# An Identity involving Tribonacci Numbers

Suaib Lateef
lateefholuyinka@gmail.com

**Abstract**
In this paper, we present an identity involving Tribonacci numbers. We prove the identity by extending the numbers of variables of Candido's identity to three.

Keywords: Candido's Identity, Binomial expansion, Tribonacci sequence.

## 1 Introduction

The Fibonacci numbers, commonly known as $F_n$, form a sequence, called the Fibonacci sequence, such that each number is a sum of the two preceeding ones, starting from 0 and 1. The Fibonacci sequence is denoted as

$$F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1.$$

The first few Fibonacci numbers are given by:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...$$

Many identities involving Fibonacci numbers have been discovered over time and one prime example is Candido's identity. Candido[1] (1871-1941) proved that

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) = (F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2, \tag{1.1}$$

by showing that

$$2(a^4 + b^4 + (a+b)^4) = (a^2 + b^2 + (a+b)^2)^2. \tag{1.2}$$

In 2005, R. B. Nelsen[3] gave a proof of (1.2) without words. Also, Darko Veljan[2] proved (1.2) using Heron's formula.

The Tribonacci sequence, commonly known as $T_n$, is a generalization of Fibonacci sequence where each number is a sum of the three preceeding ones, starting from 0, 1, and 1. The Tribonacci sequence is denoted as

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 3, T_0 = 0, T_1 = 1, T_2 = 1.$$

The first few Tribonacci numbers are given by:

$$1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ...$$

In this work, we will provide a generalization of (1.2) which will be intrumental in proving our main result. Our main result is connected to Tribonacci numbers and hence, a generalization of (1.1). In section 2, we state our main result and give a proof in section 3.

## 2 The Main Result

**Identity:** If $T_n$ is the nth Tribonacci number, then

$$2((T_n+T_{n+1})^4+(T_n+T_{n+2})^4+(T_{n+1}+T_{n+2})^4)+4(T_n^4+T_{n+1}^4+T_{n+2}^4+T_{n+3}^4) = 3(T_n^2+T_{n+1}^2+T_{n+2}^2+T_{n+3}^2)^2 \tag{2.1}$$

To prove (2.1), we need to show that if $a$, $b$, and $c$ are real numbers, then

$$2((a+c)^4+(b+c)^4+(a+b)^4)+4(a^4+b^4+c^4+(a+b+c)^4) = 3((a+c)^2+(b+c)^2+(a+b)^2)^2 \tag{2.2}$$

Setting $c = 0$ in (2.2) gives

$$2((a+0)^4+(b+0)^4+(a+b)^4)+4(a^4+b^4+0^4+(a+b+0)^4) = 3((a+0)^2+(b+0)^2+(a+b)^2)^2$$

$$2(a^4 + b^4 + (a + b)^4) + 4(a^4 + b^4 + (a + b)^4) = 3(a^2 + b^2 + (a + b)^2)^2$$

$$6(a^4 + b^4 + (a + b)^4) = 3(a^2 + b^2 + (a + b)^2)^2$$

$$2(a^4 + b^4 + (a + b)^4) = (a^2 + b^2 + (a + b)^2)^2 \tag{2.3}$$

We can see that (2.2) is equal to (1.2)

## 3 The Proof

Let

$$m = a + b \tag{3.1}$$

Now, from (2.2), let

$$X_1 = (a + c)^4 + (b + c)^4 + (a + b)^4$$

$$X_1 = (a + c)^4 + (b + c)^4 + m^4 \tag{3.2}$$

$$X_2 = a^4 + b^4 + c^4 + (a + b + c)^4$$

$$X_2 = a^4 + b^4 + c^4 + (m + c)^4 \tag{3.3}$$

$$X_3 = (a + c)^2 + (b + c)^2 + (a + b)^2$$

$$X_3 = (a + c)^2 + (b + c)^2 + m^2 \tag{3.4}$$

So, we have from (2.2) that

$$2X_1 + 4X_2 = 3{X_3}^2 \tag{3.5}$$

We can see that to prove (2.2), it suffices to show that (3.5) is true.
We know from Binomial expansion that

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$a^2 + b^2 = (a+b)^2 - 2ab$$

$$a^2 + b^2 = m^2 - 2ab \tag{3.6}$$

Also, we know that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$(a+b)^3 = (a+b)^3 - 3ab(a+b)$$

$$a^3 + b^3 = m^3 - 3abm \tag{3.7}$$

Also, we know that

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$m^4 = a^4 + b^4 + 4ab(a^2+b^2) + 6a^2b^2 \tag{3.8}$$

Putting (3.6) in (3.8) gives

$$m^4 = a^4 + b^4 + 4ab(m^2 - 2ab) + 6a^2b^2$$

$$m^4 = a^4 + b^4 + 4abm^2 - 8a^2b^2 + 6a^2b^2$$

$$m^4 = a^4 + b^4 + 4abm^2 - 2a^2b^2$$

$$a^4 + b^4 = m^4 - 4abm^2 + 2a^2b^2 \tag{3.9}$$

Now, from (3.2), we see that

$$X_1 = (a+c)^4 + (b+c)^4 + m^4$$

$$X_1 = a^4 + 4a^3c + 6a^2c^2 + 4ac^3 + c^4 + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4 + m^4$$

$$X_1 = a^4 + b^4 + 2c^4 + 4a^3c + 4b^3c + 4ac^3 + 4bc^3 + 6a^2c^2 + 6b^2c^2 + m^4$$

$$X_1 = m^4 + (a^4 + b^4) + 2c^4 + 4c(a^3 + b^3) + 4c^3(a+b) + 6c^2(a^2+b^2) \tag{3.10}$$

Putting (3.1), (3.6), (3.7), and (3.9) in (3.10), we have

$$X_1 = m^4 + (m^4 - 4abm^2 + 2a^2b^2) + 2c^4 + 4c(m^3 - 3abm) + 4c^3m + 6c^2(m^2 - 2ab)$$

$$X_1 = 2m^4 + 2c^4 - 4abm^2 + 2a^2b^2 - 12abmc + 4c^3m + 6m^2c^2 - 12abc^2 \tag{3.11}$$

Multiplying both sides of (3.11) by 2 gives

$$2X_1 = 4m^4 + 4c^4 - 8abm^2 + 4a^2b^2 - 24abmc + 8m^3c + 12c^2m^2 - 24abc^2 \tag{3.12}$$

From (3.3), we see that

$$X_2 = a^4 + b^4 + c^4 + (m + c)^4$$

$$X_2 = (a^4 + b^4) + c^4 + m^4 + 4m^3c + 6m^2c^2 + 4mc^3 + c^4 \tag{3.13}$$

Putting (3.9) in (3.13), we have

$$X_2 = (m^4 - 4abm^2 + 2a^2b^2) + c^4 + m^4 + 4m^3c + 6m^2c^2 + 4mc^3 + c^4$$

$$X_2 = 2m^4 + 2c^4 - 4abm^2 + 2a^2b^2 + 4m^3c + 6m^2c^2 + 4mc^3 \tag{3.14}$$

Multiplying both sides of (3.14) by 4 gives

$$4X_2 = 8m^4 + 8c^4 - 16abm^2 + 8a^2b^2 + 16m^3c + 24m^2c^2 + 16mc^3$$

$$4X_2 = 8m^4 + 8c^4 - 16abm^2 + 8a^2b^2 + 16mc^3 + 16m^3c + 24m^2c^2 \tag{3.15}$$

Adding (3.12) and (3.15) gives

$$2X_1 + 4X_2 = 12m^4 + 12c^4 - 24abm^2 + 12a^2b^2 + 24mc^3 + 24m^3c + 36m^2c^2 - 24abmc - 24abc^2$$

$$2X_1 + 4X_2 = 12(m^4 + c^4 - 2abm^2 + a^2b^2 + 2mc^3 + 2m^3c + 3m^2c^2 - 2abmc - 2abc^2) \tag{3.16}$$

From (3.4), we see that

$$X_3 = m^2 + (a + c)^2 + (b + c)^2$$

$$X_3 = m^2 + a^2 + 2ac + c^2 + b^2 + 2bc + c^2$$

$$X_3 = m^2 + (a^2 + b^2) + 2c^2 + 2c(a + b) \tag{3.17}$$

Putting (3.1) and (3.6) in (3.17) gives

$$X_3 = m^2 + (m^2 - 2ab) + 2c^2 + 2mc$$

$$X_3 = 2m^2 + 2c^2 + 2mc - 2ab$$

$$X_3 = 2(m^2 + c^2 + mc - ab) \tag{3.18}$$

Squaring both sides of (3.18) gives

$${X_3}^2 = 4(m^2 + c^2 + mc - ab)^2$$

$${X_3}^2 = 4(m^4 + c^4 + m^2c^2 + a^2b^2 + 2m^2c^2 + 2m^3c - 2abm^2 + 2mc^3 - 2abc^2 - 2abmc)$$

$${X_3}^2 = 4(m^4 + c^4 + 3m^2c^2 + a^2b^2 + 2m^3c - 2abm^2 + 2mc^3 - 2abc^2 - 2abmc)$$

$${X_3}^2 = 4(m^4 + c^4 - 2abm^2 + a^2b^2 + 2mc^3 + 2m^3c + 3m^2c^2 - 2abmc - 2abc^2) \tag{3.19}$$

Multiplying both sides of (3.19) by 3 gives

$$3{X_3}^2 = 12(m^4 + c^4 - 2abm^2 + a^2b^2 + 2mc^3 + 2m^3c + 3m^2c^2 - 2abmc - 2abc^2) \tag{3.20}$$

Since (3.16) equals (3.20), then (2.2) is true.
We know that if $T_k$ is the kth Tribonacci number, then

$$T_k = T_{k-1} + T_{k-2} + T_{k-3}$$

Now, if we let

$$a = T_{k-1}, \tag{3.21}$$

$$b = T_{k-2}, \tag{3.22}$$

$$c = T_{k-3}, \tag{3.23}$$

we see that

$$a + b + c = T_k \tag{3.24}$$

From (2.2), we see that

$$(a+c)^2 + (b+c)^2 + (a+b)^2 = a^2 + 2ac + c^2 + b^2 + 2bc + c^2 + a^2 + 2ab + b^2$$

$$(a+c)^2 + (b+c)^2 + (a+b)^2 = 2a^2 + 2b^2 + 2c^2 + 2ab + 2ac + 2bc$$

$$(a+c)^2 + (b+c)^2 + (a+b)^2 = (a^2 + b^2 + c^2) + (a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)$$

$$(a+c)^2 + (b+c)^2 + (a+b)^2 = (a^2 + b^2 + c^2 + (a+b+c)^2)$$

So, (2.2) becomes

$$2((a+c)^4+(b+c)^4+(a+b)^4)+4(a^4+b^4+c^4+(a+b+c)^4) = 3((a^2+b^2+c^2+(a+b+c)^2)^2 \tag{3.25}$$

Putting (3.21), (3.22), (3.23) and (3.24) in (3.25) gives

$$2((T_{k-1}+T_{k-3})^4+(T_{k-2}+T_{k-3})^4+(T_{k-1}+T_{k-2})^4)+4(T_{k-1}^4+T_{k-2}^4+T_{k-3}^4+T_k^4) = 3(T_{k-1}^2+T_{k-2}^2+T_{k-3}^2+T_k^2)^2 \tag{3.26}$$

Letting $k = n + 3$ in (3.26), we have

$$2((T_{n+2}+T_n)^4(T_{n+1}+T_n)^4+(T_{n+2}+T_{n+1})^4)+4(T_{n+2}^4+T_{n+1}^4+T_n^4+T_{n+3}^4) = 3(T_{n+2}^2+T_{n+1}^2+T_n^2+T_{n+3}^2)^2$$

$$2((T_n+T_{n+1})^4+(T_n+T_{n+2})^4+(T_{n+1}+T_{n+2})^4)+4(T_n^4+T_{n+1}^4+T_{n+2}^4+T_{n+3}^4) = 3(T_n^2+T_{n+1}^2+T_{n+2}^2+T_{n+3}^2)^2 \tag{3.27}$$

We can see that (3.27) equals (2.1), which completes the proof.

# 4 Conclusion

In this paper, we have proved an identity involving Tribonacci numbers by extending the number of variables of Candido's identity to three.

# References

[1] G. Candido, *A Relationahip Between the fourth powers of the Terms of the Fibonacci Series*, Scripta Mathematica, **17** (1951), 230.

[2] Darko Veljan, *A note on Candido's identity and Heron's formula*, `https://doi.org/10.5592/CO/CCD.2016.08`.

[3] R. B. Nelsen, *Proof without words:Candido's Identity*, Mathematics Magazine, **78** (2005), 131.

AMS Mathematical Number: 11B39