceiling[gridjηBarMax[[2]] * (360. / (2. π))] - 1,
     dηContourPlot}], ColorFunction → "TemperatureMap", PlotRange → {{-4.0, 3.5}, 7.7/11. {-3, 3}},
  Axes -> False, Frame → False, PlotLegends → Placed[BarLegend[Automatic, LegendMargins → {{0, 0},
       {10, 5}}, LegendLabel → "η̅(H), °", LabelStyle → {Plain, FontFamily → "Times"}], Right]] ;
```

In[563]:=
```
(*Construct the map of η̄(H).*)
mapOfηBar =
  Show[{listCP, Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLAT, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}
      (*{23,11,0}*),MeshStyle→Thick,*)PlotPoints → 60], {gLON, -180, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}(*{23,11,0}*),
      MeshStyle→Thick,*)PlotPoints → 60], {gLAT, -60, 60, 30}],
    Graphics[{PointSize[0.004], Text[StyleForm["N", FontSize -> 14, FontWeight -> "Plain"],
       {0, 1.85}], (*Sources S:*)Green, Point[ xyAitoffSources ], Gray,
      PointSize[0.003], Point[ xyAitoffGreatMin ], Point[ xyAitoffGreatMax ], Black,
      Text[StyleForm["Hmin", FontSize → 12, FontWeight -> "Bold"], {-3.3, -1.0}],
      {Arrow[BezierCurve[{{-3.3, -1.2}, {-2.3, -2.0}, {xHGal[gLONHminDegrees + 180,
            -gLATHminDegrees], yHGal[gLONHminDegrees + 180, -gLATHminDegrees]}}]]},
      Text[StyleForm["Hmax", FontSize → 12, FontWeight -> "Bold"], {3.3, -1.0}],
      {Arrow[BezierCurve[{{3.3, -1.2}, {2.3, -2.0}, {xHGal[gLONHmaxDegrees, gLATHmaxDegrees],
           yHGal[gLONHmaxDegrees, gLATHmaxDegrees]}}]]},
      Text[StyleForm["Hmax", FontSize → 12, FontWeight -> "Bold"], {-3.3, 1.0}],
      {Arrow[BezierCurve[{{-3.3, 1.2}, {-2.3, 2.0}, {xHGal[gLONHmaxDegrees + 180,
            -gLATHmaxDegrees], yHGal[gLONHmaxDegrees + 180, -gLATHmaxDegrees]}}]]},
      Text[StyleForm["Hmin", FontSize → 12, FontWeight -> "Bold"], {3.3, 1.0}] ,
      {Arrow[BezierCurve[{{3.3, 1.2}, {2.3, 2.0}, {xHGal[gLONHminDegrees, gLATHminDegrees],
           yHGal[gLONHminDegrees, gLATHminDegrees]}}]]}
     }]}, ImageSize → 1.2 × 432];
```

5c. Section Summary

In[564]:=
```
mapOfηBar
```

Out[564]=

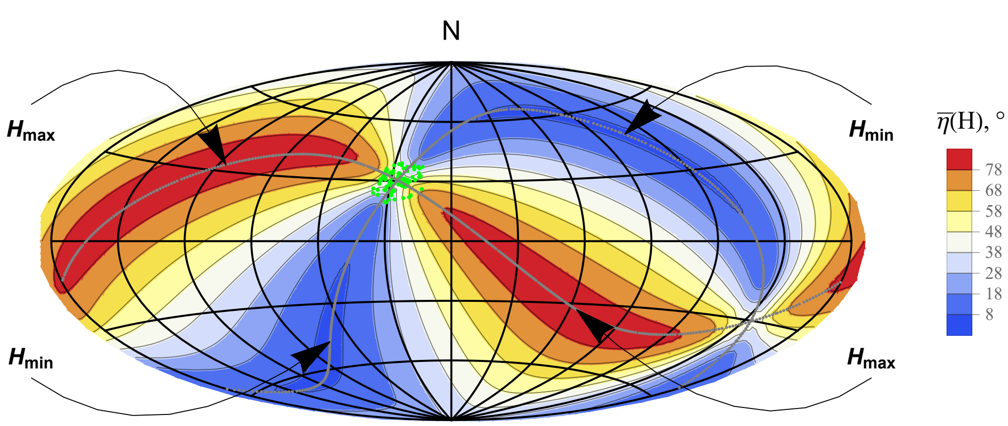


Figure 13: The alignment function $\overline{\eta}(H)$, Eq. (2). The map is centered on (gLON,gLAT) = (0°,0°), with East to the left as if one were looking up at the sky.

The sources are located at the dots, shaded ■ .

The smallest alignment angle is $\overline{\eta}_{min}$ = 7°, located in the areas shaded ■ at (gLON,gLAT) = {-107, 48} and {73, -48}, in degrees, which are the alignment hubs $H_{min}$ and $-H_{min}$ .

The arc on the sphere from the sample's center and the alignment hub $H_{min}$ is 88.6862°.

The largest avoidance angle is $\overline{\eta}_{max}$ = 83°, located in the areas shaded ■ at (gLON,gLAT) = {116, 33} and at {-64, -33}, in degrees, which are the avoidance hubs $H_{max}$ and $-H_{max}$ .

The arc on the sphere from the sample's center and the avoidance hub $H_{max}$ is 74°.

To guide the eye, two Great Circles are plotted, one through the sources' center and the avoidance hubs $H_{max}$ and $-H_{max}$. The other connects the center of the sources' locations with the alignment hubs $H_{min}$ and $-H_{min}$. The Great Circles are shaded Gray, ■ .

The angle between the planes of the two great circles is 90°.

Notes: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the function $\overline{\eta}(H)$ is symmetric across diameters. Diametrically opposite points -H and H have the same alignment angle $\overline{\eta}(H)$.

In[575]:=
```
(* Local contour plot of the alignment function ηBar(H). *)
frameticks = {{{ {yHGal[60, 30], 30 °}, {yHGal[60, 0], 0 °}}, None},
   {{{xHGal[0, 0], 0 °}, {xHGal[30, 3], 30 °}, {xHGal[60, 3], 60 °}}, None}};
listCPlocal = ListContourPlot[Union[xyηBarAitoffTable(*,
     {{xHGal[gLONHminDegrees,gLATHminDegrees],yHGal[gLONHminDegrees,gLATHminDegrees],
      ηBarMin*(360./(2.π))-1.0}},{{xHGal[gLONHmaxDegrees,gLATHmaxDegrees],
      yHGal[gLONHmaxDegrees,gLATHmaxDegrees],ηBarMax*(360./(2.π))+1.0}}*)],
   AspectRatio → 1., Contours → Table[η, {η, Floor[gridjηBarMin[[2]] * (360. / (2. π))] + 1,
      Ceiling[gridjηBarMax[[2]] * (360. / (2. π))] - 1, dηContourPlot}], ColorFunction →
    "TemperatureMap", PlotRange → {{xHGal[60, 0], xHGal[0, 0]}, {yHGal[0, 0], yHGal[0, 60]}},
   Axes -> False, Frame → True, FrameLabel → {"gLON", "gLAT", "Close-Up View"},
   FrameTicks → frameticks,
   PlotLegends → Placed[BarLegend[Automatic, LegendMargins → {{0, 0}, {10, 5}},
      LegendLabel → "η̅(H), °", LabelStyle → {Plain, FontFamily → "Times"}], Right]] ;
```

In[577]:=
```
(*Plot polarization directions*)
rPlusψ[i_, d_] :=
 (rSrc[[i]] + d vψSrc[[i]]) / ((rSrc[[i]] + d vψSrc[[i]]).(rSrc[[i]] + d vψSrc[[i]]))^(1/2)
polarLines[d_] :=
 Table[Line[{{xHGal[gLONFROMr[ rPlusψ[i, d]] (360./(2. π)), gLATFROMr[ rPlusψ[i, d]] (360./(2. π))],
     yHGal[gLONFROMr[ rPlusψ[i, d]] (360./(2. π)), gLATFROMr[ rPlusψ[i, d]] (360./(2. π))]},
    {xHGal[gLONFROMr[ rPlusψ[i, -d]] (360./(2. π)), gLATFROMr[ rPlusψ[i, -d]] (360./(2. π))], yHGal[
      gLONFROMr[ rPlusψ[i, -d]] (360./(2. π)), gLATFROMr[ rPlusψ[i, -d]] (360./(2. π))]}}], {i, nSrc}]
```

In[579]:=
```
(*Construct the map of η̄(H).*)
mapOfηBarLocal =
  Show[{listCPlocal, Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, -20, 90},
      PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60], {gLON, -60, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -60, 180},
      PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60], {gLAT, -30, 90, 30}],
    Graphics[{PointSize[0.009], Black, {Thick, polarLines[0.03]}, (*Sources S:*)
      Green, PointSize[0.012], Point[ xyAitoffSources ], Gray,
      PointSize[0.008], Point[ xyAitoffGreatMin ], Point[ xyAitoffGreatMax ]
        }]}, ImageSize → 0.9 × 432];
```

In[580]:=
```
mapOfηBarLocal
```

Out[580]=

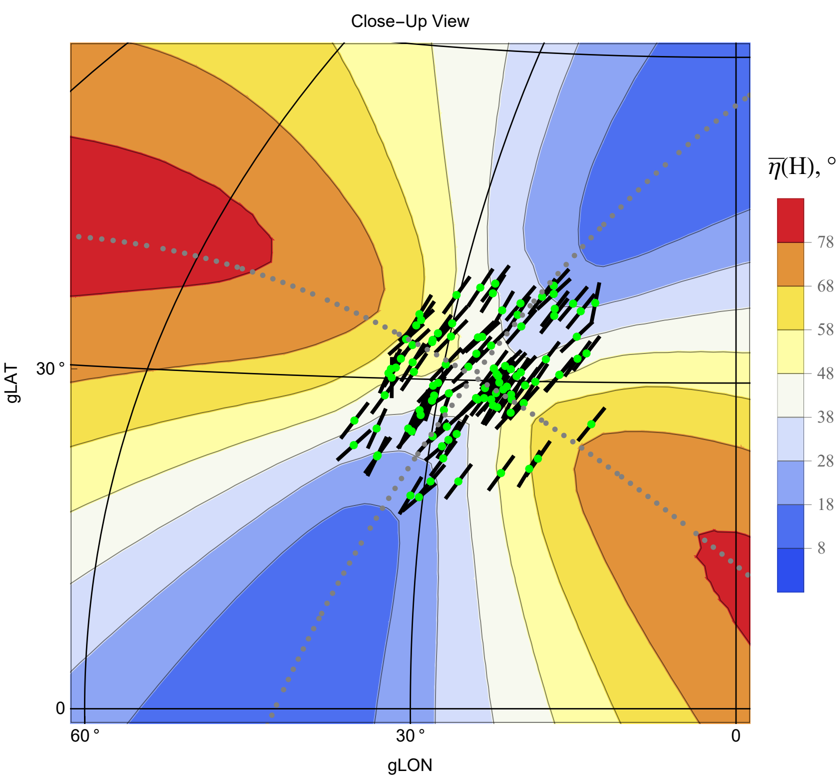



Figure 14: Map of the alignment angle function $\overline{\eta}\langle H\rangle$ in the neighborhood of the sources. The green dots locate the sources and the short line segments show the polarization directions. The average polarization position angle is $\psi$ = 149 °, which is measured counterclockwise from North with East to the Left.

## 6. Uncertainty Runs

### 6a. Creating and Storing Uncertainty Runs

For each “uncertainty run”, the polarization direction $\psi$ for each source is allowed to differ from the best value $\psi$Src by an amount $\delta\psi$ chosen according to a Gaussian distribution with mean (best) value $\psi$Src and half-width $\sigma\psi$, $\psi = \psi\text{Src} + \delta\psi$. Both values $\psi$Src and $\sigma\psi$ are taken from the catalogs.

Definitions:

rSrcxrGrid unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector $S_i$ to a source with the radial vector $H_j$ to a grid point
$\mu$ the mean value $\mu$ of the measurement Gaussian for $\psi$
$\sigma$ the uncertainty of the measured polarization position angle $\psi$
$\psi$DataU polarization directions $\psi = \psi\text{Src} + \delta\psi$
runDataU collection of data to save from the uncertainty runs, see below for content list
nRunPrintU dummy index controlling when current TimeUsed and MemoryInUse are printed
$\psi$SrcU the polarization direction $\psi$ for the run.
rSrcx$\psi$SrcU unit vector, $S_i \times \psi_i$, cross product of the radial vector $S_i$ to the source with the vector $\hat{v}_\psi$ in the direction of the polarization
j$\eta$BarToGridU {j, $\overline{\eta}(H_j)$}, where j is the index for the grid point $H_j$ and $\overline{\eta}(H_j)$ is the alignment angle function, (1), at $H_j$
sortj$\eta$BarToGrid sort {j, $\overline{\eta}(H_j)$}, with the smaller angle $\overline{\eta}$(H) first.
j$\eta$BarMinUU {$j$,$\overline{\eta}$(H)} for the smallest value of $\overline{\eta}$(H) , best alignment
j$\eta$BarMaxUU {$j$,$\overline{\eta}$(H)}, for the largest value of $\overline{\eta}$(H) , most avoided

$\eta$BarMinUData values of $\overline{\eta}_{\min}$ from uncertainty runs, alignment
$\eta$BarMaxUData values of $\overline{\eta}_{\max}$ from uncertainty runs, avoidance
HmingLONData values of gLON = gLON for hub $H_{\min}$ from uncertainty runs, alignment
HmingLATData values of gLAT = gLAT for hub $H_{\min}$ from uncertainty runs, alignment
HmaxgLONData values of gLON = gLON for hub $H_{\max}$ from uncertainty runs, avoidance
HmaxgLATData values of gLAT = gLAT for hub $H_{\max}$ from uncertainty runs, avoidance

Tables:
$\psi$DataU entries: 1. Run # 2. $\psi$SrcU, list of polarization position angles $\psi$
runDataU entries: 1. Run # 2. {$\overline{\eta}_{\min}$, {gLON,gLAT} at $H_{\min}$} 3. {$\overline{\eta}_{\max}$, {gLON,gLAT} at $H_{\max}$}

To create Uncertainty Runs, first calculate “rSrcxrGridU” and then evaluate the “For” statement in the following two cells. One can save the results with the “Put[]” statements.
Once saved, there is no need to repeat the runs. Comment out the “rSrcxrGridU” and “For” statements by enclosing each in (*c-omment brackets*). The data can be retrieved with the “Get” statements.

In[582]:=
```
(* Evaluate this cell for the notebook .nb version *)
 (*
nR=2;
t1=TimeUsed[];
rSrcxrGridU1=Table[ Cross[  rSrc[[i]],rGrid[[j]]  ] ,    {i,nSrc},{j,nGrid}];
(*first step: gLONw cross product, not unit vectors*)
rSrcxrGridU=Table[ rSrcxrGridU1[[i,j]]/
   (rSrcxrGridU1[[i,j]].rSrcxrGridU1[[i,j]]+ 0.000001)^1/2. ,    {i,nSrc},{j,nGrid}];
Clear[rSrcxrGridU1];
*)
```

In[583]:=
```
(*
gridDataU={};ψDataU={};funcDataU={};nRunPrint=0;
*)
```

In[584]:=
```
(*
For[nRun=1,nRun≤nR,nRun++,
  If[nRun>nRunPrint,Print["At the start of run ",nRun,", the time is ",
    TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];
   nRunPrint=nRunPrint+100];
     ψSrcU=Table[RandomVariate[NormalDistribution[ψSrc[[i]],σψSrc[[i]]]],{i,nSrc}];
  (*table of PPA angles ψ for the sources in region j0, in radians*)
  rSrcxψSrcU = Table[ Sin[ψSrcU[[i]]]eNSrc[[i]]-
     Cos[ψSrcU[[i]]] eESrc[[i]],    {i,nSrc}];
  (*table of the cross product of rSrc and vector in direction of ψSrcU,
  a unit vector*)jηBarToGridU = Table[{j,(1/nSrc)Sum[ArcCos[  Abs[
          rSrcxψSrcU[[i]].rSrcxrGridU[[i,j]]  ] - 0.000001  ],{i,nSrc}]},{j,nGrid}];
  (*
  {grid point #, value of the alignment angle ηnHj[j] averaged over all sources,
   in radians}*) sortjηBarToGridU=Sort[jηBarToGridU,#1[[2]]<#2[[2]]&];
  (*jηBarToGridU, {j,ηj}, but sorted with the smallest alignment angles first
  *)
  jηBarMinU=sortjηBarToGridU[[1]]; (*  {j,ηj}, at the grid point Hj with minimum η̅*)
  jηBarMaxU=sortjηBarToGridU[[-1]]; (*  {j,ηj},
  at the grid point Hj with maximum η̅*)AppendTo[ψDataU,{nRun,ψSrcU}];
  AppendTo[gridDataU,{nRun,{ jηBarMinU[[2]],{gLONGrid [ [ jηBarMinU[[1]]  ]],
      gLATGrid [[  jηBarMinU[[1]]  ]]}},{ jηBarMaxU[[2]],
     {gLONGrid [[  jηBarMaxU[[1]]  ]],gLATGrid [[  jηBarMaxU[[1]]  ]]}}} ];
  (*collect discrete (on-grid) data*)
     ηmingLONgLATHU=FindMinimum[ηBarAtHwithAnyψ[gLONH,gLATH,ψDataU[[nRun,2]]],
    {{gLONH,gridDataU[[nRun,2,2,1]]},{gLATH,gridDataU[[nRun,2,2,2]]}}];
   ηmaxgLONgLATHU=
   FindMaximum[ηBarAtHwithAnyψ[gLONH,gLATH,ψDataU[[nRun,2]]],
    {{gLONH,gridDataU[[nRun,3,2,1]]},{gLATH,gridDataU[[nRun,3,2,2]]}}];
  AppendTo[funcDataU,{nRun,{ ηmingLONgLATHU[[1]],{gLONH,gLATH}/.ηmingLONgLATHU[[2]]},
    {  ηmaxgLONgLATHU[[1]],{gLONH,gLATH}/.ηmaxgLONgLATHU[[2]]}} ]
   (*collect continuous (function-based) data*)   ]
 t2=TimeUsed[];
Print["Time used to compute ψDataU, gridDataU, and funcDataU: t2 - t1 = ",t2-t1]
*)
```

Hint: You can save memory if you do not get the “ψDataU”. The table ψDataU is needed to reconstruct the exact values of the runDataU table, but it is not needed in any following calculation.

In[585]:=
```
SetDirectory[homeDirectory];(*Save memory space; ψDataU is not used below.*)
(*
Put[ψDataU,"20210627PsiUDataLon30Lat30offDiskHB5000b.dat"  ]
*)(*
Put[gridDataUn,"20211005gridDataLon30Lat30offDiskHB5000b.dat"  ]
*)(*
Put[runDataU,"20210928runDataULon30Lat30offDiskHB5000b.dat"  ]
*)(*
Put[funcDataU,"20211005funcDataLon30Lat30offDiskHB5000b.dat"  ]
*)
```

Hint: Saving “runDataU” to a file avoids the time it takes to complete the “For” statement. Make the above “For” statement into a remark so that it doesn’t evaluate.

In[586]:=
```
SetDirectory[homeDirectory];
(*
ψDataU500=Get["20211018PsiUDataLon30Lat30offDiskHB500z.dat"];
ψDataU4500=Get["20211019PsiUDataLon30Lat30offDiskHB4500z.dat"];
ψDataU5000=Get["20211019PsiUDataLon30Lat30offDiskHB5000z.dat"];
*)(*
gridDataU500=Get["20211018gridDataULon30Lat30offDisk500z.dat"];
gridDataU4500=Get["20211019gridDataULon30Lat30offDisk4500z.dat"];
gridDataU5000=Get["20211019gridDataULon30Lat30offDisk5000z.dat"];
*)

funcDataU500 = Get["20211018funcDataLon30Lat30offDisk500z.dat"];
funcDataU4500 = Get["20211019funcDataLon30Lat30offDisk4500z.dat"];
funcDataU5000 = Get["20211019funcDataLon30Lat30offDisk5000z.dat"];
```

In[590]:=
```
(*If needed, edit the following to collect data files together.*)

funcDataU = Join[funcDataU500, funcDataU4500, funcDataU5000];
```

In[591]:=
```
(*nR may not be previously defined, depending on what cells have been processed.*)
(*Define nR for the pdf version:*)

nR = Length[funcDataU]
```

Out[591]= 10 000

In[592]:=
```
(*Define quantities based on the off-grid results.*)
(*"func" indicates the function "ηBarAtHwithAnyψ" was used to go off-grid.*)
ηBarMinfunDataU = Table[funcDataU[[i1, 2, 1]] , {i1, Length[funcDataU]}];
ηBarMaxfunDataU = Table[funcDataU[[i1, 3, 1]] , {i1, Length[funcDataU]}];
HmingLONfunDataU = Table[ funcDataU[[i1, 2, 2, 1]] , {i1, Length[funcDataU]}];
HmingLATfunDataU = Table[funcDataU[[i1, 2, 2, 2]], {i1, Length[funcDataU]}];
HmaxgLONfunDataU = Table[ funcDataU[[i1, 3, 2, 1]] , {i1, Length[funcDataU]}];
HmaxgLATfunDataU = Table[funcDataU[[i1, 3, 2, 2]], {i1, Length[funcDataU]}];
```

In[598]:=
```
(*ListPlot[
  {Table[{-HmingLONfunDataU[[i]],HmingLATfunDataU[[i]]},{i,Length[HmingLATfunDataU]}],
   Table[{-HmaxgLONfunDataU[[i]],HmaxgLATfunDataU[[i]]},
    {i,Length[HmaxgLATfunDataU]}] },PlotRange→All,
  PlotStyle→{{Blue,PointSize[0.01]},{Red,PointSize[0.01]}},
  PlotLabel→"The hubs from the uncertainty runs",
  AxesLabel→{"-gLON (rad)","gLAT (rad)Change TICKS"}]
 Print["Figure NN: Uncertainty run hubs. The alignment hubs Hmin are in blue, ",
  Blue," The avoidance hubs Hmax are in ",Red,
  ". Symmetry across a diameter means there are hubs
    diametrically opposed to these. Including any diametrically
    opposed hubs would ruin the statistical calculations for hubs."]*)
```

6b. The Effects of Uncertainty on the Smallest Alignment Angle $\overline{\eta}_{\min}$

This section fits a Gaussian distribution to the $\overline{\eta}_{\min}$ from the uncertainty runs.

Definitions

| | |
|---|---|
| sort$\eta$BarMin | sort the list of $\overline{\eta}_{\min}$ from the uncertainty runs |
| $\eta$0minU | estimated mean of the Gaussian fit |
| $\sigma$minU | estimated half-width of the Gaussian fit |
| hlminU0, hlminU | histogram {$\eta$, bin height} tables needed to set up the NonlinearModelFit |
| nlmminU | non-linear model fit of a Gaussian to the $\overline{\eta}_{\min}$ histogram |
| showNLMB | plot of Gaussian and histogram |
| pTableNLMminU | table of parameter attributes, including standard error |

$\sigma\eta$BarminUFit, $\eta$BarminUFit - half-width, and mean of the Gaussian fit

In[599]:=
```
Print["The number of uncertainty runs is ", Length[funcDataU], "."]
```

```
The number of uncertainty runs is 10000.
```

In[600]:=
```
sortηBarMinU = Sort[ηBarMinfunDataU];
η0minU = mean[ηBarMinfunDataU ]; (*Guess the mean for the Gaussian. *)
σminU = stanDev[ηBarMinfunDataU ];(*Guess the half-width.*)
hlminU0 = HistogramList[sortηBarMinU, {η0minU - 5 σminU, η0minU + 5 σminU, 0.4 σminU}];
hlminU = Table[{(1/2) (hlminU0[[1, i1]] + hlminU0[[1, i1 + 1]]), hlminU0[[2, i1]]},
   {i1, Length[ hlminU0[[2]] ]}];
nlmminU = NonlinearModelFit[hlminU, a Exp[-(1/2.) ((x - x0) / b)^2],
   {{a, Length[sortηBarMinU / 6]}, {b, σminU}, {x0, η0minU}}, x];(*x is ηBarMin*)
```

In[605]:=
```
pTableNLMminU = nlmminU["ParameterTable"]
{σηBarminUFit, ηBarminUFit} = {b, x0} /. nlmminU["BestFitParameters"];(*radians*)
```

Out[605]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1589.93 | 6.35719 | 250.099 | $1.70755\times10^{-39}$ |
| b | 0.00500679 | 0.0000231162 | 216.592 | $4.03796\times10^{-38}$ |
| x0 | 0.126606 | 0.0000231162 | 5476.96 | $5.55364\times10^{-69}$ |

In[607]:=
```
showNLMB = Show[{Histogram[sortηBarMinU, {η0minU - 5 σminU, η0minU + 5 σminU, 0.4 σminU},
     PlotLabel → "Uncertainty run η̄min ", AxesLabel → {"η̄min, radians", "ΔR"}],
   Plot[Normal[nlmminU], {x, η0minU - 5 σminU, η0minU + 5 σminU}, PlotLabel → "η̄min"],
   ListPlot[hlminU, PlotLabel → "η̄min"] }];
```

In[608]:=
```
(*showNLMB
 Print[
  "Figure NN: The Gaussian fit to the alignment angle η̄min histogram. The height is the
    number of runs ΔR in each bin. Note how nicely symmetric this is."]
 Print["The total number of runs is R = Σ(ΔR) = ",Length[funcDataU],"."]*)
```

6c. The Effects of Uncertainty on the Largest Avoidance Angle $\overline{\eta}_{\max}$

This section fits a Gaussian distribution to the $\overline{\eta}_{\max}$ returned by the uncertainty runs.

Definitions: Similar to the definitions in Sec. 6b.

In[609]:=
```
sortηBarMaxU = Sort[ηBarMaxfunDataU];
η0maxU = mean[ηBarMaxfunDataU ]; (*Guess the mean for the Gaussian. *)
σmaxU = stanDev[ηBarMaxfunDataU ];(*Guess the half-width.*)
histogramrangemaxU = {η0maxU - 5 σmaxU, η0maxU + 5 σmaxU, 0.4 σmaxU};
hl0maxU = HistogramList[sortηBarMaxU, histogramrangemaxU];
hlmaxU = Table[{(1/2) (hl0maxU[[1, i1]] + hl0maxU[[1, i1 + 1]]), hl0maxU[[2, i1]]},
  {i1, Length[ hl0maxU[[2]] ]}];
nlmmaxU = NonlinearModelFit[hlmaxU, a Exp[- (1/2.) ((x - x0) / b)^2],
  {{a, 300.}, {b, σmaxU}, {x0, η0maxU}}, x];(*x is ηBarmaxU *)
```

In[615]:=
```
pTableNLMmaxU = nlmmaxU["ParameterTable"]
{σηBarmaxUFit, ηBarmaxUFit} = ParametersNLMmaxU = {b, x0} /. nlmmaxU["BestFitParameters"];
(*radians*)
```

Out[615]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1592.59 | 10.3845 | 153.362 | $7.98694\times10^{-35}$ |
| b | 0.00492273 | 0.0000370645 | 132.815 | $1.88487\times10^{-33}$ |
| x0 | 1.44504 | 0.0000370645 | 38 987.1 | $9.82146\times10^{-88}$ |

In[617]:=
```
showNLMmaxU = Show[{Histogram[sortηBarMaxU,
     histogramrangemaxU, PlotLabel → "η̄max", AxesLabel → {"η̄max, radians", "ΔR"}],
   Plot[Normal[nlmmaxU], {x, η0maxU - 5 σmaxU, η0maxU + 5 σmaxU}, PlotLabel → "η̄max"] ,
   ListPlot[hlmaxU, PlotLabel → "η̄max"]}];
```

In[618]:=
```
GraphicsRow[{showNLMB, showNLMmaxU}]
```

Out[618]=

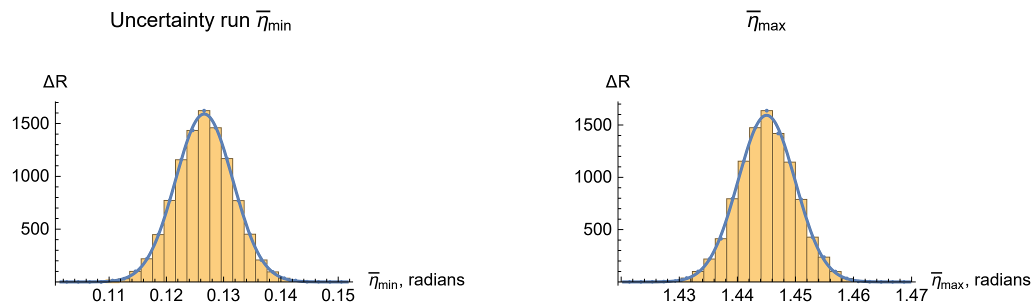



Figure 15: The Gaussian fits to the (*Left*) alignment angle $\overline{\eta}_{\min}$ and the (*Right*) avoidance angle $\overline{\eta}_{\max}$ histograms. The distributions for the pdf version are nicely symmetric. The pdf version is based on 10,000 uncertainty runs. Those who upload the Mathematica notebook in Ref. 1 and generate at least 500 runs should see a rough resemblance to the 10,000 run result.

6d. The Effects of Uncertainty on the Locations (gLON,gLAT) of the Alignment Hubs $H_{\min}$

Each uncertainty run returns an alignment hub $H_{\min}$. In this section, we investigate the distribution of the locations the alignment Hubs $H_{\min}$.

There are two hubs, $H_{\min}$ and $-H_{\min}$ for each uncertainty run, by the symmetry across a diameter. So we collect the data together by moving the $-H_{\min}$ hubs across a diameter to join the $H_{\min}$ hubs.

Definitions

| | |
|---|---|
| HmingLON | gLON in radians for $H_{\min}$ |
| HmingLAT | gLAT in radians for $H_{\min}$ |
| $\sigma$gLONMinFit1 | half-width for gLON uncertainty runs |
| gLONMinFit1 | mean for gLON uncertainty runs |
| $\sigma$gLATMinFit1 | half-width for gLAT uncertainty runs |
| gLATMinFit1 | mean for gLAT uncertainty runs |
| HmingLONAVE | average over all uncertainty runs of gLON for $H_{\min}$ |
| HmingLONgLAT | (gLON,gLAT) table for ListPlot |
| lpHmin | plot Hmin hubs from uncertainty runs |
| gLON1,2Min1 | values needed for framing the most likely hubs |
| gLAT1,2Min1 | ditto for latitude |

In[620]:=
```
HmingLONU = {}; HmingLATU = {};
For[i = 1, i ≤ Length[HmingLONfunDataU], i++,
 If[HmingLONfunDataU[[i]] < 0, AppendTo[HmingLONU, HmingLONfunDataU[[i]] + π];
  AppendTo[HmingLATU, -HmingLATfunDataU[[i]]],
  AppendTo[HmingLONU, HmingLONfunDataU[[i]]];
  AppendTo[HmingLATU, HmingLATfunDataU[[i]]]]
]
lpHminU = ListPlot[Table[{-HmingLONU[[i]], HmingLATU[[i]]}, {i, Length[HmingLATU]}],
   PlotRange → All, PlotStyle → {Blue, PointSize[0.01]},
   PlotLabel → "The alignment hubs from the uncertainty runs",
   AxesLabel → {"gLON (rad)", "gLAT (rad)"}];
```

In[623]:=
```
sortHmingLON = Sort[HmingLONU];
x0Hmin = mean[HmingLONU];(*Guess the mean for the Gaussian. *)
dx0Hmin = stanDev[HmingLONU];(*Guess the half-width.*)
histogramrangeRAHminU = {x0Hmin - 5 dx0Hmin, x0Hmin + 5 dx0Hmin, 0.4 dx0Hmin};
hl0xHmin = HistogramList[sortHmingLON, histogramrangeRAHminU];
hlxHmin = Table[{(1/2) (hl0xHmin[[1, i1]] + hl0xHmin[[1, i1 + 1]]), hl0xHmin[[2, i1]]},
  {i1, Length[ hl0xHmin[[2]] ]}];
nlmxHmin = NonlinearModelFit[hlxHmin, a Exp[-(1/2.) ((x - x0) / b)^2],
  {{a, Length[sortHmingLON / 6]}, {b, dx0Hmin}, {x0, x0Hmin}}, x];(*x is Hminα*)
```

In[629]:=
```
pTablenlmxHmin = nlmxHmin["ParameterTable"]
{σHmingLONFit, HmingLONFit} =
  ParametersnlmxHmin = {b, x0} /. nlmxHmin["BestFitParameters"];(*radians*)
expOfnlmxHmin[x_] := -(1/2.) ((x - x0) / b)^2 /. nlmxHmin["BestFitParameters"]
Print["The fitting function for the gLON of H_min is aⅇ^A = ",
 Normal[nlmxHmin], ", with the following function in the exponential, A = ",
 expOfnlmxHmin[x], ". Here, 'x' stands for gLON in radians."]
```

Out[629]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1660.04 | 164.284 | 10.1047 | $9.98397\times10^{-10}$ |
| b | 0.111447 | 0.0127354 | 8.75094 | $1.29001\times10^{-8}$ |
| x0 | 1.3273 | 0.0127354 | 104.221 | $3.87389\times10^{-31}$ |

The fitting function for the gLON of $H_{min}$ is $a e^{A}$ = 
 $1660.04\, e^{-40.2564\,(-1.3273+x)^2}$, with the following function in the exponential, A = 
 $-40.2564\,(-1.3273+x)^2$. Here, 'x' stands for gLON in radians.

In[633]:=
```
shownlmxHmin = Show[{Histogram[sortHmingLON, histogramrangeRAHminU,
     PlotLabel → "gLON, Hmin ", AxesLabel → {"gLON, radians", "ΔR"}, PlotRange → All],
    Plot[Normal[nlmxHmin], {x, 0.8, 1.9}, PlotRange → All, PlotLabel → "gLON, Hmin"],
    ListPlot[hlxHmin, PlotLabel → "gLON, Hmin"] }];
```

In[634]:=
```
sortHmingLAT = Sort[HmingLATU];
y0Hmin = mean[HmingLATU ];(*Guess the mean for the Gaussian. *)
dy0Hmin = stanDev[HmingLATU ];(*Guess the half-width.*)
histogramrangeDecHminU = {y0Hmin - 5 dy0Hmin, y0Hmin + 5 dy0Hmin, 0.4 dy0Hmin};
hl0yHmin = HistogramList[sortHmingLAT, histogramrangeDecHminU];
hlyHmin = Table[{(1/2) (hl0yHmin[[1, i1]] + hl0yHmin[[1, i1 + 1]]), hl0yHmin[[2, i1]]},
   {i1, Length[ hl0yHmin[[2]] ]}];
nlmyHmin = NonlinearModelFit[hlyHmin, a Exp[- (1/2.) ((y - y0) / b)^2],
   {{a, Length[sortHmingLAT / 6]}, {b, dy0Hmin}, {y0, y0Hmin}}, y];(*y is Hminδ*)
```

In[640]:=
```
pTablenlmyHmin = nlmyHmin["ParameterTable"]
{σHmingLATFit, HmingLATFit} =
  ParametersnlmyHmin = {b, y0} /. nlmyHmin["BestFitParameters"];(*radians*)
Normal[nlmyHmin];
expOfnlmyHmin[y_] := - (1/2.) ((y - y0) / b)^2 /. nlmyHmin["BestFitParameters"]
expOfnlmyHmin[y];
Print["The fitting function for the gLAT of H_min is aℯ^B = ",
 Normal[nlmyHmin], ", with the following function in the exponential, B = ",
 expOfnlmyHmin[y] , ". Here, 'y' stands for gLAT in radians."]
```

Out[640]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1549.32 | 143.035 | 10.8317 | $2.76734 \times 10^{-10}$ |
| b | 0.070105 | 0.00747344 | 9.38055 | $3.81407 \times 10^{-9}$ |
| y0 | −0.872899 | 0.00747344 | −116.8 | $3.17184 \times 10^{-32}$ |

The fitting function for the gLAT of $H_{min}$ is $a e^{B}$ = 1549.32 $e^{-101.735\,(0.872899+y)^2}$, with the following function in the exponential, B = $-101.735\,(0.872899 + y)^2$. Here, 'y' stands for gLAT in radians.

In[646]:=
```
shownlmyHmin = Show[{Histogram[sortHmingLAT, histogramrangeDecHminU,
     PlotLabel → "gLAT, Hmin ", AxesLabel → {"gLAT, radians", "ΔR"}, PlotRange → All],
    Plot[Normal[nlmyHmin], {y, -1.25, -0.55}, PlotRange → All, PlotLabel → "gLAT, Hmin"],
    ListPlot[hlyHmin, PlotLabel → "gLAT, Hmin"] }];
```

In[647]:=
```
GraphicsRow[{shownlmxHmin, shownlmyHmin}]
```

Out[647]=

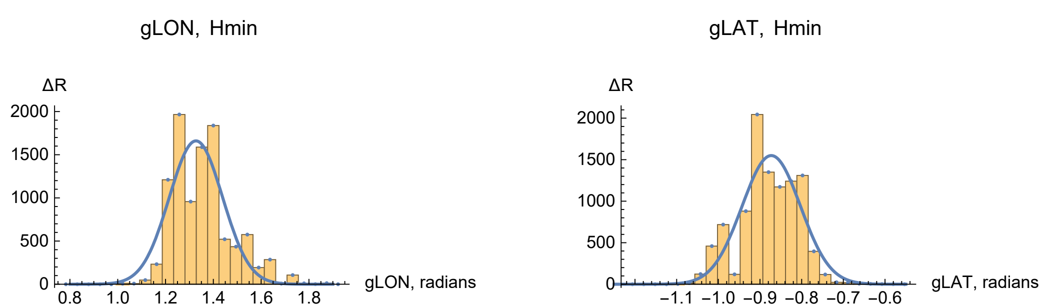



Figure 16: The Gaussian fits to the Hmin gLON and gLAT histograms.
Uncertainty runs may bring out local minima. Clearly, the gLON distribution on the Left suggests that there are two or more competing local minima.

In[649]:=
```
expoHminU[x_, y_] := -(expOfnlmxHmin[x] + expOfnlmyHmin[y])
Print["The exponent of the probability distribution for a particular combination
   (gLON,gLAT) for Hmin, i.e. the negative log of the distribution, -(A + B) = ",
 expoHminU[gLON, gLAT], ", where 'x' stands for gLON and 'y' stands for gLAT."]
```

The exponent of the probability distribution for a particular combination (gLON,gLAT) for $H_{min}$, *i.e.* the negative log of the distribution, $-(A + B) = 101.735\,(0.872899 + gLAT)^2 + 40.2564\,(-1.3273 + gLON)^2$, where 'x' stands for gLON and 'y' stands for gLAT.

In[651]:=
```
p3DHminU =
  Plot3D[{expoHminU[x, y], 0.5}, {x, x0 - 0.15, x0 + 0.15} /. nlmxHmin["BestFitParameters"],
   {y, y0 - 0.15, y0 + 0.15} /. nlmyHmin["BestFitParameters"],
   PlotLabel → "Negative log of the probability of (gLON,gLAT) for Hmin",
   AxesLabel → {"gLON (rad)", "gLAT (rad)", "-(A+B)"}];
```

In[652]:=
```
p3DHminU
```

Out[652]=

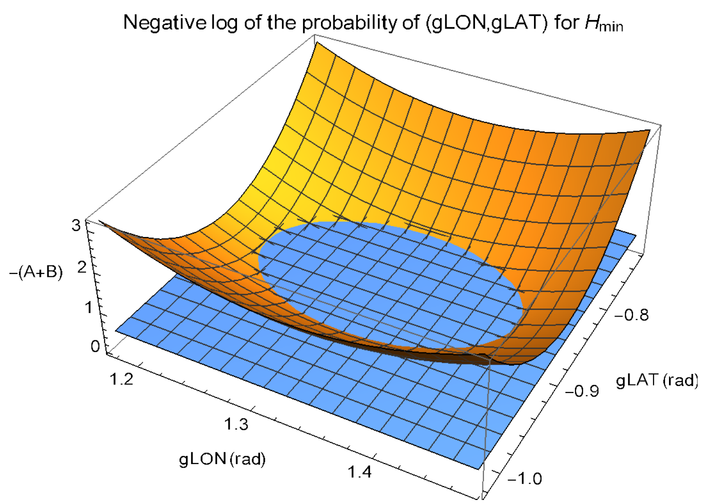



Figure 17: The negative log of the likelihood of (gLON,gLAT) for $H_{min}$, as a function of gLON and gLAT. Where the likelihood is down by a factor $e^{-1/2}$, the negative log is $-(A + B) = +0.5$ and that defines the half-width $\sigma$ of the distribution.

In[654]:=
```
(*Find the curve for the intersection in Fig. 17*)
frθHmin[r_, θ_] :=
 Simplify[(expoHminU[x, y]) - 0.5 /. {x → HmingLONFit + r Cos[θ], y → HmingLATFit + r Sin[θ]}]
frθHmin[r, θ];
solverHminθ[θ_] := Solve[frθHmin[r, θ] == 0, r];
solverHminθ[θ];
rHminθ[θ_] := Abs[r /. solverHminθ[θ][[2]]]
rHminθ[θ];
rHminθ[0.8];
Plot[rHminθ[θ], {θ, -π, π}];
```

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

In[662]:=
```
(*Show[
  {lpHminU,ParametricPlot[{-HmingLONFit-rHminθ[θ]Cos[θ],HmingLATFit+rHminθ[θ]Sin[θ]},
    {θ,0,2.π},PlotStyle→Orange,PlotRange→All(*{{3.12,3.14},{0.84,0.90}}*)]}]
 Print["Figure NN: All of the alignment hubs Hmin from uncertainty runs. The ellipse
    encloses the most likely locations of the hubs. Symmetry across diameters
    means there is another set diametrically opposite those displayed here."]*)
```

6d. The Effects of Uncertainty on the Locations (gLON,gLAT) of the Avoidance Hubs $H_{\max}$

Each uncertainty run returns an alignment hub $H_{\max}$. In this section, we investigate the distribution of the locations the alignment Hubs $H_{\max}$.

There are two hubs, $H_{\max}$ and $-H_{\max}$ for each uncertainty run, by the symmetry across a diameter. So we collect the data together by moving the $-H_{\max}$ hubs across a diameter to join the $H_{\max}$ hubs.

Definitions

| | |
|---|---|
| HmingLON | gLON in radians for $H_{\max}$ |
| HmaxgLAT | gLAT in radians for $H_{\max}$ |
| σgLONMinFit1 | half-width for gLON uncertainty runs |
| gLONMinFit1 | mean for gLON uncertainty runs |
| σgLATMinFit1 | half-width for gLAT uncertainty runs |
| gLATMinFit1 | mean for gLAT uncertainty runs |
| HmaxgLONAVE | average over all uncertainty runs of gLON for $H_{\max}$ |
| HmaxgLONgLAT | (gLON,gLAT) table for ListPlot |
| lpHmax | plot Hmax hubs from uncertainty runs |
| gLON1,2Min1 | values needed for framaxg the most likely hubs |
| gLAT1,2Min1 | ditto for latitude |

In[663]:=
```
HmaxgLONU = {}; HmaxgLATU = {};
For[i = 1, i ≤ Length[HmaxgLONfunDataU], i++,
 If[HmaxgLONfunDataU[[i]] < 0, AppendTo[HmaxgLONU, HmaxgLONfunDataU[[i]] + π];
  AppendTo[HmaxgLATU, -HmaxgLATfunDataU[[i]]],
  AppendTo[HmaxgLONU, HmaxgLONfunDataU[[i]]];
  AppendTo[HmaxgLATU, HmaxgLATfunDataU[[i]]]]
]
(*lpHmaxU=ListPlot[Table[{-HmaxgLONU[[i]],HmaxgLATU[[i]]},{i,Length[HmaxgLATU]}],
  PlotRange→All,PlotStyle→{Red,PointSize[0.01]},
  PlotLabel→"The alignment hubs from the uncertainty runs",
  AxesLabel→{"gLON (rad)","gLAT (rad)"}]*)
```

In[665]:=
```
sortHmaxgLON = Sort[HmaxgLONU];
x0Hmax = mean[HmaxgLONU];(*Guess the mean for the Gaussian. *)
dx0Hmax = stanDev[HmaxgLONU ];(*Guess the half-width.*)
histogramrangeRAHmaxU = {x0Hmax - 5 dx0Hmax, x0Hmax + 5 dx0Hmax, 0.5 dx0Hmax};
hl0xHmax = HistogramList[sortHmaxgLON, histogramrangeRAHmaxU];
hlxHmax = Table[{(1/2) (hl0xHmax[[1, i1]] + hl0xHmax[[1, i1 + 1]]), hl0xHmax[[2, i1]]},
   {i1, Length[ hl0xHmax[[2]] ]}];
nlmxHmax = NonlinearModelFit[hlxHmax, a Exp[- (1/2.) ((x - x0) / b)^2],
   {{a, Length[sortHmaxgLON / 6]}, {b, dx0Hmax}, {x0, x0Hmax}}, x];(*x is Hmaxα*)
```

In[671]:=
```
pTablenlmxHmax = nlmxHmax["ParameterTable"]
{σHmaxgLONFit, HmaxgLONFit} =
  ParametersnlmxHmax = {b, x0} /. nlmxHmax["BestFitParameters"];(*radians*)
expOfnlmxHmax[x_] := - (1/2.) ((x - x0) / b)^2 /. nlmxHmax["BestFitParameters"]
Print["The fitting function for the gLON of Hmax is ae^A = ",
 Normal[nlmxHmax], ", with the following function in the exponential, A = ",
 expOfnlmxHmax[x], ". Here, 'x' stands for gLON in radians."]
```

Out[671]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1891.43 | 76.6461 | 24.6775 | $9.41322\times10^{-15}$ |
| b | 0.0909869 | 0.00425743 | 21.3713 | $1.00971\times10^{-13}$ |
| x0 | 2.1058 | 0.00425743 | 494.618 | $8.63599\times10^{-37}$ |

The fitting function for the gLON of $H_{max}$ is $a e^{A}$ = $1891.43\, e^{-60.3965\,(-2.1058+x)^2}$, with the following function in the exponential, A = $-60.3965\,(-2.1058+x)^2$. Here, 'x' stands for gLON in radians.

In[675]:=
```
shownlmxHmax =
  Show[{Histogram[sortHmaxgLON, histogramrangeRAHmaxU, PlotLabel → "gLONHmax ",
     AxesLabel → {"gLONHmax, radians", "ΔR"}, PlotRange → All],
    Plot[Normal[nlmxHmax], {x, 1.7, 2.6}, PlotRange → All, PlotLabel → "gLONHmax"],
    ListPlot[hlxHmax, PlotLabel → "gLONHmax"] }];
```

In[676]:=
```
sortHmaxgLAT = Sort[HmaxgLATU];
y0Hmax = mean[HmaxgLATU ];(*Guess the mean for the Gaussian. *)
dy0Hmax = stanDev[HmaxgLATU ];(*Guess the half-width.*)
histogramrangeDecHmaxU = {y0Hmax - 5 dy0Hmax, y0Hmax + 5 dy0Hmax, 0.45 dy0Hmax};
hl0yHmax = HistogramList[sortHmaxgLAT, histogramrangeDecHmaxU];
hlyHmax = Table[{(1/2) (hl0yHmax[[1, i1]] + hl0yHmax[[1, i1 + 1]]), hl0yHmax[[2, i1]]},
   {i1, Length[ hl0yHmax[[2]] ]}];
nlmyHmax = NonlinearModelFit[hlyHmax, a Exp[- (1/2.) ((y - y0) / b)^2],
   {{a, Length[sortHmaxgLAT / 6]}, {b, dy0Hmax}, {y0, y0Hmax}}, y];(*y is Hmaxδ*)
```

In[682]:=
```
pTablenlmyHmax = nlmyHmax["ParameterTable"]
{σHmaxgLATFit, HmaxgLATFit} =
  ParametersnlmyHmax = {b, y0} /. nlmyHmax["BestFitParameters"];(*radians*)
expOfnlmyHmax[y_] := - (1/2.) ((y - y0) / b)^2 /. nlmyHmax["BestFitParameters"]
Print["The fitting function for the gLAT of Hmax is ae^B = ",
 Normal[nlmyHmax], ", with the following function in the exponential, B = ",
 expOfnlmyHmax[y] , ". Here, 'y' stands for gLAT in radians."]
```

Out[682]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1898.72 | 161.501 | 11.7567 | $3.65903\times10^{-10}$ |
| b | 0.0380521 | 0.00373735 | 10.1816 | $3.93552\times10^{-9}$ |
| y0 | 0.547723 | 0.00373735 | 146.554 | $1.7684\times10^{-30}$ |

The fitting function for the gLAT of $H_{max}$ is $a e^{B}$ = $1898.72\, e^{-345.312\,(-0.547723+y)^2}$, with the following function in the exponential, B = $-345.312\,(-0.547723+y)^2$. Here, 'y' stands for gLAT in radians.

In[686]:=
```
shownlmyHmax =
  Show[{Histogram[sortHmaxgLAT, histogramrangeDecHmaxU, PlotLabel → "gLATHmax ",
     AxesLabel → {"gLATHmax, radians", "ΔR"}, PlotRange → All],
    Plot[Normal[nlmyHmax], {y, 0.33, 0.74}, PlotRange → All, PlotLabel → "gLATHmax"],
    ListPlot[hlyHmax, PlotLabel → "gLATHmax"] }];
```

In[687]:=
```
GraphicsRow[{shownlmxHmax, shownlmyHmax}]
```

Out[687]=

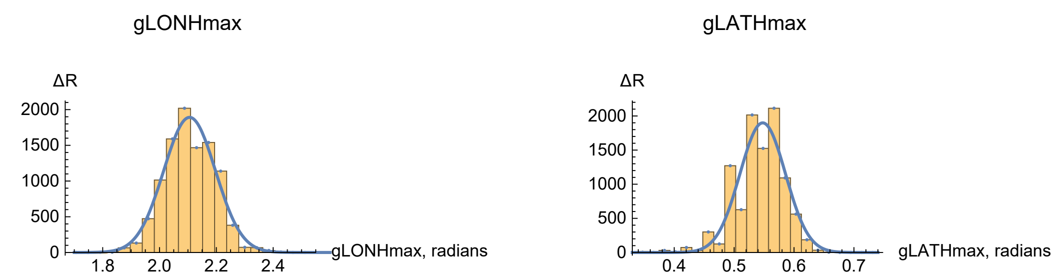



Figure 18: The Gaussian fits to the Hmax gLON and gLAT histograms, where the height is the number of runs ΔR in each bin.

In[689]:=
```
expoHmaxU[x_, y_] := - (expOfnlmxHmax[x] + expOfnlmyHmax[y])
Print["The exponent of the probability distribution for a particular combination
   (gLON,gLAT) for Hmax, i.e. the negative log of the distribution, -(A + B) = ",
 expoHmaxU[gLON, gLAT], ", where 'x' stands for gLON and 'y' stands for gLAT."]
```

The exponent of the probability distribution for a particular combination (gLON,gLAT) for $H_{max}$, *i.e.* the negative log of the distribution, $-(A + B)$ = $345.312\,(-0.547723+\text{gLAT})^2 + 60.3965\,(-2.1058+\text{gLON})^2$, where 'x' stands for gLON and 'y' stands for gLAT.

In[691]:=
```
(*Plot3D[{expoHmaxU[x,y],0.5},{x,x0-0.15,x0+0.15}/.nlmxHmax["BestFitParameters"],
  {y,y0-0.12,y0+0.12}/.nlmyHmax["BestFitParameters"],
  PlotLabel→"Negative log of the probability of (gLON,gLAT) for Hmax",
  AxesLabel→{"gLON (rad)","gLAT (rad)","-(A+B)"}]
 Print["Figure NN: The negative log of the likelihood of (gLON,gLAT)
    for Hmax, as a function of gLON and gLAT. Where the likelihood
    is down by a factor e^-1/2, the negative log is -(A + B) = +0.5
    and that defines the half-width σ of the distribution."]*)
```

In[692]:=
```
(*Find the curve for the intersection in Fig. NN just above.*)
frθHmax[r_, θ_] :=
 Simplify[(expoHmaxU[x, y]) - 0.5 /. {x → HmaxgLONFit + r Cos[θ], y → HmaxgLATFit + r Sin[θ]}]
frθHmax[r, θ];
solverHmaxθ[θ_] := Solve[frθHmax[r, θ] == 0, r];
solverHmaxθ[θ];
rHmaxθ[θ_] := Abs[r /. solverHmaxθ[θ][[2]]]
rHmaxθ[θ];
rHmaxθ[0.8];
Plot[rHmaxθ[θ], {θ, -π, π}];
```

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

... Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

In[700]:=
```
(*Show[
  {lpHmaxU,ParametricPlot[{-HmaxgLONFit-rHmaxθ[θ]Cos[θ],HmaxgLATFit+rHmaxθ[θ]Sin[θ]},
    {θ,0,2.π},PlotStyle→Orange,PlotRange→All(*{{3.12,3.14},{0.84,0.90}}*)]}]
 Print["Figure NN: Avoidance hubs Hmax from uncertainty runs. The ellipse encloses
    the most likely locations of the hubs. Symmetry across diameters means
    there is another set diametrically opposite those displayed here."]*)
```

6f. The Effects of Uncertainty on the angle between the Great Circles from the Sample Center to the hubs $H_{\min}$ and the hubs $H_{\max}$

These are the Gray dotted lines in Figs. 5,6,13,14,20.

Definitions:

| | |
|---|---|
| “uRuns” prefix | results from the uncertainty runs |
| uRunsCrossMin | unit vector normal to the Great Circle connecting the center of the source region with the alignment hub $H_{\min}$ |
| uRunsCrossMax | unit vector normal to the Great Circle connecting the center of the source region with the alignment hub $H_{\max}$ |
| uRuns$\theta$minmaxUgreatcircles | angle between the two normals in degrees |
| sort$\theta$minmaxU | sort “uRuns$\theta$minmaxUgreatcircles”, smallest $\theta$ first |

See Definitions above in Secs. 6a,6b for other quantities below. There you should find similarly named quantities.

In[701]:=
```
uRunsCrossUMin0 = Table[
   Cross[er[HmingLONU[[i]], HmingLATU[[i]]], sourceCenter], {i, Length[HmingLONU]}];
uRunsCrossUMin = Table[ uRunsCrossUMin0[[i]] /
    (uRunsCrossUMin0[[i]].uRunsCrossUMin0[[i]])^(1/2.), {i, Length[HmingLONU]}];
uRunsCrossUMax0 = Table[Cross[er[HmaxgLONU[[i]], HmaxgLATU[[i]]], sourceCenter],
   {i, Length[HmaxgLONU]}];
uRunsCrossUMax = Table[ uRunsCrossUMax0[[i]] /
    (uRunsCrossUMax0[[i]].uRunsCrossUMax0[[i]])^(1/2.), {i, Length[HmaxgLONU]}];
uRunsθminMAXUgreatcircles = Table[ArcCos[uRunsCrossUMax[[i]].uRunsCrossUMin[[i]]]
    (360. / (2. π)), {i, Length[HmaxgLONU]}];
```

In[706]:=
```
sortθminMAXU = Sort[uRunsθminMAXUgreatcircles];
η0θU = mean[uRunsθminMAXUgreatcircles]; (*Guess the mean for the Gaussian. *)
σθU = stanDev[uRunsθminMAXUgreatcircles];(*Guess the half-width.*)
histogramrange = {η0θU - 5 σθU, η0θU + 5 σθU, 0.4 σθU};
hl0 = HistogramList[sortθminMAXU, histogramrange];
hl =
 Table[{(1/2) (hl0[[1, i1]] + hl0[[1, i1 + 1]]), hl0[[2, i1]]}, {i1, Length[ hl0[[2]] ]}];
nlmθU = NonlinearModelFit[hl, a Exp[- (1/2.) ((x - x0) / b)^2],
  {{a, Length[sortθminMAXU] / 6}, {b, σθU}, {x0, η0θU}}, x];(*x is θminMAXU*)
```

In[712]:=
```
pTableNLMθU = nlmθU["ParameterTable"]
{σθminMAXUFit, θminMAXUFit} = {b, x0} /. nlmθU["BestFitParameters"];(*degrees*)
```

Out[712]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1637.41 | 12.0696 | 135.664 | $1.18243\times10^{-33}$ |
| b | 0.374277 | 0.00318565 | 117.488 | $2.78793\times10^{-32}$ |
| x0 | 90.0005 | 0.00318565 | 28 251.8 | $1.17334\times10^{-84}$ |

In[714]:=
```
showNLMθ = Show[{Histogram[sortθminMAXU, histogramrange,
     PlotLabel → "Angle θ between the Two Gray Great Circles in Figs. 5,6,13,14,20",
     AxesLabel → {"θ, degrees", "ΔR"}],
    Plot[Normal[nlmθU], {x, η0θU - 5 σθU, η0θU + 5 σθU}], ListPlot[hl] }];
```

In[715]:=
```
showNLMθ
```

Out[715]=

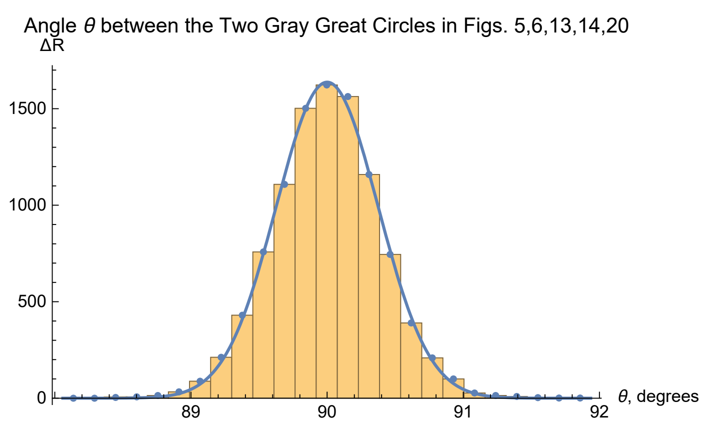



Figure 19: The Gaussian fit to the angle θ histogram. The two gray circles are perpendicular where they cross at the sample, within about 0.5°.

6g. Map of the Uncertainty Run Hubs

In this subsection, we map the locations of the many alignment hubs $H_{\min}$ and the locations of the avoidance hubs $H_{\max}$ that are found in the uncertainty runs.

Definitions:

| | |
|---|---|
| xyAitoffHmin | Aitoff coordinates for the alignment hubs $H_{\min}$ from the uncertainty runs |
| xyAitoffHmax | Aitoff coordinates for the avoidance hubs $H_{\max}$ from the uncertainty runs |
| xyAitoffOppositeHmin | Aitoff coordinates for the $-H_{\min}$ |
| xyAitoffOppositeHmax | Aitoff coordinates for the $-H_{\max}$ |
| mapOf$\sigma\psi$HminHmax | plot of the alignment and avoidance hubs $H_{\min}$, $-H_{\min}$, $H_{\max}$, and $-H_{\max}$ |

In[717]:=
```
(*The Aitoff coordinates for the hubs Hmin locations.*)
xyAitoffUHmin = Table[{xHGal[ HmingLONU [[n]] (360/ (2 π)), HmingLATU[[n]] (360/ (2 π)) ],
    yHGal[ HmingLONU [[n]] (360/ (2 π)), HmingLATU[[n]] (360/ (2 π)) ]}, {n,
    Length[HmingLATU ]}];
(*The Aitoff coordinates for the hubs Hmax locations.*)
xyAitoffUHmax = Table[{xHGal[ HmaxgLONU [[n]] (360/ (2 π)), HmaxgLATU[[n]] (360/ (2 π)) ],
    yHGal[ HmaxgLONU [[n]] (360/ (2 π)), HmaxgLATU[[n]] (360/ (2 π)) ]}, {n,
    Length[HmingLATU ]}];
(*The Aitoff coordinates for the hubs -Hmin locations.*)
xyAitoffUOppositeHmin = Table[{xHGal[ If[0 ≤ HmingLONU [[n]] (360/ (2 π)) < +180,
      HmingLONU [[n]] (360/ (2 π)) - 180, If[0 > HmingLONU [[n]] (360/ (2 π)) > -180,
       HmingLONU [[n]] (360/ (2 π)) + 180]], -HmingLATU[[n]] (360/ (2 π)) ],
    yHGal[ If[0 ≤ HmingLONU [[n]] (360/ (2 π)) < +180, HmingLONU [[n]] (360/ (2 π)) - 180,
      If[0 > HmingLONU [[n]] (360/ (2 π)) > -180, HmingLONU [[n]] (360/ (2 π)) + 180]],
     -HmingLATU[[n]] (360/ (2 π)) ]}, {n, Length[HmingLATU ]}];
(*The Aitoff coordinates for the hubs -Hmax locations.*)
xyAitoffUOppositeHmax = Table[
   {xHGal[ If[0 ≤ HmaxgLONU [[n]] (360/ (2 π)) < +180, HmaxgLONU [[n]] (360/ (2 π)) - 180,
      If[0 > HmaxgLONU [[n]] (360/ (2 π)) > -180, HmaxgLONU [[n]] (360/ (2 π)) + 180]],
     -HmaxgLATU[[n]] (360/ (2 π)) ],
    yHGal[ If[0 ≤ HmaxgLONU [[n]] (360/ (2 π)) < +180, HmaxgLONU [[n]] (360/ (2 π)) - 180,
      If[0 > HmaxgLONU [[n]] (360/ (2 π)) > -180, HmaxgLONU [[n]] (360/ (2 π)) + 180]],
     -HmaxgLATU[[n]] (360/ (2 π)) ]}, {n, Length[HmaxgLATU ]}];
```

In[721]:=
```
(*Construct the map of uncertainty run Hmin and Hmax hubs with ± regions indicated.*)
mapOfσψHminHmaxU =
  Show[{Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLAT, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60,
      PlotRange → {{-7, 7}, {-3, 3}}, Axes → False], {gLON, -180, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60], {gLAT, -60, 60, 30}],
    Graphics[{PointSize[0.007], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],
       {0, 1.85}], LightBlue, (*Hmin:*)Point[ xyAitoffUHmin ], (*-Hmin:*)
      Point[ xyAitoffUOppositeHmin ], LightRed, (*Hmax:*)Point[ xyAitoffUHmax ],
      (*-Hmax:*)Point[ xyAitoffUOppositeHmax ]    }], ParametricPlot[
    {xHGal[(HmingLONFit + rHminθ[θ] Cos[θ]) (360./(2. π)), (HmingLATFit + rHminθ[θ] Sin[θ]) (360./(2. π))],
     yHGal[(HmingLONFit + rHminθ[θ] Cos[θ]) (360./(2. π)), (HmingLATFit + rHminθ[θ] Sin[θ]) (360./(2. π))]},
    {θ, 0., 2. π}, PlotStyle → {Orange, Thickness[0.0005]}], ParametricPlot[
    {xHGal[(HmaxgLONFit + rHmaxθ[θ] Cos[θ]) (360./(2. π)), (HmaxgLATFit + rHmaxθ[θ] Sin[θ]) (360./(2. π))],
     yHGal[(HmaxgLONFit + rHmaxθ[θ] Cos[θ]) (360./(2. π)), (HmaxgLATFit + rHmaxθ[θ] Sin[θ]) (360./(2. π))]},
    {θ, 0., 2. π}, PlotStyle → {Orange, Thickness[0.0005]}],
   ParametricPlot[{xHGal[(HmingLONFit + rHminθ[θ] Cos[θ]) (360./(2. π)) - 180.,
      -(HmingLATFit + rHminθ[θ] Sin[θ]) (360./(2. π))], yHGal[
      (HmingLONFit + rHminθ[θ] Cos[θ]) (360./(2. π)) - 180., -(HmingLATFit + rHminθ[θ] Sin[θ]) (360./(2. π))]},
    {θ, 0., 2. π}, PlotStyle → {Orange, Thickness[0.0005]}],
   ParametricPlot[{xHGal[(HmaxgLONFit + rHmaxθ[θ] Cos[θ]) (360./(2. π)) - 180.,
      -(HmaxgLATFit + rHmaxθ[θ] Sin[θ]) (360./(2. π))], yHGal[
      (HmaxgLONFit + rHmaxθ[θ] Cos[θ]) (360./(2. π)) - 180., -(HmaxgLATFit + rHmaxθ[θ] Sin[θ]) (360./(2. π))]},
    {θ, 0., 2. π}, PlotStyle → {Orange, Thickness[0.0005]}]}, ImageSize → 2 × 432,
  PlotLabel → "The Hubs Found from the Uncertainty Runs"];
```

6h. Section Summary

In[722]:=
```
Print["To estimate the effects of experimental uncertainty, there were ",
 Length[funcDataU], " uncertainty runs."]
Print["The pdf version has 10,000 uncertainty runs."]
Print["Uncertainty runs have polarization directions ψ = ψSrc + δψ, ",
 "where δψ is chosen with a normal
   distribution of half-width σψ about the best value ψSrc."]
Print["The uncertainty runs determine the smallest alignment angle to be  η̄min = ",
 ηBarminUFit (360. / (2. π)), "° ± ", σηBarminUFit (360. / (2. π)), "°."]
Print["The uncertainty runs determine the largest avoidance angle to be  η̄max = ",
 ηBarmaxUFit (360. / (2. π)), "° ± ", σηBarmaxUFit (360. / (2. π)), "°."]
Print["The uncertainty runs determine the angle Θ between the two grey
   Great Circles to be  Θ = ", ΘminMAXUFit, "° ± ", σΘminMAXUFit, "°."]
```

To estimate the effects of experimental uncertainty, there were 10 000 uncertainty runs.

The pdf version has 10,000 uncertainty runs.

Uncertainty runs have polarization directions $\psi = \psi\mathrm{Src} + \delta\psi$,
where $\delta\psi$ is chosen with a normal distribution of half-width $\sigma\psi$ about the best value $\psi\mathrm{Src}$.

The uncertainty runs determine the smallest alignment angle to be $\bar{\eta}_{min}$ = 7.25402° ± 0.286868°.

The uncertainty runs determine the largest avoidance angle to be $\bar{\eta}_{max}$ = 82.7945° ± 0.282052°.

The uncertainty runs determine the angle $\Theta$ between the two grey Great Circles to be $\Theta$ =
90.0005° ± 0.374277°.

In[728]:=
```
(*mapOfσψHminHmaxU
 Print["Figure NN: The ",Length[funcDataU]," sets of hubs found for the uncertainty runs."]
 Print["The alignment hubs Hmin and -Hmin are plotted as light blue dots, ",LightBlue,". "]
 Print["The avoidance hubs Hmax and -Hmax are plotted as pink dots, ", LightRed,"."]
 Print["The most likely locations of the hubs are outlined in orange, ", Orange,"."]*)
```

As a final image, we superimpose the map of the uncertainty run hubs $H_{min}$, $-H_{min}$, $H_{max}$, and $-H_{max}$ on the graph of the alignment angle function $\bar{\eta}(H)$, Figs. 5,13.

In[729]:=
```
Show[{mapOfηBar, mapOfσψHminHmaxU}]
```

Out[729]=

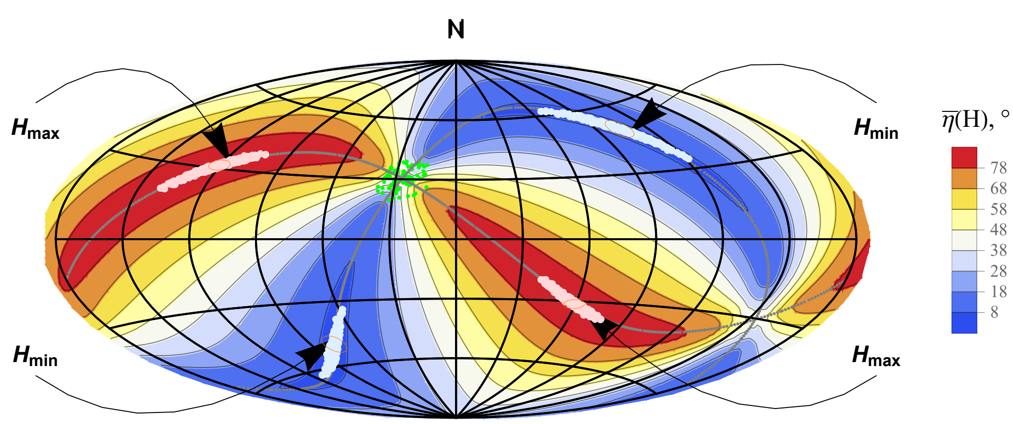


Figure 20: Note that the light blue alignment hubs from the uncertainty runs closely follow the areas of convergence (blue) for the best values ψSrc. And the pink avoidance hubs follow the areas of extreme divergence (red). The shifted hubs favor areas, in blue and red, that are close to the extremes for the alignment function $\overline{\eta}(H)$ in Figs. 5 and 13.

In[731]:=
```
(* vψ unit vectors pointing along the polarization direction,
have an experimental uncertainty. These are their plus/minus values. *)
vψSrcBig = Table[Cos[ (ψSrc[[i]] + σψSrc[[i]]) ] eN[ gLONSrc[[i]], gLATSrc[[i]] ] +
    Sin[ (ψSrc[[i]] + σψSrc[[i]]) ] eE[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];
vψSrcSmall = Table[Cos[ (ψSrc[[i]] - σψSrc[[i]]) ] eN[ gLONSrc[[i]], gLATSrc[[i]] ] +
    Sin[ (ψSrc[[i]] - σψSrc[[i]]) ] eE[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];
```

In[733]:=
```
(*Plot polarization direction Uncertainty in Sec. 6*)
rPlusψBig[i_, d_] := (rSrc[[i]] + d vψSrcBig[[i]]) /
  ((rSrc[[i]] + d vψSrcBig[[i]]).(rSrc[[i]] + d vψSrcBig[[i]]))^(1/2)
polarLinesBig[d_] := Table[
  Line[{{xHGal[gLONFROMr[ rPlusψBig[i, d]] (360./(2. π)), gLATFROMr[ rPlusψBig[i, d]] (360./(2. π))],
     yHGal[gLONFROMr[ rPlusψBig[i, d]] (360./(2. π)), gLATFROMr[ rPlusψBig[i, d]] (360./(2. π))]},
    {xHGal[gLONFROMr[ rPlusψBig[i, -d]] (360./(2. π)), gLATFROMr[ rPlusψBig[i, -d]] (360./(2. π))],
     yHGal[gLONFROMr[ rPlusψBig[i, -d]] (360./(2. π)),
      gLATFROMr[ rPlusψBig[i, -d]] (360./(2. π))]}}], {i, nSrc}]
```

In[735]:=
```
(*Plot polarization direction Uncertainty in Sec. 6*)
rPlusψSmall[i_, d_] := (rSrc[[i]] + d vψSrcSmall[[i]]) /
  ((rSrc[[i]] + d vψSrcSmall[[i]]).(rSrc[[i]] + d vψSrcSmall[[i]]))^(1/2)
polarLinesSmall[d_] := Table[Line[
   {{xHGal[gLONFROMr[ rPlusψSmall[i, d]] (360./(2. π)), gLATFROMr[ rPlusψSmall[i, d]] (360./(2. π))],
     yHGal[gLONFROMr[ rPlusψSmall[i, d]] (360./(2. π)), gLATFROMr[ rPlusψSmall[i, d]] (360./(2. π))]},
    {xHGal[gLONFROMr[ rPlusψSmall[i, -d]] (360./(2. π)), gLATFROMr[ rPlusψSmall[i, -d]] (360./(2. π))],
     yHGal[gLONFROMr[ rPlusψSmall[i, -d]] (360./(2. π)),
      gLATFROMr[ rPlusψSmall[i, -d]] (360./(2. π))]}}], {i, nSrc}]
```

In[737]:=
```
(* Local contour plot of the alignment angle function η̄(H) on the grid. *)
(*dηContourPlot = 6 ;*) (*, in degrees. *)
frameticks = {{{ {yHGal[45, 30], 30 °}}, None}, {{{xHGal[30, 15], 30 °}}, {None}}};
listCPlocalU = Show[ {Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, -10, 90},
     PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60, PlotRange → {{xHGal[0, 15],
        xHGal[45, 15]}, {yHGal[45, 15], yHGal[45, 45]}}, Axes -> False, Frame → True, FrameLabel →
      {"gLON", "gLAT", "Close-Up View"}, FrameTicks → frameticks], {gLON, 0, 60, 30}], Table[
    ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -10, 60}, PlotStyle → {Black,
       Thickness[0.002]}, PlotPoints → 60], {gLAT, -30, 90, 30}], Graphics[{PointSize[0.01], Gray,
     {Thick, polarLines[0.03]}, {Thick, polarLinesBig[0.03]}, {Thick, polarLinesSmall[0.03]},
     (*Sources S:*)Green, PointSize[0.012], Point[ xyAitoffSources ], Gray, PointSize[0.005]
        }], ParametricPlot[{xHGal[(HmingLONFit + rHminθ[θ] Cos[θ]) (360./(2. π)),
     (HmingLATFit + rHminθ[θ] Sin[θ]) (360./(2. π))],
     yHGal[(HmingLONFit + rHminθ[θ] Cos[θ]) (360./(2. π)), (HmingLATFit + rHminθ[θ] Sin[θ]) (360./(2. π))]},
    {θ, 0., 2. π}, PlotStyle → {Orange, Thickness[0.01]}]}, ImageSize → 0.9 × 432   ];
```

In[738]:=
```
(*listCPlocalU
 Print["Figure NN: Uncertainty plot. The sources are shaded green, ",
  Green,". Polarization directions for the reported value
    ψ, and the one-sigma values ψ ± σψ are plotted as gray, ",Gray,
  ", line segments through the sources. All of the alignment hubs Hmin from the uncertainty
    runs are plotted as overlapping blue, ",Blue,", dots, with the orange, ",Orange,
  ", spot denoting the tiny ellipse of highest hub density. Many of the avoidance red dots, ",
  Red,", for the Hmax are off-graph. The big orange
    ellipse encloses the likely locations for avoidance hubs. "]*)
```

7. Probability and Significance

The problem of “significance” is to determine the likelihood that random polarizations directions would produce better alignment or avoidance than the observed polarization directions.

To determine the probability distributions and related formulas, we made many runs with random data and fit the results. In this effort, as has occurred previously elsewhere, one finds that the probability distributions for the smallest alignment angle $\overline{\eta}_{\min}$ and the largest avoidance angle $\overline{\eta}_{\max}$ are not well-described by Gaussian functions. Better fits have the Gaussian multiplied by a step-function. By including a step-function, the fitting functions are based on the following distribution,

$$f(y) = \frac{\text{Norm}}{(2\pi)^{1/2}} \left(1 + e^{4(y-1)}\right)^{-1} e^{-\frac{y^2}{2}}, \tag{4}$$

where “Norm” is a constant needed to make the total probability equal to unity. More discussion appears below when the function (4) is needed.

For example, random polarization directions are well-fit by a probability distribution for the smallest alignment angle $\overline{\eta}_{\min}$ that takes the form

$$P_{\min}(\eta) = \left(\frac{\text{norm}}{\sigma\,(2\,\pi)^{1/2}}\right)\left(1 + e^{4\,\frac{(\eta-\eta 0-\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta 0}{\sigma}\right)^{2}}, \qquad (5)$$

where norm makes the integral of distribution equal to unity, $\eta 0$ and $\sigma$ are parameters that are adjusted to fit the random run results.

The significance of an alignment angle $\overline{\eta}_{\min}$ is the integral from -∞ to $\overline{\eta}_{\min}$ of the probability distribution, *i.e.* Sig( $\overline{\eta}_{\min}$) = $\int_{-\infty}^{\eta} P_{\min}(\eta)\, d\eta$.

The significance of an avoidance angle $\overline{\eta}_{\max}$ is the integral from $\overline{\eta}_{\max}$ to +∞ of the probability distribution, *i.e.* Sig( $\overline{\eta}_{\max}$) = $\int_{\eta}^{\infty} P_{\max}(\eta)\, d\eta$.

The distributions run to ±∞, while the alignment angles are acute, $0 \le \eta \le \pi/2 = 90°$. The issue is considered unimportant for samples with 7 or more sources, becasue it is judged sufficiently unlikely that 7 or more randomly directed sources would be non-acute according to the probability distributions.

The significance signiMIN0[$\eta$, $\eta 0$, $\sigma$] is the Integral of probMIN0, i.e. signiMIN0 = $\int_{-\infty}^{\eta} P_{MIN}(\eta)\, d\eta$.

The significance signiMAX0[$\eta$, $\eta 0$, $\sigma$] is the Integral of probMAX0, i.e. signiMAX0 = $\int_{\eta}^{\infty} P_{MAX}(\eta)\, d\eta$.

7a. Probability and Significance Formulas

Definitions:

| | |
|---|---|
| norm | a constant used to normalize the distribution so the integral of probability is 1. |
| probMIN0, probMAX0 | probability distributions for alignment (MIN) and avoidance (MAX), functions of $\eta$, $\eta_0$, $\sigma$ |
| signiMIN0, signiMAX0 | significance as a function of ($\eta$, $\eta_0$, $\sigma$) |

In[739]:=
```
(* y = ((η - η0)/σ); dy = dη/σ *)
(* The normalization factor "norm" is needed for the probability density *)
```
$$\text{norm} = \left(\frac{1}{(2\,\pi)^{1/2}}\,\text{NIntegrate}\left[\left(1 + e^{4\,(y-1)}\right)^{-1} e^{-\frac{y^2}{2}},\ \{y, -\infty, \infty\}\right]\right)^{-1};$$
```
norm;(*Constant needed to make the integral
 of the probability distribution equal to unity.*)
```

In[741]:=
$$\text{probMIN0}[\eta\_, \eta 0\_, \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\,\pi)^{1/2}}\right)\left(1 + e^{4\,\frac{(\eta-\eta 0-\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta 0}{\sigma}\right)^{2}}$$
$$\text{signiMIN0}[\eta\_, \eta 0\_, \sigma\_] := \text{NIntegrate}[\text{probMIN0}[\eta 1, \eta 0, \sigma], \{\eta 1, -\infty, \eta\}]$$
$$\text{probMAX0}[\eta\_, \eta 0\_, \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\,\pi)^{1/2}}\right)\left(1 + e^{-4\,\frac{(\eta-\eta 0+\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta 0}{\sigma}\right)^{2}}$$
$$\text{signiMAX0}[\eta\_, \eta 0\_, \sigma\_] := \text{NIntegrate}[\text{probMAX0}[\eta 1, \eta 0, \sigma], \{\eta 1, \eta, \infty\}]$$

7b. Generating random $\psi$ runs

The notebook .nb version generates new random runs. The pdf version uses old random runs that are uploaded from previously saved files that are not publically available. Thus both versions have some cells commented out: (* comments are not processed by Mathematica*).

Definitions:

nRunMax number of random runs to be generated
$\rho$RgnRadius distance to furthest source from sourceCenter, radians
minGridCenterToHmin, max - minimum number of grid spaces between Hmin, Hmax and sources' center
gridj$\eta$BarMinRand
iSminmas parameters for center to hub distance
nRunPrint dummy index to control printing frequency
rSrcxrGrid unit vector perpendicular to the plane of rSrc for $S_i$ and rGrid to point $H_j$
$\psi$SrcRand random polarization directions for the sources
rSrcx$\psi$Src cross product of rSrc and the vector in direction of $\psi$SrcR, both are unit vectors
j$\eta$BarToGrid {j, $\overline{\eta}_j$} = {grid point #, value of the alignment angle Eq. (2) averaged over all sources $S_i$, in radians}
sortj$\eta$BarToGrid - sort j$\eta$BarToGrid, smallest alignment angles $\overline{\eta}_j$ first
gridj$\eta$BarMinRand - {j,$\eta_j$} for the grid point $H_j$ with the smallest alignment angle $\overline{\eta}_j$, not counting $H_j$ that are too close to the sample
gridj$\eta$BarMaxRand - {j,$\eta_j$} for the grid point $H_j$ with the largest avoidance angle $\overline{\eta}_j$, not counting $H_j$ that are too close to the sample

niSnrData 1. run # 2. iSmin 3. iSmax 4. nSrc 6. $\rho$RgnRadius
$\psi$DataRand 1. run # 2. $\psi$SrcRand table
runData 1. run # 2. sourceCenter 3. {j, $\overline{\eta}$} at point $H_j$ where smallest $\overline{\eta}$ 4. {j, $\overline{\eta}$} at point $H_j$ where largest $\overline{\eta}$ 5. nSrc 6. $\rho$RgnRadius

Definitions:

nSrc Number of sources in the region
rgnRadius radius of the circular region in degrees
$\rho$rgn Region radius in radians
j0 grid number at the center of the region
rgnj0GridPoints Set of grid point #s in $j_0$ region
randomIntGrid `nSrc random integers to select gridpoints for sources in region` $j_0$
jSrc table of grid point #s at the location of the sources in region j0
randomIntPPA `nSrc random integers to choose PPA angles for sources in region` $j_0$
$\psi$Src table of PPA angles $\psi$ for the sources in region j0, in radians
rSrc table of unit vectors from origin to sources $S_i$
eNSrc table of unit vectors in direction of local North at sources $S_i$
eESrc table of unit vectors in direction of local East at sources $S_i$

rSrcx$\psi$Src table of the cross product of rSrc and the vector in direction of $\psi$Src, a unit vector

rSrcxrGrid table of the unit vectors perpendicular to the plane of the great circle containing the source $S_i$ and the grid point H

$\eta\psi$SrcToGrid table of the alignment angle $\eta_{ij}$ in radians between the PPA direction $\psi_i$ and the great circle toward Hj in the tangent space of the source $S_i$

j$\eta$BarToGrid table of {j,$\eta_j$} = {grid point #, value of the alignment angle $\eta_j$ averaged over all sources $S_i$, in radians}

sortj$\eta$BarToGrid j$\eta$BarToGrid, {j,$\eta_j$}, but sorted with the smallest alignment angles $\eta_j$ first

j$\eta$BarMin {j,$\eta_j$} for the grid point $H_j$ with the minimum average $\eta_j$

j$\eta$BarMax {j,$\eta_j$} for the grid point $H_j$ with the maximum average $\eta_j$

Tables:

niSnrData

1. nRun 2. iSmin 3. iSmax 4. nSrc 5. radius $\rho$RMS

runData

1. nRun 2. $\hat{r}$ at Region Center 3a. grid data for Hmin 3b. $\overline{\eta}_{\min}$ 4a. grid data for Hmax 4b. $\overline{\eta}_{\max}$ 5. nSrc 6. radius $\rho$RMS

In[745]:=

```
(*Remove comment marks, "(*" and "*)", below to generate your own table "runData". *)
(* Evaluate this cell for the notebook .nb version *)
(*
nRunMax=500;
niSnrData={};
ψDataRand={};
runData={};
times={};
(*Set up the For statement.*)
nRunPrint=0;
minGridCenterToHmin = 2;
(*minimum number of grid spaces between Hmin and sources' center*)
minGridCenterToHmax = 2;
(*minimum number of grid spaces between Hmax and sources' center*)
*)
```

In[746]:=

```
(* Evaluate this cell for the notebook .nb version *)
(*You may have found rSrcxrGrid already with uncertainty. Here it is again:*)
(*
rSrcxrGrid1 =Table[ Cross[  rSrc[[i]],rGrid[[j]]  ] ,    {i,nSrc},{j,nGrid}]
(*first step: raw cross product, not unit vectors*);
rSrcxrGrid=Table[ rSrcxrGrid1[[i,j]]/
   (rSrcxrGrid1[[i,j]].rSrcxrGrid1[[i,j]]+ 0.000001)^(1/2.) ,    {i,nSrc},{j,nGrid}];
t[1]=TimeUsed[];
*)
```

In[747]:=
```
(*
For[nRun=1,nRun≤nRunMax,nRun++,
   If[nRun>nRunPrint,Print["At the start of run ",nRun,", the time is ",
    TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];
   nRunPrint=nRunPrint+100];

  ψSrcRand=Table[RandomReal[{0.001,π-0.001}],{i,nSrc}];
  (*table of PPA angles ψ for the sources, in radians*)
  rSrcxψSrc = Table[ Sin[ψSrcRand[[i]]]eNSrc[[i]]-
     Cos[ψSrcRand[[i]]]eESrc[[i]],    {i,nSrc}];
  (*table of the cross product of rSrc and vector in direction of ψSrcRand,
  a unit vector*)
  jηBarToGrid = Table[{j,(1/nSrc)Sum[ ArcCos[
        Abs[  rSrcxψSrc[[i]].rSrcxrGrid[[i,j]]  ] - 0.000001  ],{i,nSrc}]},{j,nGrid}];
  (*
  {grid point #,
   value of the alignment angle ηnHj[j] averaged over all sources, in radians}*)
  sortjηBarToGrid=Sort[jηBarToGrid,#1[[2]]<#2[[2]]&];
  (*jηBarToGrid, {j,ηj}, but sorted with the smallest alignment angles first
  *)iSmin=
   Catch[Do[If[ArcCos[sourceCenter.rGrid[[sortjηBarToGrid[[i,1]]  ]] -0.000001 ]/dθ1≥
       minGridCenterToHmin,Throw[i]],{i,100}]];
  gridjηBarMinRand=sortjηBarToGrid[[iSmin]]; (*  {j,ηj},
  at the grid point Hj with minimum η̅,not counting the center j0*)iSmax=
   Catch[Do[If[ArcCos[sourceCenter.rGrid[[sortjηBarToGrid[[-i,1]]  ]] -0.000001 ]/dθ1≥
       minGridCenterToHmax,Throw[i]],{i,100}]];
  gridjηBarMaxRand=sortjηBarToGrid[[-iSmax]]; (*  {j,ηj},
  at the grid point Hj with maximum η̅, not counting the center j0*)
  AppendTo[niSnrData,{nRun,iSmin,iSmax,nSrc,ρRgnRadius}];
  AppendTo[ψDataRand,{nRun,ψSrcRand}];
  AppendTo[runData,
   {nRun,sourceCenter,{grid[[  gridjηBarMinRand[[1]]  ]], gridjηBarMinRand[[2]]},
    {grid[[  gridjηBarMaxRand[[1]]  ]], gridjηBarMaxRand[[2]]},nSrc,ρRgnRadius   } ]
   (*collect data for saving in a file.*)  ]  ;
*)
```

In[748]:=
```
(* Evaluate this cell for the notebook .nb version *)
(*
t[2]=TimeUsed[];
Print["Computer time needed to generate random runs: ",t[2]-t[1]," seconds."]
*)
```

In[749]:=
```
(*Save a new table*)
SetDirectory[homeDirectory];
(*Put[niSnrData,"20211012niSnrDataLon30Lat30offDiskHB5000b.dat"  ]*)
(*Put[ψDataRand,"20211012ψDataLon30Lat30offDiskHB5000b.dat"  ]*)
(*Put[runData,"20211012runDataLon30Lat30offDiskHB5000b.dat"]*)
```

In[750]:=
```
(*Get an previously saved table*)
SetDirectory[homeDirectory];
(*niSnrData=Get["20210917niSnrDataLon30Lat30offDiskHB5000b.dat"]*)
(*ψDataRand=Get["20210917niSnrDataLon30Lat30offDiskHB5000b.dat"]*)
(*Get the runData files for the pdf version:*)

runData5000a = Get["20210927runDataStarsN99Random5000a.dat"];
runData5000b = Get["20210927runDataStarsN99Random5000b.dat"];
```

In[753]:=
```
(*Edit the following statements to Join separate data files, if needed*)
(*Join the runData files for the pdf version:*)

runData = Join[runData5000a, runData5000b];
nRunMax = Length[runData]
```

Out[754]= 10 000

7c. Analyzing the random $\psi$ runs

Definitions:

| | |
|---|---|
| $\eta$BarminData | $\overline{\eta}_{\min}$ in order of random runs |
| sort$\eta$Barmin | sorted |
| $\eta$0Bmin, $\sigma$Bmin | mean and standard deviation of $\eta$BarminData |
| hlmin, hlmin0 | histogram data |
| nlmBmin | fit to $\overline{\eta}_{\min}$ histogram |
| {a,b,x0} | best fit parameters |
| showNlmBmin | figure displaying the fit to the $\overline{\eta}_{\min}$ from random runs |
| nlmBminPtable | Parameter table for the fit |

| | |
|---|---|
| $\eta$BarmaxData | $\overline{\eta}_{\max}$ |
| sort$\eta$Barmax | sorted |
| $\eta$0Bmax, $\sigma$Bmax | mean and standard deviation of $\eta$BarmaxData |
| hlmax, hlmax0 | histogram data |
| nlmBmax | fit to $\overline{\eta}_{\max}$ histogram |
| {a,b,x0} | best fit parameters |
| showNlmBmax | figure displaying the fit to the $\overline{\eta}_{\max}$ from random runs |
| nlmBmaxPtable | Parameter table for the fit |

| | |
|---|---|
| rHminR | rGrid at $H_{\min}$ |
| anglerHminToCenter | $\theta$ from $H_{\min}$ to sourceCenter |

$\theta$rHminToCenter, $\sigma\theta$rHminToCenter - mean and standard deviation of $\theta$

rHmaxR rGrid at $H_{max}$
anglerHmaxToCenter $\theta$ from $H_{max}$ to sourceCenter
$\theta$rHmaxToCenter, $\sigma\theta$rHmaxToCenter - mean and standard deviation of $\theta$

“fitData” table

1a. nSrc, number of sources 1b. rgnRadius, nominal radius of region 1c. RMS radius
2a. x0min: x0 = $\eta$0 align (min) 2b. dx0min error: dx0 - $\sigma$ for x0 = $\eta$0 align (min)
3a. bmin: b = $\sigma$ align (min) 3b. dbmin: err: db - $\sigma$ for b = $\sigma$ align (min)
4a. amin: a = Amplitude align (min) 4b. damin: err: da - $\sigma$ for a = Amplitude align (min)
5a. x0max: x0 = $\eta$0 avoid (max) 5b. dx0maxx0max: err: dx0 - $\sigma$ for x0 = $\eta$0 avoid (max)
6a. bmax: b = $\sigma$ avoid (max) 6b. dbmax: err: db - $\sigma$ for b = $\sigma$ avoid (max)
7a. amax: a = Amplitude avoid (max)7b. damax: err: da - $\sigma$ for a = Amplitude avoid (max)
8a. $\sigma\theta$rHminToCenter: stanDev[anglerHminToCenter] - $\sigma$ for $\theta$ to H 8b. $\theta$rHminToCenter: mean[anglerHminToCenter] - $\theta$ to H
9a. $\sigma\theta$rHmaxToCenter: stanDev[anglerHmaxToCenter] - $\sigma$ for $\theta$ to H 9b. $\theta$rHmaxToCenter: mean[anglerHmaxToCenter] - $\theta$ to H

In[755]:=
```
Print["There are ", Length[runData], " random runs to analyze."]
```

There are 10 000 random runs to analyze.

In[756]:=
```
ηBarminData = Table[runData[[i1, 3, 2]] , {i1, Length[runData]}];
ηBarmaxData = Table[runData[[i1, 4, 2]] , {i1, Length[runData]}];
rHminR = Table[runData[[i1, 3, 1, 6]] , {i1, Length[runData]}] ;
rHmaxR =  Table[runData[[i1, 4, 1, 6]] , {i1, Length[runData]}];
sortηBarmin = Sort[ηBarminData];
η0Bmin = mean[ηBarminData ] ;(*Guess the mean for the Gaussian. *)
σBmin = stanDev[ηBarminData ];(*Guess the half-width.*)
hlmin0 = HistogramList[sortηBarmin, {η0Bmin - 5 σBmin, η0Bmin + 5 σBmin, 0.4 σBmin}];
hlmin = Table[{(1/2) (hlmin0[[1, i1]] + hlmin0[[1, i1 + 1]]), hlmin0[[2, i1]]},
   {i1, Length[ hlmin0[[2]] ]}];
nlmBmin = NonlinearModelFit[hlmin, {a (1 + e^(4 (x-x0-b)/b))^-1 Exp[-1/2. ((x - x0)/b)^2] (*,b>0*)},
   {{a, Length[runData]/12}, {b, σBmin}, {x0, η0Bmin}}, x];
```

In[766]:=
```
{amin, bmin, x0min} = {a, b, x0} /. nlmBmin["BestFitParameters"];
{damin, dbmin, dx0min} = nlmBmin["ParameterErrors"]; (*x is ηBarmin*)
```

In[768]:=
```
sortηBarmax = Sort[ηBarmaxData];
η0Bmax = mean[ηBarmaxData ]; (*Guess the mean for the Gaussian. *)
σBmax = stanDev[ηBarmaxData ];(*Guess the half-width.*)
hlmax0 = HistogramList[sortηBarmax, {η0Bmax - 5 σBmax, η0Bmax + 5 σBmax, 0.4 σBmax}];
hlmax = Table[{(1/2) (hlmax0[[1, i1]] + hlmax0[[1, i1 + 1]]), hlmax0[[2, i1]]},
  {i1, Length[ hlmax0[[2]] ]}];
nlmBmax = NonlinearModelFit[hlmax, {a (1 + e^(-4 (x-x0+b)/b))^-1 Exp[- 1/2. ((x - x0)/b)^2] (*,b>0*)},
  {{a, nRunMax/12}, {b, σBmax}, {x0, η0Bmax}}, x];
```

In[773]:=
```
{amax, bmax, x0max} = {a, b, x0} /. nlmBmax["BestFitParameters"];
{damax, dbmax, dx0max} = nlmBmax["ParameterErrors"];(*x is ηBarmax*)
```

In[775]:=
```
anglerHminToCenter =
  Table[ArcCos[Abs[rHminR[[i]].sourceCenter] - 0.00001], {i, Length[rHminR]}];
θrHminToCenter = mean[anglerHminToCenter];
σθrHminToCenter = stanDev[anglerHminToCenter];
anglerHmaxToCenter =
  Table[ArcCos[Abs[rHmaxR[[i]].sourceCenter] - 0.00001], {i, Length[rHmaxR]}];
θrHmaxToCenter = mean[anglerHmaxToCenter];
σθrHmaxToCenter = stanDev[anglerHmaxToCenter]; t[6] = TimeUsed[];
fitData = {{nSrc,
    ρRgnRadius, ρRMS}, {x0min, dx0min}, {bmin, dbmin}, {amin, damin},
   {x0max, dx0max}, {bmax, dbmax}, {amax, damax}, {σθrHminToCenter,
    θrHminToCenter}, {σθrHmaxToCenter,
    θrHmaxToCenter}} ;(*collect data for saving in a file.*)
```

In[782]:=
```
ListPlot[{sortηBarmin, sortηBarmax}];
ListPlot[hlmin];
Normal[nlmBmin];
Print["The parameter table for the fit to η̄min: "]
nlmBminPtable = nlmBmin["ParameterTable"]
Normal[nlmBmax];
Print["The parameter table for the fit to η̄max: "]
nlmBmaxPtable = nlmBmax["ParameterTable"]
```

The parameter table for the fit to $\bar{\eta}_{min}$:

Out[786]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1626.95 | 17.5493 | 92.7072 | $5.06111 \times 10^{-30}$ |
| b | 0.0303192 | 0.00036554 | 82.9434 | $5.8161 \times 10^{-29}$ |
| x0 | 0.697206 | 0.000305889 | 2279.28 | $1.3209 \times 10^{-60}$ |

The parameter table for the fit to $\bar{\eta}_{max}$:

Out[789]=

| | Estimate | Standard Error | t-Statistic | P-Value |
|---|---|---|---|---|
| a | 1619.23 | 10.3053 | 157.125 | $4.68757 \times 10^{-35}$ |
| b | 0.0308047 | 0.000219132 | 140.576 | $5.41144 \times 10^{-34}$ |
| x0 | 0.873952 | 0.000183368 | 4766.11 | $1.18259 \times 10^{-67}$ |

In[790]:=
```
showNlmBmin =
  Show[{Histogram[Sort[ηBarminData], {η0Bmin - 5 σBmin, η0Bmin + 5 σBmin, 0.4 σBmin},
     PlotLabel → "Histogram for η̄min, random runs",
     PlotRange → {{0., 0.8}, Automatic}, AxesLabel → {"η̄min, radians", "ΔR"}],
    Plot[Normal[nlmBmin], {x, 0.4(*η0Bmin-5σBmin*), η0Bmin + 5 σBmin},
     PlotRange → {{0., 0.8}, Automatic}, PlotPoints → 200], ListPlot[hlmin],
    Graphics[{Blue, Arrow[{{ηBarMinfunDataObs, nRunMax/24}, {ηBarMinfunDataObs, 5.}}]}] }];
```

In[791]:=
```
showNlmBminA =
  Show[{Histogram[Sort[ηBarminData], {η0Bmin - 5 σBmin, η0Bmin + 5 σBmin, 0.4 σBmin},
     PlotLabel → "Histogram for η̄min, random runs",
     PlotRange → {{0.55, 0.8}, Automatic}, AxesLabel → {"η̄min, radians", "ΔR"}],
    Plot[Normal[nlmBmin], {x, (*η0Bmin-5σBmin*)0.52, η0Bmin + 5 σBmin}], ListPlot[hlmin],
    Graphics[{Blue, Arrow[{{ηBarMinfunDataObs, nRunMax/24}, {ηBarMinfunDataObs, 5.}}]}] }];
```

In[792]:=
```
GraphicsRow[{showNlmBminA, showNlmBmin}]
```

Out[792]=

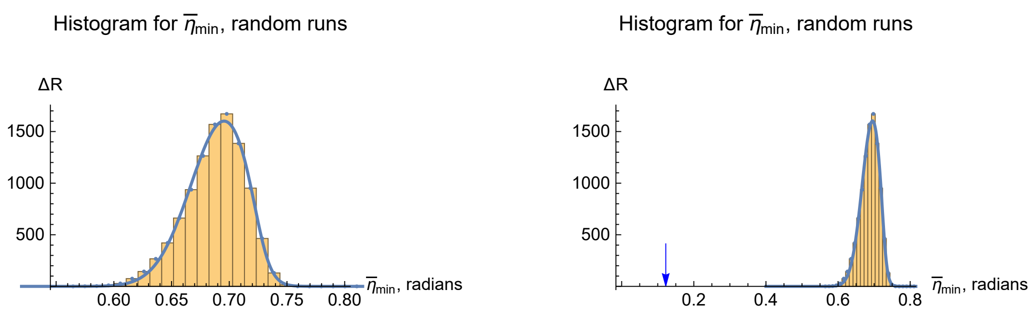



Figure 21: *Left*. A detailed look at the histogram. Note the steeper slope on the $\eta = \pi/4 = 0.785$ radian side. *Right*. The observed polarization directions yield an angle $\overline{\eta}_{min}$ that is far down the tail of the distribution.

In[794]:=
```
showNlmBmax =
  Show[{Histogram[Sort[ηBarmaxData], {η0Bmax - 5 σBmax, η0Bmax + 5 σBmax, 0.4 σBmax},
     PlotLabel → "Histogram for η̄max, random runs", PlotRange → {{0.75, 1.50}, Automatic},
     AxesLabel → {"η̄max, radians", "ΔR"}], Plot[Normal[nlmBmax],
     {x, (*η0Bmax-5σBmax*)0.7, (*η0Bmax+5σBmax*)1.1}], ListPlot[hlmax],
    Graphics[{Red, Arrow[{{ηBarMaxfunDataObs, nRunMax/24}, {ηBarMaxfunDataObs, 5.}}]}] }];
```

In[795]:=
```
showNlmBmaxA =
  Show[{Histogram[Sort[ηBarmaxData], {η0Bmax - 5 σBmax, η0Bmax + 5 σBmax, 0.4 σBmax},
     PlotLabel → "Histogram for η̄max, random runs", PlotRange → {{0.75, 1.05}, Automatic},
     AxesLabel → {"η̄max, radians", "ΔR"}], Plot[Normal[nlmBmax],
     {x, (*η0Bmax-5σBmax*)0.74, (*η0Bmax+5σBmax*)1.06}], ListPlot[hlmax],
    Graphics[{Red, Arrow[{{ηBarMaxfunDataObs, nRunMax/24}, {ηBarMaxfunDataObs, 5.}}]}] }];
```

In[796]:= **GraphicsRow[{showNlmBmaxA, showNlmBmax}]**

Out[796]=

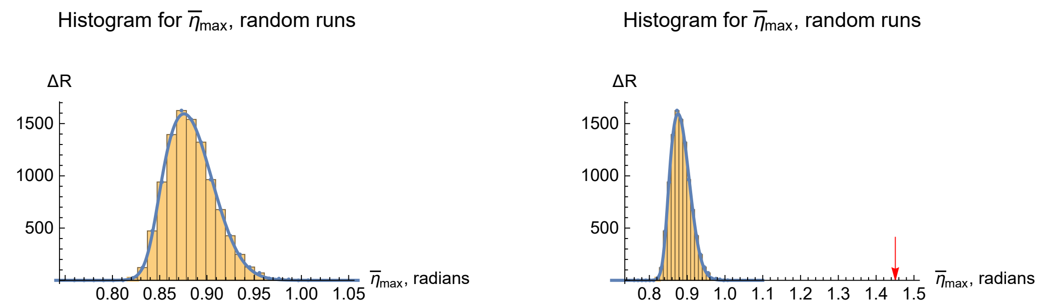



Figure 22: Random run results for largest avoidance angle $\overline{\eta}_{max}$. *Left*. Note that the curve is steeper toward the midway acute angle value, $\eta = \pi/4 = 45°$. That requires non-Gaussian fitting functions in the 'NonlinearModelFit' statement above. *Right*. The observed polarization directions give the result indicated by the arrow.

7d. Significance of the results for the sample studied in this work

Definitions

fitting function parameters from random runs:

$\eta$0min mean of probability distribution for smallest alignment angle $\overline{\eta}_{min}$
d$\eta$0min standard error in the mean as reported by Mathematica
$\sigma$min half-width of probability distribution for smallest alignment angle $\overline{\eta}_{min}$
d$\sigma$min standard error in the half-width as reported by Mathematica

$\eta$0max mean of probability distribution for largest avoidance angle $\overline{\eta}_{max}$
d$\eta$0max standard error in the mean as reported by Mathematica
$\sigma$max half-width of probability distribution for largest avoidance angle $\overline{\eta}_{max}$
d$\sigma$max standard error in the half-width as reported by Mathematica

probmin probability distribution for smallest alignment angle $\overline{\eta}_{min}$ . This depends on the random runs.
signimin significance, integral of probmin over smaller values of $\overline{\eta}_{min}$

probmax probability distribution for largest avoidance angle $\overline{\eta}_{max}$
signimax significance, integral of probmax over larger values of $\overline{\eta}_{max}$

sig$\eta$BarMinfunDataObs Significance of the smallest alignment angle $\overline{\eta}_{min}$
sigrange$\eta$BarMinfunDataObs standard errors in $\eta$0min and $\sigma$min, *i.e.* d$\eta$0min and d$\sigma$min, give the significances plus/minus values
sigSmall$\eta$BarMinfunDataObs, Big extremes of significance assuming one standard error

sig$\eta$BarMaxfunDataObs Significance of the largest avoidance angle $\overline{\eta}_{max}$

sigrange$\eta$BarMaxfunDataObs standard errors in $\eta$0max and $\sigma$max, *i.e.* d$\eta$0max and d$\sigma$max, give the significances plus/minus values

sigSmall$\eta$BarMaxfunDataObs, Big extremes of significance assuming one standard error

In[798]:=
```
(*Parameters η0 and σ from random runs, together with their standard errors.*)
η0min = x0min; dη0min = dx0min;
η0max = x0max; dη0max = dx0max;
σmin = bmin; dσmin = dbmin;
σmax = bmax; dσmax = dbmax;
```

In[802]:=
```
probmin[η_] := probMIN0[ η, η0min, σmin ]
signimin[η_] := signiMIN0[η, η0min, σmin]
probmax[η_] := probMAX0[ η, η0max, σmax]
signimax[η_] := signiMAX0[η, η0max, σmax]
```

In[806]:=
```
Print["For this sample, but with random polarization directions ψ, the
   random runs give the mean value η0min and the half-width σmin of the
   fitting function of random runs for the smallest alignment angle η̄min:"]
Print[" η0min =  ", η0min (360./(2. π)), "° ± ", dη0min (360./(2. π)), "° and σmin  = ",
  σmin (360./(2. π)), "° ± ", dσmin (360./(2. π)), "°. (Random ψ distribution)"]
```

For this sample, but with random polarization directions $\psi$, the random runs give the mean value $\eta$0min and the half-width $\sigma$min of the fitting function of random runs for the smallest alignment angle $\overline{\eta}_{\text{min}}$:

$\eta$0min = 39.947° ± 0.0175261° and $\sigma$min = 1.73716° ± 0.0209439°. (Random $\psi$ distribution)

In[808]:=
```
Print[
  "For this sample, but with random polarization directions ψ, the random runs give the
   mean η0max  and the half-width σmax for the distributions for avoidance angle :"]
Print["   η0max =  ", η0max (360./(2. π)), "° ± ", dη0max (360./(2. π)), "° and σmax  = ",
  σmax (360./(2. π)), "° ± ", dσmax (360./(2. π)), "°. (Random ψ distribution)"]
```

For this sample, but with random polarization directions $\psi$, the random runs give the mean $\eta$0max and the half-width $\sigma$max for the distributions for avoidance angle :

$\eta$0max = 50.0738° ± 0.0105062° and $\sigma$max = 1.76498° ± 0.0125553°. (Random $\psi$ distribution)

In[810]:=
```
(*Significance of the smallest alignment angle η̄min .*)
sigηBarMinfunDataObs = signimin[ηBarMinfunDataObs];
sigrangeηBarMinfunDataObs =
  Sort[Partition[Flatten[Table[{signiMIN0[ηBarMinfunDataObs, η0min + γ1 dη0min,
        σmin + γ2 dσmin], γ1, γ2}, {γ1, -1, 1}, {γ2, -1, 1}] ], 3 ]  ];
{sigrangeηBarMinfunDataObs[[1]], sigrangeηBarMinfunDataObs[[-1]]};
sigSmallηBarMinfunDataObs = sigrangeηBarMinfunDataObs[[1, 1]];
sigBigηBarMinfunDataObs = sigrangeηBarMinfunDataObs[[-1, 1]];
```

In[815]:=
```
(*Experimental uncertainties and the
 Significance of the smallest alignment angle η̄min .*)
(*sigηBarMinfunDataObs=signimin[ηBarMinfunDataObs];*)
sigηBarMinfunDataObs;
sigrangeηBarMinfunDataObsU = Sort[Table[
    {signiMIN0[ηBarMinfunDataObs + γ1 σηBarminUFit, η0min, σmin], γ1}, {γ1, -1, 1}]  ];
sigSmallηBarMinfunDataObsU = sigrangeηBarMinfunDataObsU[[1, 1]];
sigBigηBarMinfunDataObsU = sigrangeηBarMinfunDataObsU[[-1, 1]];
```

In[819]:=
```
(*Significance of the largest avoidance angle η̄max .*)
sigηBarMaxfunDataObs = signimax[ηBarMaxfunDataObs];
sigrangeηBarMaxfunDataObs =
  Sort[Partition[Flatten[Table[{signiMAX0[ηBarMaxfunDataObs, η0max + γ1 dη0max,
        σmax + γ2 dσmax], γ1, γ2}, {γ1, -1, 1}, {γ2, -1, 1}] ], 3 ]  ];
{sigrangeηBarMaxfunDataObs[[1]], sigrangeηBarMaxfunDataObs[[-1]]};
sigSmallηBarMaxfunDataObs = sigrangeηBarMaxfunDataObs[[1, 1]];
sigBigηBarMaxfunDataObs = sigrangeηBarMaxfunDataObs[[-1, 1]];
```

In[824]:=
```
(*Experimental uncertainties and the
 Significance of the smallest alignment angle η̄max .*)
(*sigηBarMaxfunDataObs=signimax[ηBarMaxfunDataObs];*)
sigηBarMaxfunDataObs;
sigrangeηBarMaxfunDataObsU = Sort[Table[
    {signiMAX0[ηBarMaxfunDataObs + γ1 σηBarmaxUFit, η0max, σmax], γ1}, {γ1, -1, 1}]  ];
sigSmallηBarMaxfunDataObsU = sigrangeηBarMaxfunDataObsU[[1, 1]];
sigBigηBarMaxfunDataObsU = sigrangeηBarMaxfunDataObsU[[-1, 1]];
```

In[828]:=
```
(*The names "gridjηBarMinRan", "jηBarMax" are, or perhaps were,
similar to quantities below, so save the current values labeled by "Best".*)
(* jηBar entries: 1. grid point # , 2. alignment angle .*)
{jηBarMinBest, jηBarMaxBest} = {ηBarMinfunDataObs, ηBarMaxfunDataObs} ;
```

In[829]:=
```
Print["The smallest alignment angle is ηmin = ", ηBarMinfunDataObs * (360. / (2. π)),
  "° , which has a significance of sig. = ", sigηBarMinfunDataObs,
  ",  plus/minus = + ", sigBigηBarMinfunDataObs - sigηBarMinfunDataObs, " and - ",
  sigηBarMinfunDataObs - sigSmallηBarMinfunDataObs, " , giving a range from sig. = ",
  sigSmallηBarMinfunDataObs, " to ", sigBigηBarMinfunDataObs, " ."]
Print["The largest avoidance angle is ηmax = ", ηBarMaxfunDataObs * (360. / (2. π)),
  "° , which has a significance of sig. = ", sigηBarMaxfunDataObs,
  ",  plus/minus = + ", sigBigηBarMaxfunDataObs - sigηBarMaxfunDataObs, " and - ",
  sigηBarMaxfunDataObs - sigSmallηBarMaxfunDataObs, " , giving a range from sig. = ",
  sigSmallηBarMaxfunDataObs, " to ", sigBigηBarMaxfunDataObs, " ."]
Print["These ± values are due to the standard errors
    for the parameters in the fit to the random runs."]
```

The smallest alignment angle is $\eta$min = 7.00685° , which has a significance of sig. = $2.14201\times10^{-80}$,  plus/minus = + $1.82565\times10^{-78}$ and - $2.12098\times10^{-80}$ , giving a range from sig. = $2.10346\times10^{-82}$ to $1.84707\times10^{-78}$ .

The largest avoidance angle is $\eta$max = 83.1219° , which has a significance of sig. = $1.91177\times10^{-78}$,  plus/minus = + $2.34425\times10^{-77}$ and - $1.77554\times10^{-78}$ , giving a range from sig. = $1.36235\times10^{-79}$ to $2.53543\times10^{-77}$ .

These ± values are due to the standard errors for the parameters in the fit to the random runs.

In[832]:=
```
Print["Given experimental uncertainties in the
    measured data, the most likely smallest alignment angle is ηmin = ",
  ηBarminUFit (360. / (2. π)), "° ± ", σηBarminUFit (360. / (2. π)),
  "°., which has a significance range from sig. = ",
  signimin[ηBarminUFit - σηBarminUFit], " to ", signimin[ηBarminUFit + σηBarminUFit], " ."]
Print["The most likely smallest alignment angle is ηmax = ", ηBarmaxUFit (360. / (2. π)),
  "° ± ", σηBarmaxUFit (360. / (2. π)), "°., which has a significance range from sig. = ",
  signimax[ηBarmaxUFit + σηBarmaxUFit], " to ", signimax[ηBarmaxUFit - σηBarmaxUFit], " ."]
Print["These uncertainties are due to the experimental
    uncertainty in the observed polarization directions."]
```

Given experimental uncertainties in the measured data, the most likely smallest alignment angle is $\eta$min = 7.25402° ± 0.286868 °., which has a significance range from sig. = $1.38678\times10^{-80}$ to $7.06336\times10^{-78}$ .

The most likely smallest alignment angle is $\eta$max = 82.7945° ± 0.282052 °., which has a significance range from sig. = $3.09421\times10^{-78}$ to $1.17836\times10^{-75}$ .

These uncertainties are due to the experimental uncertainty in the observed polarization directions.

In[835]:=
```
Print["More Statistics of the Alignment Function η̄(H) :"]
Print[" "]
Print["The observed data produce a smallest alignment angle ηmin = ",
 ηBarMinfunDataObs * (360. / (2. π)), "°, which is Δη = ",
 (η0min - ηBarMinfunDataObs) * (360. / (2. π)), "° below the most likely value, ",
 η0min * (360. / (2. π)), "°, for random runs."]
Print["Since the half-width σ is ", σmin * (360. / (2. π)),
 "°, the difference, Δη = ", (η0min - ηBarMinfunDataObs) * (360. / (2. π)),
 "° makes ηmin separated from the most likely random run value by ",
 (η0min - ηBarMinfunDataObs) / σmin, "σs."]
Print["Thus, the smallest alignment angle η̄min is " , (η0min - ηBarMinfunDataObs) / σmin,
 "σs below the most likely random run value. (Very Significant)"]
```

```
More Statistics of the Alignment Function η̄(H) :


The observed data produce a smallest alignment angle ηmin = 7.00685
 °, which is Δη = 32.9401° below the most likely value, 39.947°, for random runs.

Since the half-width σ is 1.73716°, the difference, Δη = 32.9401
 ° makes ηmin separated from the most likely random run value by 18.9621σs.

Thus, the smallest alignment angle η̄min is 18.9621
 σs below the most likely random run value. (Very Significant)
```

In[840]:=
```
Print["The observed data yield a largest avoidance angle ηmax = ",
 ηBarMaxfunDataObs * (360. / (2. π)), "°, which is Δη = ",
 - (η0max - ηBarMaxfunDataObs) * (360. / (2. π)), "° above the most likely value, ",
 η0max * (360. / (2. π)), "°, for random runs."]
Print["Since the half-width σ is ", σmax * (360. / (2. π)),
 "°, the difference Δη = ", - (η0max - ηBarMaxfunDataObs) * (360. / (2. π)),
 "° makes ηmax separated from the most likely random run value by  ",
 - (η0max - ηBarMaxfunDataObs) / σmax, "σs."]
Print["Thus, the largest avoidance angle η̄max is " , - (η0max - ηBarMaxfunDataObs) / σmax,
 "σs above the most likely random run value. (Very significant)"]
```

```
The observed data yield a largest avoidance angle ηmax = 83.1219
 °, which is Δη = 33.0481° above the most likely value, 50.0738°, for random runs.

Since the half-width σ is 1.76498°, the difference Δη = 33.0481
 ° makes ηmax separated from the most likely random run value by  18.7244σs.

Thus, the largest avoidance angle η̄max is 18.7244
 σs above the most likely random run value. (Very significant)
```

In[843]:=
```
Print["The computer time spent on this program: ", TimeUsed[], ", in seconds."]
```

```
The computer time spent on this program: 167.996, in seconds.
```