A Geometric Variational Problem on a Periodic Domain

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Abstract of Dissertation

A Geometric Variational Problem on a Periodic Domain

A diblock copolymer melt is a soft material, characterized by fluid-like disorder on the molecular scale and a high degree of order at a longer length scale. A molecule in a diblock copolymer is a linear sub-chain of A-monomers grafted covalently to another sub-chain of B-monomers. The Ohta-Kawasaki density functional theory of diblock copolymer gives rise to a non local free boundary problem. We will work on a periodic lattice in $\mathbb{C}$ generated by two complex numbers and we will assume periodic boundary conditions. In this thesis we will find two stationary sets of the energy functional of the problem. The first set is a perturbation of a round disk in $\mathbb{C}$. More specifically, we will perturb a round disk under the polar coordinates. The radius of this perturbed disk will be sufficiently small. Also, we will minimize the energy of this stationary set with respect to the shape and size of the lattice. Additionally, we will show that for every $K \geq 2$, $K \in \mathbb{N}$ there exists a stationary set of the free energy functional that is the union of K disjoint perturbed disks in $\mathbb{C}$. Later, we will assume that $K = 2$ and we will deal with the problem of finding the centers of these two perturbed disks. We will show that the centers of these disks are close to a global minimum of the Green's function of the problem. We will minimize the Green's function of the problem looking at some special cases of lattice structures. These lattices are the hexagonal lattice and a family of rectangular lattices.

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Chapter 1

Introduction

1.1 Diblock Copolymer Problem

Following [2], [11] and [30] a diblock copolymer melt is a soft material, characterized by fluid-like disorder on the molecular scale and a high degree of order at a longer length scale. A molecule in a diblock copolymer is a linear sub-chain of A-monomers grafted covalently to another sub-chain of B-monomers. For more information about block copolymers see [29]. The Ohta Kawasaki density functional theory of diblock copolymers gives rise to a non local free boundary problem.The triblock copolymer system is studied in [22] and [27].

The articles [2], [18], [20] and [14] deal with this problem assuming Neumann boundary conditions. In this article we deal with the problem assuming periodic boundary conditions. This changes the Green's function of the problem. To see this change, the Green's function of the problem assuming Neumann boundary conditions satisfies $R(x,x) \to \infty$ as x approaches the boundary of the domain, by [21]. Here, R is the regular part of the Green's function. In periodic boundary conditions we have that $R(x,x)$ is a constant. The Green's function on Neumann boundary conditions can be found in [24]. For a treatment using the theory of Γ-Convergence see [8], [25] and [10]. For the definition of Γ-Convergence see [28]. We will study the problem in two dimensions. The article [17] mentions the problem in one dimension. Let Λ be a lattice generated by two complex number a and b, where $Im(\frac{b}{a}) > 0$. We have $\Lambda = \{na + mb : m, n \in \mathbb{Z}\}$. For more information about lattices see [33]. Also let $D = D_\Lambda = \{ta + sb : t, s \in (0,1)\}$. Following [4] and [44] and based on a density functional theory of Ohta and Kawasaki, the free energy of the diblock copolymer system on a Λ- periodic lattice is

$$J(\Omega) = P_D(\Omega) + \frac{\gamma}{2}\int_D |\nabla I_\Lambda(\Omega)(x)|^2 dx \tag{1.1}$$

for $\Omega \subseteq D$ under the constraint $|\Omega| = \omega|D|$, where $I_\Lambda(\Omega)$ is the Λ-periodic solution of Poisson's equation

$$-\Delta I_\Lambda(\Omega)(x) = \chi_\Omega(x) - \omega, x \in \mathbb{C}, \int_D I_\Lambda(\Omega)(y)dy = 0, \tag{1.2}$$

where χ_Ω is the characteristic function of Ω and γ and ω are parameters of the problem. Here $|\Omega|$ stands for the 2-dimensional Lebesgue measure of Ω. Also by following [5] $P_D(\Omega)$ is the perimeter of the set Ω in D, defined by

$$P_D(\Omega) = \sup\{\int_\Omega div\phi : \phi \in C_c^1(D,\mathbb{C}), |\phi| \leq 1\}. \tag{1.3}$$

The perimeter is closely related with functions of bounded variation. For more details about functions of bounded variation, see [15] and [45]. Poisson's equation is the prototype of an Elliptic Partial Differential Equation. For more information on Poisson's equation and Elliptic Partial Differential Equations see [16], [47], [36], [41] and [32]. For the free energy functional on Neumann boundary conditions see [23]. As far as the perimeter part is concerned see [15]. In our problem, if Ω is compactly contained in D and the boundary of Ω is a smooth curve or a union of smooth curves in $\mathbb{C}$, then the perimeter of Ω in D defined in Equation (1.3) is simply the length of the boundary of Ω. By [9] one calls $\partial\Omega \cap D$ the interface of Ω because it seperates Ω and $D - \Omega$. Also, for the Ohta Kawasaki model on Neumann boundary conditions see [13] and [26].

The Green's function G_Λ associated to the above Poisson equation of I_Λ can be found in [1]. We have

$$G_\Lambda(x, y) = G(x - y),$$

where

$$G(z) = -\frac{1}{2\pi}\log\left(\frac{2\pi|z|}{\sqrt{|D|}}\right) + \frac{|z|^2}{4|D|} + H(z), z \in \mathbb{C} - \Lambda, \tag{1.4}$$

where H is a harmonic function on $(\mathbb{C} - \Lambda) \cup \{0\}$. given by

$$H(z) = -\frac{1}{2\pi}\log\Big|e\left(\frac{\bar{a}z^2}{4a|D|i} - \frac{z}{2a} + \frac{b}{12a}\right)\frac{(1 - e(\frac{z}{a}))\sqrt{|D|}}{2\pi z}\times$$

$$\times\prod_{n=1}^{\infty}\left[\left(1 - e\left(\frac{nb + z}{a}\right)\right)\left(1 - e\left(\frac{nb - z}{a}\right)\right)\right]\Big|. \tag{1.5}$$

For harmonic functions see [39]. The infinite product on the formula for H converges absolutely for $z \in \mathbb{C} - \Lambda$. For more information about infinite products see [40]. Here, we define $e(w) = e^{2\pi i w}$ for $w \in \mathbb{C}$ and $\bar{a}$ is the conjugate of the complex number a. With the help of the Green's function, we write the function I_Λ as

$$I_\Lambda(\Omega)(x) = \int_\Omega G(x - y)dy. \tag{1.6}$$

For an alternative treatment using periodic boundary conditions see [12]. Here, we note that all integrals discussed are of Lebesgue type. For more information about Lebesgue integration theory, see [5], [42], [43] and [38].

1.2 Perturbed Disks

This dissertation can be divided into two parts. The first part (also called the first problem) is to find a stationary set of the energy functional J which is a perturbed disk in D. The second part (also called the second problem) is to prove that for every $K \geq 2$ there exists a stationary set of J which is the union of K disjoint perturbed disks in D. Consider the problem with the one disk first. Let $B = B(\xi, r)$ be a round disk in $\mathbb{C}$ that is compactly contained in D. We perturb the disk B under the polar coordinates using a function ϕ and we form the set

$$E_\phi = \{\xi + we^{i\theta} : \theta \in [0, 2\pi], w \in [0, \sqrt{r^2 + \phi(\theta)})\} \tag{1.7}$$

under the constrains $\phi \in H^2(S^1)$ and $\int_0^{2\pi} \phi(\theta) d\theta = 0$. We can view S^1 as the interval $[0, 2\pi]$ with the endpoints identified. Here, $H^2(S^1)$ is the Sobolev space consisting of all functions $\phi \in L^2(S^1)$ and two times weakly differentiable such that the weak derivatives belong to $L^2(S^1)$. Here, we are considering weak derivatives of functions. For weak derivatives see [37]. Here, we require $\int_0^{2\pi} \phi(\theta) d\theta = 0$ in order to fix the area of E_ϕ to πr^2. We set $\omega = \frac{\pi r^2}{|D|}$ and from now on r replaces ω.

Definition 1.1. *The set E_ϕ defined in Equation (1.7) is a stationary set of the energy functional J if and only if $\frac{dJ(\phi+\epsilon\psi)}{d\epsilon}|_{\epsilon=0} = 0$ for every $\psi \in H^2(S^1)$ with $\int_0^{2\pi} \psi(\theta) d\theta = 0$. Here,* *we write* $J(\phi+\epsilon\psi)=J(E_{\phi+\epsilon\psi})$.

Note also, that by [3] $H^2(S^1)\subseteq C^{1,\frac{1}{2}}(S^1)$, where $C^{1,\frac{1}{2}}(S^1)$ is the Holder space consisting of all functions that they belong to $C^1(S^1)$ and their first derivatives are Holder continuous with exponent $\frac{1}{2}$. From now on, we will identify the function $\phi\in H^2(S^1)$ with its $C^{1,\frac{1}{2}}(S^1)$ version. Also, calculations show the following lemma.

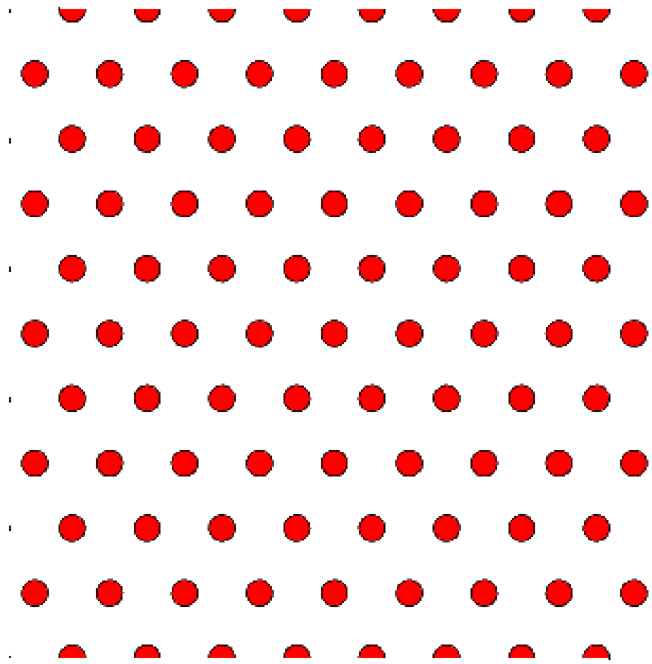

Figure 1.1: One disk on the Hexagonal lattice

Lemma 1.2. *For a function* $\phi\in H^2(S^1)$ *with* $\int_0^{2\pi}\phi(\theta)d\theta=0$*, we have* $\frac{dJ(\phi+\epsilon\psi)}{d\epsilon}|_{\epsilon=0}=\frac{1}{2}\int_0^{2\pi}S(\phi)(\theta)\psi(\theta)d\theta$ *for all* $\psi\in H^2(S^1)$ *such that* $\int_0^{2\pi}\psi(\theta)d\theta=0$*. Here, J is defined in Equation (1.1) and*

$$S(\phi)(\theta)=W(\phi)(\theta)+\gamma\int_{E_\phi}G(\xi+\sqrt{r^2+\phi(\theta)}e^{i\theta}-y)dy, \tag{1.8}$$

where

$$W(\phi)(\theta) = \frac{r^2 + \phi(\theta) + \frac{3\phi'(\theta)}{4(r^2+\phi(\theta))} - \frac{\phi''(\theta)}{2}}{(r^2 + \phi(\theta) + \frac{(\phi'(\theta))^2}{4(r^2+\phi(\theta))})^{\frac{3}{2}}}. \tag{1.9}$$

$W(\phi)$ *is the signed curvature of the boundary of the set* E_ϕ*.*

For the definition of signed curvature see [46].

In the light of the previous lemma we have the following definition.

Definition 1.3. *The Euler Lagrange equations of the function* ϕ *are* $S(\phi)(\theta) = \lambda$ *for every* $\theta \in [0, 2\pi]$*, where* λ *is a constant.*

It is evident that if the function ϕ satisfies the Euler Lagrange equations then by 1.1 and 1.2 the set E_ϕ is a stationary set of the functional J. The Euler-Lagrange equations assuming Neumann boundary conditions can be found in [19]. In this paper we will prove that there exists a function $\phi \in H^2(S^1)$ such that $S(\phi)(\theta) = \lambda$ for every $\theta \in [0, 2\pi]$, where λ is a constant. The H^2 norm of ϕ will be small compared to r and also r will be sufficiently small, so we can ensure that $r^2 + \phi(\theta) > 0$ for all $\theta \in [0, 2\pi]$ and also that E_ϕ is compactly contained in D. In the first problem we will assume that r is sufficiently small and $\gamma r^3 < 12 - \delta$ for some $\delta > 0$. This means that γ can be large, because we are taking $r > 0$ small enough. For a treatment where γ is small, see [31].

Remark 1.4. *In the first problem we will assume* $\xi = 0$*, since the Euler-Lagrange equations remain the same for a translation of the set* E_ϕ*, by the periodic boundary conditions.*

Now, we consider the problem with the K perturbed disks. We will prove that for every $K \geq 2$, $K \in \mathbb{N}$ there exists a stationary set of the free energy functional that is the union of K disjoint perturbed disks in D. Let ρ be a sufficiently small positive number and let $\rho_1, \rho_2, .., \rho_K$ be positive numbers so that $\sum_{i=1}^K \rho_i^2 = K\rho^2$. Let K disjoint round disks $B(\xi_1, \rho_1), ..., B(\xi_K, \rho_K)$ compactly contained in D. We perturb these disks using K functions $\varphi_1, \varphi_2, .., \varphi_K$ and we form the set $E_\varphi = \bigcup_{k=1}^K E_{\varphi_k}$, where

$$E_{\varphi_k} = \{\xi_k + we^{i\theta} : \theta \in [0, 2\pi], w \in [0, \sqrt{\rho_k{}^2 + \varphi_k(\theta)})\}. \tag{1.10}$$

Here, we set $\varphi = (\varphi_1, \varphi_2, ..., \varphi_K)$. We take ρ sufficiently small so that the K perturbed disks remain disjoint. The function φ belongs to $H^2(S^1)$ and also the H^2 norm of φ is sufficiently small compared to ρ, so we can ensure that $\rho_k{}^2 + \varphi_k(\theta) > 0$ for all $\theta \in S^1$ and also that E_φ is compactly contained in D. We also assume $\int_0^{2\pi} \varphi_k(\theta) d\theta = 0$ for $k = 1, 2, .., K$, so that $|E_\varphi| = \sum_{i=1}^K \pi\rho_i^2 = K\pi\rho^2$. .We set $\omega = \frac{K\pi\rho^2}{|D|}$ and ρ replaces ω.

In the second problem we consider the function $F(\xi) = \sum_{i\neq j} G(\xi_j - \xi_i)$ for $\xi_1, .., \xi_K \in D$ and $\xi_i \neq \xi_j$ for $i \neq j$. Let U_1 be a small open neighborhood of the set $\{\eta : F(\eta) = \min_{\xi \in D^K, \xi_i \neq \xi_j} F(\xi)\}$. Also let $U_2 = \{(\rho_1, \rho_2, .., \rho_K) : \rho_k \in ((1-\delta_1)\rho, (1+\delta_1)\rho), \sum_{k=1}^K \rho_k^2 = K\rho^2\}$, where δ_1 is sufficiently small.

Remark 1.5. *In the second problem we assume that the centers of the K perturbed disks belong to U_1, i.e. $(\xi_1, \xi_2, .., \xi_K) \in U_1$. Also we assume that $(\rho_1, \rho_2, ..., \rho_K) \in U_2$. We will show that there exist centers $(\zeta_1, \zeta_2, ..., \zeta_K) \in U_1$ and radii $(\rho_1, .., \rho_K) \in U_2$ such that with these centers and radii φ satisfies the Euler-Lagrange equations. Thus, the centers of the K perturbed disks are close to a global minimum of the function $F(\xi)$.*

Definition 1.6. *The set E_φ is a stationary set of the functional J if and only if $\frac{dJ(\varphi+\epsilon\psi)}{d\epsilon}|_{\epsilon=0} = 0$ for every $\psi = (\psi_1, \psi_2, .., \psi_K) \in H^2(S^1)$ such that $\sum_{k=1}^K \int_0^{2\pi} \psi_k(\theta) d\theta = 0$. Here we write $J(\varphi + \epsilon\psi) = J(E_{\varphi+\epsilon\psi})$.*

By a direct computation we can prove the following lemma.

Lemma 1.7. *For a function $\varphi = (\varphi_1, \varphi_2, .., \varphi_K) \in H^2(S^1)$ with $\int_0^{2\pi} \varphi_k(\theta)d\theta = 0$, we have $\frac{dJ(\phi+\epsilon\psi)}{d\epsilon}|_{\epsilon=0} = \frac{1}{2}\int_0^{2\pi} T(\phi)(\theta)\cdot\psi(\theta)d\theta$ for all $\psi = (\psi_1, \psi_2, ..., \psi_K) \in H^2(S^1)$ such that $\sum_{k=1}^K \int_0^{2\pi} \psi_k(\theta)d\theta = 0$. Here $\cdot$ denotes the dot product and*

$$T(\varphi)(\theta) = (T_1(\theta), T_2(\theta), ..., T_K(\theta)).$$

$$T_k(\theta) = W_k(\varphi_k)(\theta) + \sum_{i=1}^K \gamma \int_{E_{\varphi_i}} G(\xi_k + \sqrt{\rho_k^2 + \varphi_k(\theta)}e^{i\theta} - y)dy, \tag{1.11}$$

where $k \in \{1, ..., K\}$.

$$W_k(\varphi_k)(\theta) = \frac{\rho_k^2 + \varphi_k(\theta) + \frac{3\varphi_k'(\theta)}{4(\rho_k^2+\varphi_k(\theta))} - \frac{\varphi_k''(\theta)}{2}}{(\rho_k^2 + \varphi_k(\theta) + \frac{(\varphi_k'(\theta))^2}{4(\rho_k^2+\varphi_k(\theta))})^{\frac{3}{2}}}. \tag{1.12}$$

$W_k(\varphi_k)$ is the signed curvature of the boundary of the set E_{φ_k}.

We have the following definition.

Definition 1.8. *The Euler-Lagrange equations for the function φ are $T_k(\varphi)(\theta) = \lambda$ for all $\theta \in S^1$ and for every $k \in \{1, .., K\}$, where λ is a constant.*

By Lemma 1.7 it is evident that if the function φ satisfies the Euler-Lagrange equations then the set E_φ is a stationary set of J. In this article we will show that there exists a function φ which satisfies the Euler-Lagrange equations. The H^2 norm of φ will be small compared to ρ and also ρ will be sufficiently small, so we can ensure that $\rho_k^2 + \varphi_k(\theta) > 0$ for all $\theta \in [0, 2\pi]$ and also that E_φ is compactly contained in D.

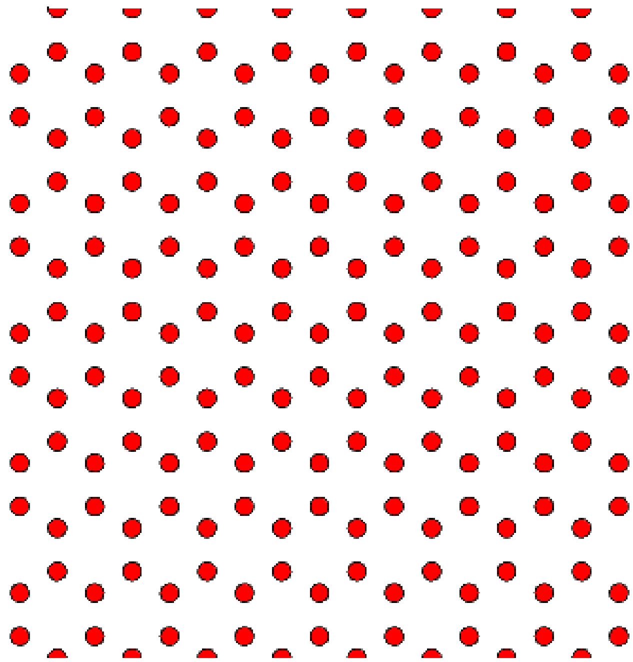

Figure 1.2: Two disks on the Hexagonal lattice

Chapter 2

Statement of the results

2.1 The Results of the First Problem

For the first problem we have the following.

We assume that r is sufficiently small and also $\gamma r^3 < 12 - \delta$, where $\delta > 0$. We will work on the Hilbert space $H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$, where

$$\{1, \cos\theta, \sin\theta\}^{\perp} = \{\phi \in L^2(S^1) : \int_0^{2\pi} \phi(\theta) d\theta = \int_0^{2\pi} \phi(\theta)\cos\theta d\theta = \int_0^{2\pi} \phi(\theta)\sin\theta d\theta = 0\}.$$

Definition 2.1. *For functions $u, v, \psi \in L^2(S^1)$ and for an operator A we define $A'(\psi)(u)$ and $A''(\psi)(u, v)$ to be the first and second Gateaux derivatives of A respectively. That is,*

$$A'(\psi)(u) = \frac{dA(\psi + \epsilon u)}{d\epsilon}|_{\epsilon=0}, A''(\psi)(u, v) = \frac{d^2 A(\psi + \epsilon u + \eta v)}{d\epsilon d\eta}|_{\epsilon=\eta=0}.$$

We can write $S(\phi) = S(0) + S'(0)(\phi) + N(\phi)$, where $N(\phi)$ is a higher order term.

Let Π be the orthogonal projection from $L^2(S^1)$ to $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$. Now, we will state the main results of the first problem.

Proposition 2.2. *Let S be the operator defined in Equation (1.8). The operator $\Pi S'(0)$: $H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^\perp \to L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^\perp$ is one to one and onto.*

By Proposition 2.2 we can write the equation $\Pi(S(\phi)) = 0$ equivalently as

$$\phi = -(\Pi S'(0))^{-1}(\Pi S(0) + \Pi N(\phi)). \tag{2.1}$$

Proposition 2.3. *There exists a function $\phi \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^\perp$ such that $\Pi S(\phi) = 0$. Equivalently, there exist constants λ, μ, ν such that $S(\phi)(\theta) = \lambda + \mu\cos\theta + \nu\sin\theta$ for all $\theta \in [0, 2\pi]$. Also, the solution ϕ satisfies $||\phi||_{H^2} = O(\gamma r^7) = O(r^4)$.*

Proposition 2.4. *The constants μ, ν in Proposition 2.3 satisfy $\mu = \nu = 0$. That is, ϕ satisfies the Euler-Lagrange equations.*

By Propositions 2.2, 2.3 and 2.4 we have proved the following Theorem.

Theorem 2.5. *1. For every $\delta > 0$ and $\xi \in D$ there exists $r_0 > 0$ such that if $\gamma r^3 < 12 - \delta$ and $r < r_0$ then there exists a solution to the Euler-Lagrange equations E_ϕ with center ξ.*
2. This solution is a perturbed disk compactly contained in D.

We denote the stationary set E_ϕ by E_Λ. Consider $J^*_\Lambda(E_\Lambda) = \frac{1}{|D_\Lambda|} J_\Lambda(E_\Lambda)$. Assume $|D_\Lambda| = 1$ and consider the set $E_{t\Lambda}$ for $t > 0$. Then we can see that $E_{t\Lambda} = tE_\Lambda$. In the next Propositions we will consider the role played by the lattice Λ.

Proposition 2.6. *Assume that the lattice Λ satisfies $|D_\Lambda| = 1$ and consider the lattice $t\Lambda$, where $t > 0$. A set E_Λ in $\mathbb{C}/\Lambda$ gives rise to tE_Λ in $\mathbb{C}/t\Lambda$. Then, the energy per cell area $\frac{J_{t\Lambda}(tE_\Lambda)}{|tD_\Lambda|}$ is minimized at $t = t_\Lambda$, where*

$$t_\Lambda = \left(\frac{P_D(E_\Lambda)}{\gamma \int_D |\nabla I_\Lambda(E_\Lambda)(x)|^2 dx}\right)^{\frac{1}{3}}. \tag{2.2}$$

In the next Proposition we will consider the function $A(\zeta) = J^*_{t_\Lambda \Lambda}(E_{t_\Lambda \Lambda})$, where $\zeta = \frac{b}{a}$.

Proposition 2.7. *Let $\zeta = \frac{b}{a}$. Then for every ζ with $Im(\zeta) > 0$ and $\zeta \neq e^{\frac{\pi i}{3}}$ we have $A(\zeta) > A(e^{\frac{\pi i}{3}})$ for r small enough.*

2.2 The Results of the Second Problem

For the second problem we have the following.

We assume that ρ is sufficiently small and also $\gamma\rho^3 < 12 - \delta$, where $\delta > 0$. We assume also that $\gamma\rho^3 \log \frac{1}{\rho} > 1 + \delta$.

We will work on the Hilbert space X, where

$$X = \{\varphi \in H^2(S^1, \mathbb{R}^K) : \int_0^{2\pi} \varphi_k(\theta) d\theta = \int_0^{2\pi} \varphi_k(\theta) \cos\theta d\theta = \int_0^{2\pi} \varphi_k(\theta) \sin\theta d\theta = 0,$$

$$k \in \{1, ..., K\}\}.$$

We define

$$Y = \{\varphi \in L^2(S^1, \mathbb{R}^K) : \int_0^{2\pi} \varphi_k(\theta) d\theta = \int_0^{2\pi} \varphi_k(\theta) \cos\theta d\theta = \int_0^{2\pi} \varphi_k(\theta) \sin\theta d\theta = 0,$$

$$k \in \{1, .., K\}\}.$$

In the space $L^2(S^1, \mathbb{R}^K)$ the inner product is defined as $< u, v >= \int_0^{2\pi} u(\theta) \cdot v(\theta) d\theta$.

Also, the inner product in the space $H^2(S^1, \mathbb{R}^K)$ is defined as

$< u, v >= \int_0^{2\pi} u \cdot v + u' \cdot v' + u'' \cdot v''$.

Let Π be the orthogonal projection from $L^2(S^1, \mathbb{R}^K)$ to Y.

We can write $T(\varphi) = T(0) + T'(0)(\varphi) + N(\varphi)$, where $N(\varphi)$ is a higher order term.

Now we can state the main results of the second problem.

Proposition 2.8. *The operator* $\Pi T'(0) : X \to Y$ *is one to one and onto.*

Thus, we can write the equation $\Pi T(\varphi) = 0$ equivalently as

$$\varphi = -(\Pi T'(0))^{-1}(\Pi T(0) + \Pi N(\varphi)). \tag{2.3}$$

Proposition 2.9. *There exists a function* $\varphi \in X$ *such that* $\Pi T(\varphi) = 0$*. Equivalently, there exist constants* λ_k, μ_k, ν_k *such that*

$$T_k(\varphi)(\theta) = \lambda_k + \mu_k \cos\theta + \nu_k \sin\theta \tag{2.4}$$

for all $k \in \{1, ..., K\}$*. Also this solution satisfies* $||\varphi||_{H^2} = O(\rho^4)$*.*

Proposition 2.10. *There exist centers* $(\zeta_1, ..., \zeta_K) \in U_1$ *for* E_φ *such that in these centers the constants in Proposition 2.9* λ_k, μ_k *and* ν_k *satisfy* $\mu_1 = ... = \mu_K = \nu_1 = ... = \nu_K = 0$ *and* $\lambda_1 = \lambda_2 = ... = \lambda_K = \lambda$*. That is, the function* φ *satisfies the Euler-Lagrange equations with centers* $(\zeta_1, .., \zeta_K)$*.*

By Propositions 2.8, 2.9 and 2.10 we have proved the following theorem.

Theorem 2.11. *Let* $K \geq 2$ *be an integer.*

1. For every $\delta > 0$ *there exists* $\rho_0 > 0$ *such that if* $\gamma\rho^3 \log\frac{1}{\rho} > 1 + \delta$*,* $\gamma\rho^3 < 12 - \delta$ *and* $\rho < \rho_0$ *then there exists a solution to the Euler-Lagrange equations* E_φ*.*

2. This solution is the union of K *disjoint perturbed disks compactly contained in* D *with centers* $(\zeta_1, .., \zeta_K) \in U_1$ *and radii* $\rho_1, .., \rho_K \in U_2$*. The radius of every perturbed disk is close to* ρ*.*

3. Let the centers of these perturbed disks be ζ_1*,..,*ζ_K*. Then* $\zeta = (\zeta_1, .., \zeta_K)$ *is close to a global minimum of the function* F*, where* $F(\xi) = \sum_{i \neq j} G(\xi_j - \xi_i)$ *for* $\xi_1, .., \xi_K \in D$*.*

For the following Proposition we assume that $K = 2$, i.e. we consider the case where we have two perturbed disks. By Theorem 2.11 we can see that the location of the centers (ζ_1, ζ_2) is close to a global minimum of the function $F(\xi_1, \xi_2) = G(\xi_2 - \xi_1)$. Therefore, to find the location of these centers we need to find the global minima of the function F. We consider the following cases. The first case is for $\zeta = \frac{b}{a} = e^{\frac{i\pi}{3}}$ and the second is $\zeta = \frac{b}{a} = ti$, for $t > \frac{4\log 2}{\pi}$.

Proposition 2.12. *Assume* $K = 2$.

1. If $\zeta = \frac{b}{a} = e^{\frac{i\pi}{3}}$, *then the difference of the centers* ζ_1, ζ_2 *of* E_φ *are close to the points* $\frac{a+b}{3}$ *and* $\frac{2(a+b)}{3}$.

2. If $\zeta = \frac{b}{a} = ti$, *where* $t > \frac{4\log 2}{\pi} = 0.8825...$, *then the difference of the centers* ζ_1, ζ_2 *of* E_φ *is close to the point* $\frac{a+b}{2}$.

Chapter 3

Proof of the main results of the first problem

3.1 Important Lemmas

We write $S(\phi)(\theta) = W(\phi)(\theta) + A(\phi)(\theta) + B(\phi)(\theta)$, where

$$A(\phi)(\theta) = -\gamma \int_{E_\phi} \frac{1}{2\pi} \log |\sqrt{r^2 + \phi(\theta)} e^{i\theta} - y| dy$$

and

$$B(\phi)(\theta) = \gamma \int_{E_\phi} R(\sqrt{r^2 + \phi(\theta)} e^{i\theta} - y) dy.$$

Here, W is the operator defined in Equation 1.9. Also, we write $G(x) = -\frac{1}{2\pi} \log |x| + R(x)$. R is the regular part of the Green's function.

Following [2] calculations show that if $u \in H^2(S^1)$ and $\int_0^{2\pi} u(\theta) d\theta = 0$, then

$$W'(0)(u) = -\frac{u'' + u}{2r^3} \tag{3.1}$$

$$A'(0)(u) = -\frac{\gamma}{8\pi}\int_0^{2\pi} u(\omega)\log(1-\cos(\theta-\omega))d\omega - \frac{\gamma u(\theta)}{4} \tag{3.2}$$

$$B'(0)(u) = \frac{\gamma}{2}\int_0^{2\pi} u(\omega)R(re^{i\theta}-re^{i\omega})d\omega + \frac{u(\theta)\gamma}{2r}\int_{B(0,r)} \nabla R(re^{i\theta}-y)e^{i\theta}dy. \tag{3.3}$$

Let $L_1(u) = W'(0)(u) + A'(0)(u)$.

Lemma 3.1. *Let* $\lambda_n = \frac{n^2-1}{2r^3} + \frac{\gamma}{4n} - \frac{\gamma}{4}$*, where* $n \geq 2$*. Then* $L_1(\cos(n\theta)) = \lambda_n \cos(n\theta)$ *and* $L_1(\sin(n\theta)) = \lambda_n \sin(n\theta)$ *for* $n \geq 2$*. The eigenvalues of* L_1 *are* λ_n*,* $n \in \mathbb{N}$*,* $n \geq 2$ *with corresponding eigenfunctions* $\cos(n\theta)$ *and* $\sin(n\theta)$*.*

Proof. Using Equations (3.1), (3.2) and (3.3) and differentiating $e^{in\theta}$ that

$$L_1(e^{in\theta}) = -\frac{1-n^2}{2r^3}e^{in\theta} - \frac{\gamma}{4}e^{in\theta} - \frac{\gamma}{8\pi}\int_0^{2\pi} e^{in\omega}\log(1-\cos(\theta-\omega))d\omega.$$

Also, calculations show that $\int_0^{2\pi} e^{in\omega}\log(1-\cos(\theta-\omega))d\omega = -\frac{2\pi}{n}e^{in\theta}$. Now by the linearity of L_1 and considering real and imaginary parts we get the lemma. □

Now we will get an estimate on $B'(0)(u)$.

Lemma 3.2. *There exists a constant* $C > 0$ *independent of* r *and* γ *such that* $||B'(0)(u)||_{L^2} \leq C\gamma r||u||_{L^2}$ *for every* $u \in H^2(S^1) \cap \{1\}^\perp$*.*

Proof. We have by following [2] that

$$||B'(0)(u)||_{L^2} \leq \frac{\gamma}{2}||\int_0^{2\pi} u(\omega)R(re^{i\theta}-re^{i\omega})d\omega||_{L^2} + \frac{\gamma}{2r}||u(\theta)\int_{B(0,r)} \nabla R(re^{i\theta}-y)e^{i\theta}dy||_{L^2}.$$

By Taylor's expansion we have that $R(re^{i\theta}-re^{i\omega}) = R(0) + O(r)$. Thus,

$$\int_0^{2\pi} u(\omega)R(re^{i\theta}-re^{i\omega})d\omega = \int_0^{2\pi} u(\omega)O(r)d\omega$$

since $u \in \{1\}^{\perp}$. This shows that

$$||\int_0^{2\pi} u(\omega)R(re^{i\theta} - re^{i\omega})d\omega||_{L^2}^2 = \int_0^{2\pi} |\int_0^{2\pi} u(\omega)O(r)d\omega|^2 d\theta.$$

Also, we have

$$|\int_0^{2\pi} u(\omega)O(r)d\omega| \leq Cr\int_0^{2\pi} |u(\omega)|d\omega.$$

Thus,

$$|\int_0^{2\pi} u(\omega)O(r)d\omega|^2 \leq Cr^2(\int_0^{2\pi} |u(\omega)|d\omega)^2.$$

By Holder's inequality we get

$$(\int_0^{2\pi} |u(\omega)|d\omega)^2 \leq C||u||_{L^2}^2.$$

Thus, finally

$$\frac{\gamma}{2}||\int_0^{2\pi} u(\omega)R(re^{i\theta} - re^{i\omega})d\omega||_{L^2} \leq C\gamma r||u||_{L^2}.$$

We will prove now that

$$\frac{\gamma}{2r}||u(\theta)\int_{B(0,r)} \nabla R(re^{i\theta} - y)e^{i\theta}dy||_{L^2} \leq C\gamma r||u||_{L^2}.$$

We have

$$|\int_{B(0,r)} \nabla R(re^{i\theta} - y)e^{i\theta}dy| \leq Cr^2$$

by the smoothness of R near 0. Thus,

$$|\frac{u(\theta)}{2r}\int_{B(0,r)} \nabla R(re^{i\theta} - y)e^{i\theta}dy|^2 \leq Cr^2|u(\theta)|^2.$$

Therefore,

$$\int_0^{2\pi} |\frac{u(\theta)}{2r}\int_{B(0,r)} \nabla R(re^{i\theta} - y)e^{i\theta}dy|^2 d\theta \leq Cr^2\int_0^{2\pi} |u(\theta)|^2 d\theta.$$

Then, it follows easily that

$$\frac{\gamma}{2r}||u(\theta)\int_{B(0,r)}\nabla R(re^{i\theta}-y)e^{i\theta}dy||_{L^2}\leq C\gamma r||u||_{L^2}.$$

which completes the proof of the lemma. □

The next lemma shows that the operator

$\Pi S'(0): H^2(S^1)\cap\{1,\cos\theta,\sin\theta\}^{\perp}\rightarrow L^2(S^1)\cap\{1,\cos\theta,\sin\theta\}^{\perp}$ is one to one.

Lemma 3.3. *There exists a constant $C>0$ independent of r and γ such that $||u||_{L^2}\leq Cr^3||\Pi S'(0)(u)||_{L^2}$ for every $u\in H^2(S^1)\cap\{1,\cos\theta,\sin\theta\}^{\perp}$.*

Proof. We have $< u,1 >=< u,\cos\theta >=< u,\sin\theta >= 0$, where $< . >$ is the inner product in L^2. By considering Fourier series we have

$$u=\sum_{n=2}^{\infty}a_n\cos(n\theta)+b_n\sin(n\theta).$$

Thus,

$$L_1(u)=\sum_{n=2}^{\infty}a_nL_1(\cos(n\theta))+b_nL_1(\sin(n\theta)).$$

Now, by Lemma 3.1 we get

$$L_1(u)=\sum_{n=2}^{\infty}a_n\lambda_n\cos(n\theta)+b_n\lambda_n\sin(n\theta).$$

Thus,

$$||L_1(u)||_{L^2}^2=\sum_{n=2}^{\infty}\int_0^{2\pi}a_n^2\lambda_n^2\cos^2(n\theta)+b_n^2\lambda_n^2\sin^2(n\theta)d\theta=\pi\sum_{n=2}^{\infty}a_n^2,\lambda_n^2+b_n^2\lambda_n^2,$$

where

$$\lambda_n=\frac{n^2-1}{2r^3}+\frac{\gamma}{4n}-\frac{\gamma}{4}=(n-1)(\frac{2n(n+1)-\gamma r^3}{4r^3n}). \tag{3.4}$$

Also, we have $\gamma r^3 < 12 - \delta$. This is equivalent to say that $2(n+1)n - \gamma r^3 > \epsilon n^2$ for some ϵ sufficiently small and all $n \geq 2$. Therefore, by Equation (3.4) we can see that $\lambda_n > \frac{C}{r^3}$. Thus, $\lambda_n^2 > \frac{C}{r^6}$. Also,

$$||u||_{L^2}^2 = C\sum_{n=2}^{\infty} a_n^2 + b_n^2 = Cr^6 \sum_{n=2}^{\infty} \frac{a_n^2}{r^6} + \frac{b_n^2}{r^6} \leq Cr^6 \sum_{n=2}^{\infty} a_n^2\lambda_n^2 + b_n^2\lambda_n^2 = Cr^6||L_1(u)||_{L^2}^2.$$

So, finally we have

$$||u||_{L^2} \leq Cr^3||L_1(u)||_{L^2}.$$

It is evident that $L_1(u) = \Pi L_1(u)$. To finish the proof we will show that

$$||u||_{L^2} \leq Cr^3||\Pi S'(0)(u)||_{L^2}.$$

We have

$$\Pi S'(0) = \Pi L_1 + \Pi B'(0).$$

Thus,

$$||\Pi S'(0)(u)||_{L^2} \geq ||\Pi L_1(u)||_{L^2} - ||\Pi B'(0)(u)||_{L^2} \geq ||\Pi L_1(u)||_{L^2} - C\gamma r||u||_{L^2}.$$

Also, we have

$$||u||_{L^2} \leq Cr^3||L_1(u)||_{L^2} \leq Cr^3[||\Pi S'(0)(u)||_{L^2} + C\gamma r||u||_{L^2}].$$

Thus,

$$(1 - C\gamma r^4)||u||_{L^2} \leq Cr^3||\Pi S'(0)(u)||_{L^2}.$$

But γr^4 is sufficiently small. Therefore, it is evident that

$$||u||_{L^2} \leq Cr^3||\Pi S'(0)(u)||_{L^2}.$$

$\square$

Remark 3.4. *By Lemma 3.3 it is evident that $\Pi S'(0)$ is one to one. To complete the proof of Proposition 2.2 it suffices to show that $\Pi S'(0)$ is onto $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^\perp$.*

We will now prove an improvement of Lemma 3.3.

Lemma 3.5. *There exists a constant $C > 0$ independent of r and γ such that $||u||_{H^2} \leq Cr^3||\Pi S'(0)(u)||_{L^2}$ for every $u \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^\perp$.*

Proof. We have

$$u = \sum_{n=2}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Thus,

$$u' = \sum_{n=2}^{\infty} na_n(-\sin(n\theta)) + nb_n \cos(n\theta)$$

and

$$u'' = \sum_{n=2}^{\infty} -n^2 a_n \cos(n\theta) - n^2 b_n \sin(n\theta).$$

Therefore,

$$||u||_{L^2}^2 = \pi \sum_{n=2}^{\infty} a_n^2 + b_n^2$$

$$||u'||_{L^2}^2 = \pi \sum_{n=2}^{\infty} n^2 a_n^2 + n^2 b_n^2$$

and

$$||u''||_{L^2}^2 = \pi \sum_{n=2}^{\infty} n^4 a_n^2 + n^4 b_n^2.$$

So,

$$||u||_{H^2}^2 = \pi \sum_{n=2}^{\infty} [a_n^2(1 + n^2 + n^4) + b_n^2(1 + n^2 + n^4)] \leq C \sum_{n=2}^{\infty} (a_n^2 + b_n^2) n^4.$$

Also,

$$||\Pi L_1(u)||_{L^2}^2 = \pi \sum_{n=2}^{\infty} \lambda_n^2 (a_n^2 + b_n^2).$$

We have $\gamma r^3 - 2n(n+1) < -\epsilon n^2$ for every $n \geq 2$.

Also we have $\lambda_n = (n-1)(\frac{n+1}{2r^3} - \frac{\gamma}{4n})$ = $(n-1)[\frac{2(n+1)n - \gamma r^3}{4r^3 n}]$> $\frac{(n-1)\epsilon n^2}{4r^3 n}$> $\frac{Cn^2}{r^3}$. Thus,

$$||\Pi L_1(u)||^2_{L^2} = \pi \sum_{n=2}^{\infty} \lambda_n^2 (a_n^2 + b_n^2) > \pi \sum_{n=2}^{\infty} (\frac{Cn^2}{r^3})^2 (a_n^2 + b_n^2) \geq \frac{C}{r^6} ||u||^2_{H^2}.$$

Therefore, we get

$$||u||_{H^2} \leq Cr^3 ||\Pi L_1(u)||_{L^2}.$$

Also,

$$||L_1(u)||_{L^2} = ||\Pi L_1(u)||_{L^2} = ||\Pi S'(0)(u) - \Pi B'(0)(u)||_{L^2} \leq ||\Pi S'(0)(u)||_{L^2} + C\gamma r ||u||_{L^2}.$$

Therefore, we have

$$||\Pi S'(0)(u)||_{L^2} \geq ||L_1(u)||_{L^2} - C\gamma r ||u||_{L^2} \geq \frac{C}{r^3} ||u||_{H^2} - C\gamma r ||u||_{H^2} = (\frac{C_1}{r^3} - C_2 \gamma r) ||u||_{H^2}.$$

Then, we have $\frac{C_1}{r^3} - C_2\gamma r = \frac{C_1 - C_2\gamma r^4}{r^3} \leq \frac{C}{r^3}$. Since γr^4 is small we have $\frac{C_1 - C_2\gamma r^4}{r^3} > 0$. Thus,

$$||u||_{H^2} \leq Cr^3 ||\Pi S'(0)(u)||_{L^2}.$$

which proves the lemma. $\square$

It is known that the operator $\Pi S'(0) : H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp} \to L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ is self adjoint. Also, it is known that self adjoint operators have closed graph. In the next lemma we will show that the range of $\Pi S'(0)$ is closed in $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$.

Lemma 3.6. $\Pi S'(0)(H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp})$ *is closed in* $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$.

Proof. We will show that $\Pi S'(0)(H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp})$ is closed in $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$. For this purpose and by arguing as in [2], let a sequence u_n such that $\Pi S'(0)(u_n) \to w$ in $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$.

Then, we have that $\Pi S'(0)(u_n)$ is a Cauchy sequence. Thus,

$$||\Pi S'(0)(u_n) - \Pi S'(0)(u_m)||_{L^2} \to 0$$

as $n,m \to +\infty$. Now, by Lemma 3.5 we get $||u_n - u_m||_{H^2} \to 0$ as $n,m \to +\infty$. Thus, u_n is a Cauchy sequence in H^2. Therefore, $u_n \to u$ in H^2 for some $u \in H^2$. This gives that

$$(u_n, \Pi S'(0)(u_n)) \to (u, w)$$

in the space

$$(H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}) \times (L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}).$$

This gives that $w = \Pi S'(0)(u)$, since the graph of $\Pi S'(0)$ is closed. This completes the proof. $\square$

3.2 Proof of the main results of the first problem

In this section we will use the previous lemmas to prove the Propositions 2.2, 2.3 and Proposition 2.4.

Proof of Proposition 2.2

We let $z \in L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ with $z \in R(\Pi S'(0))^{\perp}$, where $R(\Pi S'(0))$ is the range of the operator $\Pi S'(0)$. Let $x \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$. Then we have $< \Pi S'(0)(x), z >= 0$. This gives that the map $x \to < \Pi S'(0)(x), z >$ is the zero map. This means that z belongs to the domain of the adjoint of $\Pi S'(0)$. Thus, $z \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$, since $\Pi S'(0)$ is self adjoint.

This gives that $< \Pi S'(0)(z), x >= 0$.

But $H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ is dense in $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$.

Therefore, $< \Pi S'(0)(z), x >= 0$ for every $x \in L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$. Thus, $\Pi S'(0)(z) = 0$ and because $\Pi S'(0)$ is one to one we have $z = 0$. So we have showed that

$$[L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}] \cap R(\Pi S'(0))^{\perp} = \{0\}.$$

Also, since $R(\Pi S'(0))$ is closed in $L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ we have

$$L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp} = R(\Pi S'(0)) \oplus R(\Pi S'(0))^{\perp}.$$

This proves $R(\Pi S'(0)) = L^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ and $\Pi S'(0)$ is onto. Now by Remark 3.4 the proof is complete. $\square$

We consider

$$W(\phi)(\theta) = \frac{r^2 + \phi(\theta) + \frac{3\phi'(\theta)}{4(r^2+\phi(\theta))} - \frac{\phi''(\theta)}{2}}{(r^2 + \phi(\theta) + \frac{(\phi'(\theta))^2}{4(r^2+\phi(\theta))})^{\frac{3}{2}}}.$$

We define the function Φ to be $\Phi = \frac{\phi}{r^2}$. Also, we define the function

$$\widehat{W}(\Phi, \Phi', \Phi'') = \frac{1 + \Phi + \frac{3(\Phi')^2}{4(1+\Phi)} - \frac{\Phi''}{2}}{(1 + \Phi + \frac{(\Phi')^2}{4(1+\Phi)})^{\frac{3}{2}}}.$$

Then, we can see easily that

$$\widehat{W}(\Phi, \Phi', \Phi'') = rW(\phi, \phi', \phi'').$$

The next lemma is an estimate of $W''(\phi)(u, v)$, $A''(\phi)(u, v)$ and $B''(\phi)(u, v)$ for $u, v \in H^2(S^1)$.

Lemma 3.7. *(1) There exists a constant $C > 0$ independent of r and γ such that for $\phi \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ with $||\phi||_{H^2} \leq cr^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have*

$$||W''(\phi)(u, v)||_{L^2} \leq \frac{C}{r^5}||u||_{H^2}||v||_{H^2}.$$

(2) There exists a constant $C > 0$ independent of r and γ such that for $\phi \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ with $||\phi||_{H^2} \leq cr^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have

$$||A''(\phi)(u, v)||_{L^2} \leq \frac{C\gamma}{r^2}||u||_{H^2}||v||_{H^2}.$$

(3) There exists a constant $C > 0$ independent of r and γ such that for $\phi \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^{\perp}$ with $||\phi||_{H^2} \leq cr^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have

$$||B''(\phi)(u, v)||_{L^2} \leq \frac{C\gamma}{r}||u||_{H^2}||v||_{H^2}.$$

Proof. To prove (1) we argue as in [2]. We consider the function $\widehat{W}(\Phi, \Phi', \Phi'')$ defined above. By assumption we can prove that $||\Phi||_{L^\infty}$ is sufficiently small. We will prove that

$$||\widehat{W}''(\phi)(u,v)||_{L^2} \leq C||u||_{H^2}||v||_{H^2}.$$

By a direct calculation, we can prove that

$$\widehat{W}''(\Phi)(u,v) = D_{11}(\widehat{W})uv + D_{21}(\widehat{W})v'u + D_{31}(\widehat{W})v''u + D_{12}(\widehat{W})vu' + D_{22}(\widehat{W})v'u' +$$

$$+D_{32}(\widehat{W})v''u' + D_{13}(\widehat{W})vu'' + D_{23}(\widehat{W})v'u''.$$

Also, we have that

$$|(D_{ij}\widehat{W})(\Phi, \Phi', \Phi'')| \leq C$$

by the smoothness of $\widehat{W}$. Thus, we have

$$||(D_{11}\widehat{W})(\Phi, \Phi', \Phi'')uv||_{L^\infty} \leq C||u||_{L^\infty}||v||_{L^\infty} \leq C||u||_{H^2}||v||_{H^2}.$$

Also,

$$||(D_{22}\widehat{W})(\Phi, \Phi', \Phi'')u'v'||_{L^\infty} \leq C||u'||_{L^\infty}||v'||_{L^\infty} \leq C||u||_{H^2}||v||_{H^2}.$$

By arguing similarly we can prove that

$$||\widehat{W}''(\Phi)(u,v)||_{L^\infty} \leq C||u||_{H^2}||v||_{H^2}.$$

Also, we have that

$$||\widehat{W}''(\Phi)(u,v)||_{L^2} \leq C||\widehat{W}''(\Phi)(u,v)||_{L^\infty},$$

which proves that

$$||\widehat{W}''(\Phi)(u,v)||_{L^2} \leq C||u||_{H^2}||v||_{H^2}.$$

Now we use that

$$\widehat{W}(\Phi, \Phi', \Phi'') = rW(\phi, \phi', \phi'')$$

and also that $\Phi = \frac{\phi}{r^2}$ to conclude that

$$||W''(\phi)(u,v)||_{L^2} \leq \frac{C}{r^5}||u||_{H^2}||v||_{H^2}$$

which proves (1). To prove (2) we consider the function

$$\widehat{A}(\Phi)(\theta) = \int_0^{2\pi}\int_0^{\sqrt{1+\Phi(\omega)}} \log|\sqrt{1+\Phi(\theta)}e^{i\theta} - se^{i\omega}|sdsd\omega$$

with $\Phi = \frac{\phi}{r^2}$. Then, we can show that $A(\phi) = -\frac{\gamma r^2}{2\pi}\widehat{A}(\Phi)$. Arguing as in [2] we have by calculations that

$$\widehat{A}''(\Phi)(u,v) = A_1(\Phi) + A_2(\phi) + A_3(\Phi) + A_4(\Phi) + A_5(\Phi),$$

where

$$A_1(\Phi) = \frac{u(\theta)e^{i\theta}}{4\sqrt{1+\Phi(\theta)}}\int_0^{2\pi} K(\theta,\omega)v(\omega)d\omega$$

$$A_2(\Phi) = \frac{v(\theta)e^{i\theta}}{4\sqrt{1+\Phi(\theta)}}\int_0^{2\pi} K(\theta,\omega)u(\omega)d\omega$$

$$A_3(\Phi) = -\frac{1}{4}\int_0^{2\pi} \frac{K(\theta,\omega)u(\omega)v(\omega)e^{i\omega}}{\sqrt{1+\Phi(\omega)}}d\omega$$

$$A_4(\Phi) = \frac{u(\theta)v(\theta)}{4(1+\Phi(\theta))}\int_{E_\Phi} \frac{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^2 - 2(\sqrt{1+\Phi(\theta)} - e^{i\theta}y)^2}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^4}dy$$

and

$$A_5(\Phi) = -\frac{u(\theta)v(\theta)}{4(1+\Phi(\theta))^{\frac{3}{2}}}\int_{E_\Phi} \frac{(\sqrt{1+\Phi(\theta)}e^{i\theta} - y)e^{i\theta}}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^2}dy.$$

Here,

$$K(\theta,\omega) = \frac{\sqrt{1+\Phi(\theta)}e^{i\theta} - \sqrt{1+\Phi(\omega)}e^{i\omega}}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - \sqrt{1+\Phi(\omega)}e^{i\omega}|^2}.$$

Then arguing as in [2] we can prove (2). To prove (3), by a direct calculation we can show that

$$B''(\phi)(u,v) = B_1(\phi) + B_2(\phi) + B_3(\phi) + B_4(\phi) + B_5(\phi),$$

where

$$B_1(\phi) = \frac{\gamma v(\theta)}{4\sqrt{r^2+\phi(\theta)}}\int_0^{2\pi} u(\omega)\nabla R(\sqrt{r^2+\phi(\theta)}e^{i\theta} - \sqrt{r^2+\phi(\omega)}e^{i\omega})e^{i\theta}d\omega$$

$$B_2(\phi) = \frac{\gamma u(\theta)}{4\sqrt{r^2+\phi(\theta)}} \int_0^{2\pi} v(\omega) \nabla R(\sqrt{r^2+\phi(\theta)}e^{i\theta} - \sqrt{r^2+\phi(\omega)}e^{i\omega})e^{i\theta} d\omega$$

$$B_3(\phi) = \frac{\gamma u(\theta)v(\theta)}{4(r^2+\phi(\theta))} \int_{E_\phi} [(\nabla\nabla_1 R)(\sqrt{r^2+\phi(\theta)}e^{i\theta} - y)\cos\theta +$$

$$+(\nabla\nabla_2 R)(\sqrt{r^2+\phi(\theta)}e^{i\theta} - y)\sin\theta]e^{i\theta} dy$$

$$B_4(\phi) = -\frac{\gamma u(\theta)v(\theta)}{4(r^2+\phi(\theta))^{\frac{3}{2}}} \int_{E_\phi} \nabla R(\sqrt{r^2+\phi(\theta)}e^{i\theta} - y)e^{i\theta} dy$$

and

$$B_5(\phi) = \frac{\gamma}{4} \int_0^{2\pi} \frac{v(\omega)u(\omega)}{\sqrt{r^2+\phi(\omega)}} \nabla R(\sqrt{r^2+\phi(\theta)}e^{i\theta} - \sqrt{r^2+\phi(\omega)}e^{i\omega})e^{i\theta} d\omega,$$

where $\nabla_k R$ is the kth component of ∇R. Using the smoothness of R (3) follows. □

Now, we will prove a helpfull lemma.

Lemma 3.8. *Let* $m = \frac{1}{2\pi}\int_0^{2\pi} S(0)d\theta$. *Then there exists a constant* $C > 0$ *independent of* r *and* γ *such that* $||S(0) - m - \gamma\pi r^3 \nabla H(0)\cdot e^{i\theta}||_{L^2} \le C\gamma r^4$. *Also, the operator norm of* $(\Pi S'(0))^{-1}$ *satisfies* $||(\Pi S'(0))^{-1}|| \le Cr^3$ *for a constant* $C > 0$ *independent of* r *and* γ.

Proof. From lemmas 3.5 and 3.6, it is obvious that $||(\Pi S'(0))^{-1}|| \le Cr^3$. Also, it is evident that

$$S(0)(\theta) = \frac{1}{r} + \gamma \int_{B(0,r)} G(re^{i\theta} - y)dy. \tag{3.5}$$

Therefore, by Equation (1.4)

$$\int_{B(0,r)} G(x-y)dy = \int_{B(0,r)} [-\frac{1}{2\pi}\log(\frac{2\pi|x-y|}{\sqrt{|D|}}) + \frac{|x-y|^2}{4|D|} + H(x-y)]dy.$$

Also, we have that

$$\int_{B(0,r)} \log|x-y|dy = \pi r^2 \log|x| = \pi r^2 \log r$$

if $x = re^{i\theta}$. Thus,

$$\int_{B(0,r)} -\frac{1}{2\pi}\log(\frac{2\pi|x-y|}{\sqrt{|D|}})dy = -r^2(\frac{\log(2\pi)}{2} + \frac{1}{2}\log r) + \frac{r^2}{2}\log(\sqrt{|D|}).$$

Also, we have that

$$\int_{B(0,r)} \frac{|x-y|^2}{4|D|} dy = \frac{1}{4|D|} \int_{B(0,r)} |x|^2 + |y|^2 dy$$

since

$$\int_{B(0,r)} xy dy = 0.$$

By using polar coordinates and setting $x = re^{i\theta}$, we can show that

$$\frac{1}{4} \int_{B(0,r)} |x-y|^2 dy = \frac{3\pi r^4}{8}.$$

Also, since the function H defined in Equation (1.5) is harmonic we can use the mean value theorem for harmonic functions and prove that

$$\int_{B(0,r)} H(x-y) dy = \pi r^2 H(x).$$

Thus, we have by Equation (3.5) that

$$S(0)(\theta) = \frac{1}{r} + \gamma \pi r^2 H(x) + \gamma(-r^2(\frac{\log(2\pi)}{2} + \frac{1}{2}\log r)) + \gamma \frac{r^2}{2} \log(\sqrt{|D|}) + \gamma \frac{3\pi r^4}{8|D|} =$$

$$= C + \gamma \pi r^2 H(re^{i\theta}).$$

Therefore,

$$m = \frac{1}{2\pi} \int_0^{2\pi} S(0)(\omega) d\omega = C + \gamma \pi r^2 \int_0^{2\pi} H(re^{i\omega}) d\omega.$$

So,

$$S(0) - m = \gamma \pi r^2 (H(re^{i\theta}) - \frac{1}{2\pi} \int_0^{2\pi} H(re^{i\omega}) d\omega).$$

By using Taylor's formula we have that

$$H(re^{i\theta}) - \frac{1}{2\pi} \int_0^{2\pi} H(re^{i\omega}) d\omega = H(0) - \frac{1}{2\pi} \int_0^{2\pi} H(0) d\theta + r \nabla H(0) \cdot e^{i\theta} + O(r^2) =$$

$$= r \nabla H(0) \cdot e^{i\theta} + O(r^2).$$

Thus,

$$S(0) - m = \gamma\pi r^2(r\nabla H(0)e^{i\theta} + O(r^2)) = \gamma\pi r^3\nabla H(0)\cdot e^{i\theta} + O(\gamma r^4).$$

$$||S(0) - m - \gamma\pi r^3\nabla H(0)\cdot e^{i\theta}||_{L^\infty} \leq C\gamma r^4$$

which implies that

$$||S(0) - m - \gamma\pi r^3\nabla H(0)\cdot e^{i\theta}||_{L^2} \leq C\gamma r^4.$$

$\square$

Now, we are in a position to prove Proposition 2.3. Firstly, we will state and use a well known proposition about complete metric spaces.

Proposition 3.9. *(Contraction Mapping Principle) Let M be a complete metric space with metric d. Assume that we have a function $f : M \to M$ such that there exists a real number $q < 1$ such that $d(f(x), f(y)) \leq qd(x, y)$ for all $x, y \in M$(such a map is called a contraction). Then the function f admits a unique fixed point, i.e. there exists a unique $x \in M$ such that $f(x) = x$.*

We will argue as in [2] and we will use Proposition 3.9 to prove that there exists a function ϕ such that ϕ solves the Equation (2.1). We have already seen that the above equation is equivalent to $\Pi S(\phi) = 0$.

Proof of Proposition 2.3

We define an operator $T : D(T) \to D(T)$ with

$$T(\phi) = -(\Pi S'(0))^{-1}(\Pi S(0) + \Pi N(\phi)),$$

where

$$D(T) = \{\phi \in H^2(S^1) \cap \{1, \cos\theta, \sin\theta\}^\perp : ||\phi||_{H^2} \leq M\gamma r^7\}$$

and M is a constant sufficiently large to be determined. Firstly we write

$$N(\phi) = N_1(\phi) + N_2(\phi),$$

where

$$N_1(\phi) = W(\phi) - W(0) - W'(0)$$

and

$$N_2(\phi) = A(\phi) - A(0) - A'(0)(\phi) + B(\phi) - B(0) - B'(0)(\phi).$$

Let a function $\psi \in L^2$. We use the Taylor's formula from [34] to write

$$< W(t\phi), \psi >=< W(0), \psi > +t < W'(0)(\phi), \psi > +\frac{t^2}{2} < W''(\tau\phi)(\phi, \phi), \psi >$$

for some $\tau \in (0, 1)$. Thus we have by setting $t = 1$

$$| < W(\phi) - W(0) - W'(0)(\phi), \psi > | = \frac{1}{2} | < W''(\tau\phi)(\phi, \phi), \psi > |$$

$$\leq \frac{1}{2} ||W''(\tau\phi)(\phi, \phi)||_{L^2} ||\psi||_{L^2}.$$

Now, we take the supremum for all $\psi \in L^2$ with $||\psi||_{L^2} \leq 1$ and we have

$$||W(\phi) - W(0) - W'(0)(\phi)||_{L^2} \leq C||W''(\tau\phi)(\phi, \phi)||_{L^2}.$$

Here we use Lemma 3.7 part 1 to conclude that

$$||W(\phi) - W(0) - W'(0)(\phi)||_{L^2} \leq \frac{C}{r^5} ||\phi||^2_{H^2}.$$

Similarly, we can show that

$$||N(\phi)||_{L^2} \leq (\frac{C}{r^5} + \frac{C\gamma}{r^2})||\phi||^2_{H^2}.$$

Thus, by Lemma 3.8 we have

$$||(\Pi S'(0))^{-1}(\Pi N(\phi))||_{H^2} \leq Cr^3(\frac{C}{r^5} + \frac{C\gamma}{r^2})||\phi||^2_{H^2}.$$

Also,

$$||(\Pi S'(0))^{-1}(\Pi S(0))||_{H^2} = ||(\Pi S'(0))^{-1}(\Pi(S(0) - m - \gamma\pi r^3 \nabla H(0) \cdot e^{i\theta}))||_{H^2} \leq$$

$$\leq Cr^3 C\gamma r^4 = C\gamma r^7$$

by Lemma 3.8. Thus, we have that

$$||T(\phi)||_{H^2} \leq C\gamma r^7 + Cr^3(\frac{C}{r^5} + \frac{C\gamma}{r^2})||\phi||^2_{H^2} = C\gamma r^7 + (\frac{C}{r^2} + C\gamma r)||\phi||^2_{H^2}.$$

Also, we have that $\phi \in D(T)$. So, $||\phi||_{H^2} \leq M\gamma r^7$. Thus,

$$||T(\phi)||_{H^2} \leq C\gamma r^7 + C(\frac{1}{r^2} + \gamma r)(M^2\gamma^2 r^{14}).$$

Now, take M such that $M > 3C$. Then, we have that $C\gamma r^7 < \frac{M}{3}\gamma r^7$. Also, we have $CM^2\gamma^2 r^{12} = C\gamma r^7(M^2\gamma r^5)$. Also we have that γr^5 is small. So, we write $\gamma r^5 = \delta$, where δ is small. We take δ such that $M^2\delta < 1$. Then we can see easily that $CM^2\gamma^2 r^{12} = C\gamma r^7(M^2\gamma r^5) \leq C\gamma r^7 < \frac{M}{3}\gamma r^7$. Now, we have that

$$C\gamma^3 M^2 r^{15} = C\gamma r^7(M^2\gamma r^4 \gamma r^4) < C\gamma r^7(M^2\delta^2).$$

We take δ such that $M^2\delta^2 \leq 1$. Then, finally we have $C\gamma^3 M^2 r^{15} < C\gamma r^7 < \frac{M}{3}\gamma r^7$. Thus, finally we have $||T(\phi)||_{H^2} \leq M\gamma r^7$.

. Take ϕ_1 and $\phi_2 \in D(T)$. Then we have

$$T(\phi_1) = -(\Pi S'(0))^{-1}(\Pi S(0) + \Pi N(\phi_1))$$

and

$$T(\phi_2) = -(\Pi S'(0))^{-1}(\Pi S(0) + \Pi N(\phi_2)).$$

So, we have

$$T(\phi_1) - T(\phi_2) = -(\Pi S'(0))^{-1}(\Pi N(\phi_1) - \Pi N(\phi_2)).$$

We have

$$N(\phi_1) = S(\phi_1) - S(0) - S'(0)(\phi_1)$$

and

$$N(\phi_2) = S(\phi_2) - S(0) - S'(0)(\phi_2).$$

Thus,

$$N(\phi_1) - N(\phi_2) = S(\phi_1) - S(\phi_2) - S'(0)(\phi_1 - \phi_2).$$

By taking a function $\psi \in L^2$ and using Taylor's formula again we have

$$< S(t(\phi_1 - \phi_2) + \phi_2), \psi >$$

$$=< S(\phi_2), \psi > +t < S'(\phi_2)(\phi_1 - \phi_2), \psi > +\frac{t^2}{2} < S''(\tau(\phi_1 - \phi_2) + \phi_2)(\phi_1 - \phi_2, \phi_1 - \phi_2), \psi > .$$

for some $\tau \in (0, 1)$. Therefore, by setting $t = 1$ we get

$$< S(\phi_1), \psi >=< S(\phi_2), \psi > + < S'(\phi_2)(\phi_1 - \phi_2), \psi > +$$

$$\frac{1}{2} < S''(\tau(\phi_1 - \phi_2) + \phi_2)(\phi_1 - \phi_2, \phi_1 - \phi_2), \psi > .$$

Now, we take the supremum for $\psi \in L^2$ with $||\psi||_{L^2} \leq 1$ and we get

$$||S(\phi_1) - S(\phi_2) - S'(\phi_2)(\phi_1 - \phi_2)||_{L^2} \leq C(\frac{1}{r^5} + \frac{\gamma}{r^2})||\phi_1 - \phi_2||^2_{H^2}.$$

This gives,

$$||N(\phi_1) - N(\phi_2)||_{L^2} \leq C(\frac{1}{r^5} + \frac{\gamma}{r^2})||\phi_1 - \phi_2||^2_{H^2} + ||S'(\phi_2)(\phi_1 - \phi_2) - S'(0)(\phi_1 - \phi_2)||_{L^2}.$$

Similarly, we can prove that

$$||S'(\phi_2)(\phi_1 - \phi_2) - S'(0)(\phi_1 - \phi_2)||_{L^2} \leq C(\frac{1}{r^5} + \frac{\gamma}{r^2})||\phi_2||_{H^2}||\phi_2 - \phi_1||_{H^2}.$$

Thus,

$$||N(\phi_1) - N(\phi_2)||_{L^2} \leq C(\frac{1}{r^5} + \frac{\gamma}{r^2})||\phi_1 - \phi_2||^2_{H^2} + C(\frac{1}{r^5} + \frac{\gamma}{r^2})||\phi_2||_{H^2}||\phi_2 - \phi_1||_{H^2}.$$

Now we use that $||\phi_2||_{H^2} \leq M\gamma r^7$ and $||\phi_1 - \phi_2||_{H^2} \leq 2M\gamma r^7$ to conclude that

$$||N(\phi_1) - N(\phi_2)||_{L^2} \leq C||\phi_1 - \phi_2||_{H^2} C(\frac{1}{r^5} + \frac{\gamma}{r^2})(3M\gamma r^7).$$

This gives that

$$||T(\phi_1) - T(\phi_2)||_{H^2} \leq Cr^3||N(\phi_1) - N(\phi_2)||_{L^2} \leq$$

$$Cr^3(C||\phi_1 - \phi_2||_{H^2} C(\frac{1}{r^5} + \frac{\gamma}{r^2})3M\gamma r^7) = (CM\gamma r^5 + CM\gamma^2 r^8)||\phi_1 - \phi_2||_{H^2}.$$

But if we take $\gamma r^4 = \epsilon$ with ϵ small, we get

$$CM\gamma r^5 + CM\gamma^2 r^8 \leq CM\epsilon r + CM\epsilon^2 < 1$$

for ϵ small. Thus T is a contraction. Therefore, by Proposition 3.9 we have that T admits a unique fixed point. □

Now, we will use a reparametrization trick to prove Proposition 2.4.

Proof of Proposition 2.4

By Proposition 2.3 we have that $S(\phi)(\theta) = \lambda + \mu \cos\theta + \nu \sin\theta$. Let $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$. Consider the set $\tau + E_\phi = \{\tau + x : x \in E_\phi\}$. Then we have $\tau + E_\phi = \{\tau + we^{i\theta} : \theta \in [0, 2\pi], w \in [0, \sqrt{r^2 + \phi(\theta)})\}$. We reparametrize $\tau + E_\phi$ by changing the center to 0. So we let $\tau + E_\phi = E_{\psi_\tau}$ for some function $\psi_\tau \in H^2(S^1)$. Here $E_{\psi_\tau} = \{we^{i\theta} : \theta \in [0, 2\pi], w \in [0, \sqrt{r^2 + \psi_\tau(\theta)})\}$. It is easy to see that for the energy functional J we have $J(\tau + E_\phi) = J(E_\phi)$ for $|\tau|$ small enough by the periodic boundary conditions. Thus, we have

$$\frac{d}{d\tau_1} J(E_{\psi_\tau})|_{\tau_1=\tau_2=0} = 0 \tag{3.6}$$

and

$$\frac{d}{d\tau_2} J(E_{\psi_\tau})|_{\tau_1=\tau_2=0} = 0. \tag{3.7}$$

For $\theta \in [0, 2\pi]$ we have

$$\tau + \sqrt{r^2 + \phi(\theta)} e^{i\theta} = \sqrt{r^2 + \psi(\widetilde{\eta}(\theta, \tau), \tau)} e^{i\widetilde{\eta}(\theta, \tau)} \tag{3.8}$$

for some function $\widetilde{\eta}(\theta,\tau)$, where $\widetilde{\eta}(.,\tau):[0,2\pi]\to[0,2\pi]$ is one to one and onto. Let a function $f: A\subset\mathbb{R}^5\to\mathbb{R}^2$, where

$$A=\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 : x_2>-r^2, x_3\in(0,2\pi)\}$$

with

$$f(x_1,x_2,x_3,x_4,x_5)=\sqrt{r^2+x_2}e^{ix_1}-(x_4,x_5)-\sqrt{r^2+\phi(x_3)}e^{ix_3}.$$

Also, we have by Equation (3.8)

$$f(\widetilde{\eta}(\theta,\tau),\psi(\widetilde{\eta}(\theta,\tau),\tau),\theta,\tau_1,\tau_2)=0$$

for every $\theta\in(0,2\pi)$ and $|\tau|$ small. Let a function g such that

$$g(\theta,\tau_1,\tau_2)=(\widetilde{\eta}(\theta,\tau),\psi(\widetilde{\eta}(\theta,\tau),\tau)).$$

Then, we have

$$f(g(\theta,\tau_1,\tau_2),\theta,\tau_1,\tau_2)=0$$

for every $\theta\in(0,2\pi)$. Let a function Ψ such that

$$\Psi(\theta,\tau)=\psi(\widetilde{\eta}(\theta,\tau),\tau).$$

Thus, using the implicit function theorem(see [35]) we have

$$\begin{bmatrix}\frac{\partial\widetilde{\eta}(\theta,\tau_1,\tau_2)}{\partial\theta} & \frac{\partial\widetilde{\eta}(\theta,\tau_1,\tau_2)}{\partial\tau_1} & \frac{\partial\widetilde{\eta}(\theta,\tau_1,\tau_2)}{\partial\tau_2}\\ \\ \frac{\partial\Psi}{\partial\theta} & \frac{\partial\Psi}{\partial\tau_1} & \frac{\partial\Psi}{\partial\tau_2}\end{bmatrix}$$

$$=-\begin{bmatrix}-\sqrt{r^2+\Psi}\sin\widetilde{\eta} & \frac{1}{2\sqrt{r^2+\Psi}}\cos\widetilde{\eta}\\ \\ \sqrt{r^2+\Psi}\cos\widetilde{\eta} & \frac{1}{2\sqrt{r^2+\Psi}}\sin\widetilde{\eta}\end{bmatrix}^{-1}\times$$

$$\times \begin{bmatrix} -\frac{1}{2\sqrt{r^2+\phi(\theta)}}\phi'(\theta)\cos\theta + \sqrt{r^2+\phi(\theta)}\sin\theta & -1 & 0 \\ \\ -\frac{1}{2\sqrt{r^2+\phi(\theta)}}\phi'(\theta)\sin\theta - \sqrt{r^2+\phi(\theta)}\cos\theta & 0 & -1 \end{bmatrix}.$$

Evaluating at $\tau_1 = \tau_2 = 0$, we have

$$-\begin{bmatrix} -\sqrt{r^2+\Psi}\sin\widetilde{\eta} & \frac{1}{2\sqrt{r^2+\Psi}}\cos\widetilde{\eta} \\ \\ \sqrt{r^2+\Psi}\cos\widetilde{\eta} & \frac{1}{2\sqrt{r^2+\Psi}}\sin\widetilde{\eta} \end{bmatrix}^{-1}$$

$$= -\begin{bmatrix} -\sqrt{r^2+\phi(\theta)}\sin\theta & \frac{1}{2\sqrt{r^2+\phi(\theta)}}\cos\theta \\ \\ \sqrt{r^2+\phi(\theta)}\cos\theta & \frac{1}{2\sqrt{r^2+\phi(\theta)}}\sin\theta \end{bmatrix}^{-1}$$

and also

$$-\begin{bmatrix} -\sqrt{r^2+\phi(\theta)}\sin\theta & \frac{1}{2\sqrt{r^2+\phi(\theta)}}\cos\theta \\ \\ \sqrt{r^2+\phi(\theta)}\cos\theta & \frac{1}{2\sqrt{r^2+\phi(\theta)}}\sin\theta \end{bmatrix}^{-1} =$$

$$= 2\begin{bmatrix} \frac{1}{2\sqrt{r^2+\phi(\theta)}}\sin\theta & -\frac{1}{2\sqrt{r^2+\phi(\theta)}}\cos\theta \\ \\ -\sqrt{r^2+\phi(\theta)}\cos\theta & -\sqrt{r^2+\phi(\theta)}\sin\theta \end{bmatrix}.$$

Also, we have by using that $||\phi||_{H^2} = O(\gamma r^7)$

$$\sqrt{r^2+\phi(\theta)}\sin\theta - \frac{1}{2\sqrt{r^2+\phi(\theta)}}\phi'(\theta)\cos\theta = \sqrt{r^2+\phi(\theta)}\sin\theta + O(\gamma r^6)$$

and

$$-\sqrt{r^2+\phi(\theta)}\cos\theta - \frac{1}{2\sqrt{r^2+\phi(\theta)}}\phi'(\theta)\sin\theta = -\sqrt{r^2+\phi(\theta)}\cos\theta + O(\gamma r^6).$$

Thus,

$$\frac{\partial \widetilde{\eta}}{\partial \theta}|_{\tau=0} = 1 + O(\gamma r^5).$$

Also,

$$\frac{\partial \widetilde{\eta}}{\partial \tau_1}|_{\tau=0} = -\frac{\sin\theta}{\sqrt{r^2+\phi(\theta)}}$$

and

$$\frac{\partial \widetilde{\eta}}{\partial \tau_2}|_{\tau=0} = \frac{\cos\theta}{\sqrt{r^2+\phi(\theta)}}.$$

Now, using that $||\phi||_{H^2} = O(\gamma r^7)$ we get that

$$\frac{\partial \Psi}{\partial \theta}|_{\tau=0} = O(\gamma r^7).$$

Also, we have

$$\frac{\partial \Psi}{\partial \tau_1}|_{\tau=0} = 2\sqrt{r^2+\phi(\theta)}\cos\theta$$

and

$$\frac{\partial \Psi}{\partial \tau_2}|_{\tau=0} = 2\sqrt{r^2+\phi(\theta)}\sin\theta.$$

We will now prove that

$$\int_0^{2\pi} \frac{\partial \psi(\tau_1,\tau_2,\theta)}{\partial \tau_1}|_{\tau=0} d\theta = \int_0^{2\pi} \frac{\partial \psi(\tau_1,\tau_2,\theta)}{\partial \tau_2}|_{\tau=0} d\theta = 0.$$

We have

$$\int_0^{2\pi} \psi(\theta,\tau)d\theta = 0$$

because the area of E_{ψ_τ} is πr^2. Also we have that

$$\int_0^{2\pi} \phi(\theta)d\theta = 0$$

and also $\psi(\theta,0) = \phi(\theta)$. This gives that

$$\frac{\partial \int_0^{2\pi} \psi(\theta,\tau)d\theta}{\partial \tau_1}|_{\tau=0} = \frac{\partial \int_0^{2\pi} \psi(\theta,\tau)d\theta}{\partial \tau_2}|_{\tau=0} = 0.$$

Thus, by dominated convergence theorem we get

$$\int_0^{2\pi} \frac{\partial \psi(\tau_1, \tau_2, \theta)}{\partial \tau_1}|_{\tau=0} d\theta = \int_0^{2\pi} \frac{\partial \psi(\tau_1, \tau_2, \theta)}{\partial \tau_2}|_{\tau=0} d\theta = 0.$$

Also, using Lemma 1.2 we can prove that

$$\frac{\partial J(E_{\psi_\tau})}{\partial \tau_1}|_{\tau=0} = \frac{1}{2}\int_0^{2\pi} S(\phi) \frac{\partial \psi(\tau, \theta)}{\partial \tau_1}|_{\tau=0} d\theta$$

and also

$$\frac{\partial J(E_{\psi_\tau})}{\partial \tau_2}|_{\tau=0} = \frac{1}{2}\int_0^{2\pi} S(\phi) \frac{\partial \psi(\tau, \theta)}{\partial \tau_2}|_{\tau=0} d\theta.$$

Now, we will calculate

$$\frac{\partial \psi(\tau, \theta)}{\partial \tau_1}|_{\tau=0}$$

and

$$\frac{\partial \psi(\tau, \theta)}{\partial \tau_2}|_{\tau=0}.$$

Since $\widetilde{\eta}(., \tau) : [0, 2\pi] \to [0, 2\pi]$ is one to one and onto we can invert $\widetilde{\eta}$ and have

$$\widetilde{\eta}(\widetilde{\theta}(\eta, \tau), \tau) = \eta$$

for every $\eta \in [0, 2\pi]$ for some function $\widetilde{\theta} : [0, 2\pi] \to [0, 2\pi]$. Then we have

$$\psi(\eta, \tau) = \Psi(\widetilde{\theta}(\eta, \tau), \tau).$$

We have by the Chain Rule that for $\epsilon_1(\tau_1) = \widetilde{\theta}(\eta, \tau_1)$ and $\epsilon_2(\tau_1) = \tau_1$

$$\frac{\partial \Psi(\widetilde{\theta}(\eta, \tau), \tau)}{\partial \tau_1}|_{\tau=0} = \frac{\partial \Psi}{\partial \epsilon_1}|_{\epsilon_1=\eta} \frac{\partial \epsilon_1}{\partial \tau_1}|_{\tau_1=0} + \frac{\partial \Psi}{\partial \epsilon_2}|_{\epsilon_2=\eta} \frac{\partial \epsilon_2}{\partial \tau_1}|_{\tau_1=0}.$$

Also, we have

$$\frac{\partial \epsilon_2}{\partial \tau_1}|_{\tau_1=0} = 1.$$

Thus,

$$\frac{\partial \Psi(\widetilde{\theta}(\eta, \tau), \tau)}{\partial \tau_1}|_{\tau=0} = \frac{\partial \Psi}{\partial \theta}|_{\theta=\eta} \frac{\partial \widetilde{\theta}(\eta, \tau)}{\partial \tau_1}|_{\tau=0} + \frac{\partial \Psi}{\partial \tau_1}|_{\tau=0}.$$

We have that

$$\frac{\partial \Psi}{\partial \theta}|_{\theta=\eta} = O(\gamma r^7)$$

and also

$$\frac{\partial \Psi}{\partial \tau_1}|_{\tau=0} = 2\sqrt{r^2+\phi(\eta)}\cos\eta.$$

Thus,

$$\frac{\partial \psi}{\partial \tau_1}|_{\tau=0} = O(\gamma r^7)\frac{\partial \widetilde{\theta}(\eta,\tau)}{\partial \tau_1}|_{\tau=0} + 2\sqrt{r^2+\phi(\eta)}\cos\eta.$$

Now we will calculate

$$\frac{\partial \widetilde{\theta}(\eta,\tau)}{\partial \tau_1}|_{\tau=0}.$$

Differentiating the relation

$$\widetilde{\eta}(\widetilde{\theta}(\eta,\tau),\tau) = \eta$$

we get

$$(1+O(\gamma r^5))\frac{\partial \widetilde{\theta}(\eta,\tau)}{\partial \tau_1}|_{\tau=0} - \frac{\sin\eta}{\sqrt{r^2+\phi(\eta)}} = 0.$$

Thus,

$$\frac{\partial \widetilde{\theta}(\eta,\tau)}{\partial \tau_1}|_{\tau=0} = -\frac{\sin\eta}{\sqrt{r^2+\phi(\eta)}(1+O(\gamma r^5))}.$$

Thus, we have

$$\frac{\partial \psi}{\partial \tau_1}|_{\tau=0} = -\frac{O(\gamma r^7)\sin\eta}{\sqrt{r^2+\phi(\eta)}(1+O(\gamma r^5))} + 2\sqrt{r^2+\phi(\eta)}\cos\eta = 2\sqrt{r^2+\phi(\theta)}\cos\theta + O(\gamma r^6) =$$

$$= 2r\cos\theta + O(\gamma r^6).$$

Similarly, we can prove that

$$\frac{\partial \psi}{\partial \tau_2}|_{\tau=0} = 2\sqrt{r^2+\phi(\theta)}\sin\theta + O(\gamma r^6) = 2r\sin\theta + O(\gamma r^6).$$

Now, we have by Equations (3.6) and (3.7) that

$$\frac{1}{2}\int_0^{2\pi} S(\phi)\frac{\partial \psi(\tau,\theta)}{\partial \tau_1}|_{\tau=0}d\theta = \frac{1}{2}\int_0^{2\pi} S(\phi)\frac{\partial \psi(\tau,\theta)}{\partial \tau_2}|_{\tau=0}d\theta = 0.$$

From Proposition 2.3 we have that

$$S(\phi)(\theta) = \lambda + \mu\cos\theta + \nu\sin\theta$$

which gives

$$\int_0^{2\pi} (\lambda + \mu\cos\theta + \nu\sin\theta)\frac{\partial\psi}{\partial\tau_1}|_{\tau=0} d\theta = \int_0^{2\pi} (\lambda + \mu\cos\theta + \nu\sin\theta)\frac{\partial\psi}{\partial\tau_2}|_{\tau=0} d\theta = 0$$

which gives

$$\int_0^{2\pi} (\mu\cos\theta + \nu\sin\theta)\frac{\partial\psi}{\partial\tau_1}|_{\tau=0} d\theta = \int_0^{2\pi} (\mu\cos\theta + \nu\sin\theta)\frac{\partial\psi}{\partial\tau_2}|_{\tau=0} d\theta = 0$$

since

$$\int_0^{2\pi} \frac{\partial\psi(\tau_1,\tau_2,\theta)}{\partial\tau_1}|_{\tau=0} d\theta = \int_0^{2\pi} \frac{\partial\psi(\tau_1,\tau_2,\theta)}{\partial\tau_2}|_{\tau=0} d\theta = 0.$$

Also, we have

$$\frac{\partial\psi}{\partial\tau_1}|_{\tau=0} = 2r\cos\theta + O(\gamma r^6)$$

and

$$\frac{\partial\psi}{\partial\tau_2}|_{\tau=0} = 2r\sin\theta + O(\gamma r^6).$$

Therefore, we have a non-singular linear system which implies that $\mu = \nu = 0$ and the proof is complete. □

Proof of Proposition 2.6

To prove this Proposition we follow [4] and we consider the energy functional $J_{t\Lambda}$ in the new lattice $t\Lambda$. Then, we have $J_{t\Lambda}(tE_\phi) = P_{tD}(tE_\phi) + \frac{\gamma}{2}\int_{tD} |\nabla I_{t\Lambda}(tE_\phi)(x)|^2 dx$ where,

$$-\Delta I_{t\Lambda}(tE_\phi)(x) = \chi_{tE_\phi}(x) - \omega, x \in \mathbb{C}, \int_{tD} I_{t\Lambda}(tE_\phi)(y)dy = 0.$$

By [1] we have $|D| = Im(\bar{a}b)$. Thus $|tD| = t^2|D| = t^2$. Also, we have $P_{tD}(tE_\phi) = tP_D(E_\phi)$. It has been proved in [4] that

$$I_{t\Lambda}(tE_\phi)(ty) = t^2 I_\Lambda(E_\phi)(y)$$

for every $y \in D$. Therefore, by differentiating we get $\nabla I_{t\Lambda}(tE_\phi)(x) = t\nabla I_\Lambda(E_\phi)(\frac{x}{t})$ for every $x \in tD$. Thus,

$$\int_{tD} |\nabla I_{t\Lambda}(tE_\phi)(x)|^2 dx = \int_{tD} t^2 |\nabla I_\Lambda(E_\phi)(\frac{x}{t})|^2 dx = \int_D t^4 |\nabla I_\Lambda(E_\phi)(y)|^2 dy.$$

Therefore, if we set $A = P_D(E_\phi)$ and $B = \frac{\gamma}{2}\int_D |\nabla I_\Lambda(E_\phi)(x)|^2 dx$ and put it all together we get

$$\frac{J_{t\Lambda}(tE_\phi)}{|tD|} = \frac{A}{t} + Bt^2.$$

By taking the derivative of the function $g(t) = \frac{A}{t} + Bt^2$, $t > 0$ we have that g has a global minimizer for $t = t_\Lambda$, where t_Λ is defined in Equation (2.2) and the proof is complete. □

Lemma 3.10. *Denote $J(E_\phi)$ by $J(\phi)$. Then we have*

$$J(\phi) = -\frac{\gamma}{4}\log(\frac{2\pi}{\sqrt{|D|}})\pi r^4 + 2\pi r + \frac{\gamma}{8|D|}\pi^2 r^6 - \frac{\gamma}{4}\pi r^4(\log r - \frac{1}{4}) + \gamma\pi^2 r^4 R(\zeta) +$$
$$+\frac{\gamma\pi r^2}{2}\int_0^{2\pi} H(re^{i\theta})\phi(\theta)d\theta + \frac{1}{4}\int_0^{2\pi} S'(\tau\phi)(\phi)\phi(\theta)d\theta$$

for some $\tau \in (0,1)$.

Here, $R(\zeta) = \frac{H(0)}{2}$ where H is the harmonic function on $(\mathbb{C} - \Lambda) \cup \{0\}$ occuring in the formula of the Green's function G. By [1] $H(0)$ depends only on ζ and D.

Proof. By the Taylor's formula for J we get

$$J(t\phi) = J(0) + t\frac{d}{d\eta}|_{\eta=0} J(\eta\phi) + \frac{t^2}{2}\frac{d^2}{d\eta^2}|_{\eta=\tau} J(\eta\phi)$$

for some $\tau \in (0, t)$.

By setting $t = 1$, we get

$$J(\phi) = J(0) + \frac{d}{d\eta}|_{\eta=0} J(\eta\phi) + \frac{1}{2}\frac{d^2}{d\eta^2}|_{\eta=\tau} J(\eta\phi)$$

for some $\tau \in (0,1)$.

Consider $J(0)$. We have $J(0) = 2\pi r + \frac{\gamma}{2}\int_{B(0,r)}\int_{B(0,r)} G(x-y)dxdy$.

Also,

$$\int_{B(0,r)}\int_{B(0,r)} G(x-y)dxdy = \int_{B(0,r)}\int_{B(0,r)} -\frac{1}{2\pi}\log(\frac{2\pi|x-y|}{\sqrt{|D|}})dxdy +$$

$$+\int_{B(0,r)}\int_{B(0,r)} \frac{|x-y|^2}{4|D|}dxdy + \int_{B(0,r)}\int_{B(0,r)} H(x-y)dxdy.$$

It can be shown that

$$\int_{B(0,r)}\int_{B(0,r)} -\frac{1}{2\pi}\log|x-y|dxdy = -\frac{\pi}{2}r^4(\log r - \frac{1}{4})$$

and

$$\int_{B(0,r)}\int_{B(0,r)} |x-y|^2 dxdy = \pi^2 r^6.$$

Also, since H is harmonic around 0 we get

$$\int_{B(0,r)}\int_{B(0,r)} H(x-y)dxdy = \pi^2 r^4 H(0) = 2\pi^2 r^4 R(\zeta)$$

by the mean value theorem for harmonic functions. Therefore,

$$J(0) = 2\pi r + \frac{\gamma}{2}[-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi r^4] + \frac{\gamma}{2}[-\frac{1}{2}\pi r^4(\log r - \frac{1}{4})] + \frac{\gamma}{8|D|}\pi^2 r^6 +$$

$$+\gamma\pi^2 r^4 R(\zeta).$$

Now,

$$J'(0)(\phi) = \frac{1}{2}\int_0^{2\pi} S(0)(\theta)\phi(\theta)d\theta$$

and

$$S(0)(\theta) = \frac{1}{r} + \gamma\int_{B(0,r)} G(re^{i\theta}-y)dy.$$

Also,

$$\int_{B(0,r)} G(re^{i\theta}-y)dy = \int_{B(0,r)} -\frac{1}{2\pi}\log(\frac{2\pi}{\sqrt{|D|}}|re^{i\theta}-y|)dy + \int_{B(0,r)} \frac{1}{4|D|}|re^{i\theta}-y|^2 dy +$$

$$+\int_{B(0,r)} H(re^{i\theta}-y)dy.$$

Thus,

$$S(0)(\theta) = C + \pi r^2 H(re^{i\theta}),$$

where C is a number which does not depend on θ. Therefore,

$$J'(0)(\phi) = \frac{\gamma \pi r^2}{2} \int_0^{2\pi} H(re^{i\theta})\phi(\theta)d\theta,$$

because $\int_0^{2\pi} \phi(\theta)d\theta = 0$. Now, can prove easily that

$$\frac{d}{d\eta} J(\eta\phi) = J'(\eta\phi)(\phi).$$

Also,

$$\frac{d}{du}|_{u=0} J(\eta\phi + u\phi) = \frac{1}{2}\int_0^{2\pi} S(\eta\phi)(\theta)\phi(\theta)d\theta.$$

Thus,

$$\frac{d^2}{d\eta^2}|_{\eta=\tau} J(\eta\phi) = \frac{d}{d\eta}|_{\eta=\tau} \frac{1}{2}\int_0^{2\pi} S(\eta\phi)(\theta)\phi(\theta)d\theta = \frac{1}{2}\int_0^{2\pi} \frac{d}{d\eta}|_{\eta=\tau} S(\eta\phi)(\theta)\phi(\theta)d\theta$$

and also

$$\frac{d}{d\eta}|_{\eta=\tau} S(\eta\phi)(\theta) = S'(\tau\phi)(\phi).$$

If we put it all together we have the result. □

Proof of Proposition 2.7

To minimize $J(\phi)$ it is equivalent to minimize

$$\widetilde{J}(\phi) = \frac{J(\phi) + \frac{\gamma}{4}\log(\frac{2\pi}{\sqrt{|D|}})\pi r^4 - 2\pi r - \frac{\gamma}{8|D|}\pi^2 r^6 + \frac{\gamma}{4}\pi r^4(\log r - \frac{1}{4})}{\gamma r^4}.$$

Then, we have

$$\widetilde{J}(\phi) = \pi^2 R(\zeta) + \frac{\pi}{2r^2}\int_0^{2\pi} H(re^{i\theta})\phi(\theta)d\theta + \frac{1}{4\gamma r^4}\int_0^{2\pi} S'(\tau\phi)(\phi)(\theta)\phi(\theta)d\theta$$

by Lemma 3.10. Also, we have

$$|\int_0^{2\pi} H(re^{i\theta})\phi(\theta)d\theta| = |\int_0^{2\pi}(H(0) + O(r))\phi(\theta)d\theta| = |\int_0^{2\pi} O(r)\phi(\theta)d\theta| \leq ||O(r)||_{L^2}||\phi||_{L^2}$$

$$\leq Cr||\phi||_{L^2} \leq C\gamma r^8.$$

Therefore, finally

$$\frac{\pi}{2r^2}\int_0^{2\pi} H(re^{i\theta})\phi(\theta)d\theta = O(\gamma r^6) = O(r^3).$$

Consider the term

$$\frac{1}{4\gamma r^4}\int_0^{2\pi} S'(\tau\phi)(\phi)(\theta)\phi(\theta)d\theta.$$

We have

$$|\frac{1}{4\gamma r^4}\int_0^{2\pi} S'(\tau\phi)(\phi)(\theta)\phi(\theta)d\theta| \leq \frac{1}{4\gamma r^4}||S'(\tau\phi)(\phi)||_{L^2}||\phi||_{L^2} \leq \frac{C\gamma r^7}{\gamma r^4}||S'(\tau\phi)(\phi)||_{L^2} =$$

$$= Cr^3||S'(\tau\phi)(\phi)||_{L^2}.$$

I will show that $||S'(\tau\phi)(\phi)||_{L^2} = O(r)$. In the proof of Proposition 2.3 we can find that

$$||S'(\phi_2)(\phi_1-\phi_2) - S'(0)(\phi_1-\phi_2)||_{L^2} \leq C(\frac{1}{r^5}+\frac{\gamma}{r^2})||\phi_2||_{H^2}||\phi_1-\phi_2||_{H^2}$$

for $\phi_1, \phi_2 \in D(T) = \{u \in H^2(S^1) \cap \{1, cos\theta, sin\theta\}^{\perp} : ||u||_{H^2} \leq M\gamma r^7\}$. Thus, for $\phi_1 = 0$ and $\phi_2 = \tau\phi$ we get

$$||S'(\tau\phi)(\tau\phi) - S'(0)(\tau\phi)||_{L^2} \leq C(\frac{1}{r^5}+\frac{\gamma}{r^2})\tau^2||\phi||^2_{H^2}.$$

Therefore,

$$||S'(\tau\phi)(\phi) - S'(0)(\phi)||_{L^2} \leq C(\frac{1}{r^5}+\frac{\gamma}{r^2})||\phi||^2_{H^2} \leq C(\frac{1}{r^5}+\frac{\gamma}{r^2})M^2\gamma^2 r^{14} =$$

$$= C(\gamma r^3)(\gamma r^6) + C(\gamma r^3)^2\gamma r^6.$$

Here, we use that $\gamma r^3 = O(1)$. Thus, we get

$$||S'(\tau\phi)(\phi) - S'(0)(\phi)||_{L^2} = O(\gamma r^6) = O(r^3).$$

Thus, to prove that $||S'(\tau\phi)(\phi)||_{L^2} = O(r)$ it suffices to show that $||S'(0)(\phi)||_{L^2} = O(r)$. We have that $S'(0)(\phi) = L_1(\phi) + B'(0)(\phi)$. Also, by 3.2 we get

$$||B'(0)(\phi)||_{L^2} \leq C\gamma r||\phi||_{L^2} \leq C\gamma r M\gamma r^7 = C\gamma^2 r^8 = O(r^2).$$

Consider now

$$L_1(\phi) = -\frac{\phi'' + \phi}{2r^3} - \frac{\gamma\phi}{4} - \frac{\gamma}{8\pi}\int_0^{2\pi} \phi(\omega)\log(1-\cos(\theta-\omega))d\omega.$$

We can prove easily that the first two parts of the equation are of order $O(r)$.

Now,

$$||\int_0^{2\pi} \phi(\omega)\log(1-\cos(\theta-\omega))d\omega||_{L^2}^2 = \int_0^{2\pi} |\int_0^{2\pi} \phi(\omega)\log(1-\cos(\theta-\omega))d\omega|^2 d\theta \leq$$

$$\leq C\int_0^{2\pi}\int_0^{2\pi} |\phi(\omega)|^2|\log(1-\cos(\theta-\omega))|^2 d\omega d\theta \leq$$

$$\leq C||\phi||_{L^\infty}^2 \int_0^{2\pi}\int_0^{2\pi} |\log(1-\cos(\theta-\omega))|^2 d\omega d\theta =$$

$$= C||\phi||_{L^\infty}^2 \leq C||\phi||_{H^2}^2.$$

Therefore,

$$||\int_0^{2\pi} \phi(\omega)\log(1-\cos(\theta-\omega))d\omega||_{L^2} \leq C\gamma r^7.$$

Thus,

$$||\frac{\gamma}{8\pi}\int_0^{2\pi} \phi(\omega)\log(1-\cos(\theta-\omega))d\omega||_{L^2} \leq C\gamma^2 r^7 = O(r).$$

Therefore, we proved that $||L_1(\phi)||_{L^2} = O(r)$, which proves that $||S'(0)(\phi)||_{L^2} = O(r)$. Thus $||S'(\tau\phi)(\phi)||_{L^2} = O(r)$. If we put it all together we can see that

$$\frac{1}{4\gamma r^4}\int_0^{2\pi} S'(\tau\phi)(\phi)(\theta)\phi(\theta)d\theta = O(r^4).$$

Finally we get

$$\widetilde{J}(\phi) = \pi^2 R(\zeta) + O(r^3).$$

To finish the proof, we have that $R(\zeta) > R(e^{\frac{i\pi}{3}})$ for $\zeta \neq e^{\frac{i\pi}{3}}$ by [1]. Therefore, for every $\zeta \neq e^{\frac{i\pi}{3}}$ $J(\phi,\zeta) > J(\phi, e^{\frac{i\pi}{3}})$ for r small enough. □

Chapter 4

Proof of the main results of the second problem

4.1 Important Lemmas

We have seen in the introduction of this dissertation that for the second problem the Euler-Lagrange equations are $T_k(\varphi)(\theta) = \lambda$ for some constant λ, where

$$T(\varphi)(\theta) = (T_1(\varphi)(\theta), ..., T_K(\varphi)(\theta))$$
$$T_k(\varphi)(\theta) = W_k(\varphi_k)(\theta) + \sum_{i=1}^{K} \gamma \int_{E_{\varphi_i}} G(\xi_k + \sqrt{\rho_k^2 + \varphi_k(\theta)}e^{i\theta} - y)dy,$$

where $k \in \{1, ..., K\}$. Here, E_φ is defined in the Equation (1.10) and T is defined in the Equation (1.11).

Thus, we can follow [2] and write

$$T_k(\varphi)(\theta) = W_k(\varphi_k)(\theta) + I_k(\varphi_k)(\theta) + A_k(\varphi_k)(\theta) + B_k(\varphi_k)(\theta) + \sum_{l \neq k} c_{k,l}(\varphi_k, \varphi_l)(\theta), \quad (4.1)$$

where $l \in \{1, ..., K\}$ and $l \neq k$. Denote by R the regular part of the Green's function. Here, we have

$$I_k(\varphi_k)(\theta) = -\frac{\gamma \log \rho_k}{2\pi}|E_{\varphi_k}|$$

$$A_k(\varphi_k)(\theta) = -\frac{\gamma}{2\pi}\int_{E_{(0,\varphi_k)}} \log|\sqrt{1+\frac{\varphi_k(\theta)}{\rho_k^2}}e^{i\theta} - \frac{y}{\rho_k}|dy$$

and

$$B_k(\varphi_k)(\theta) = -\frac{\gamma}{2\pi}\int_{E_{(0,\varphi_k)}} R(\sqrt{\rho_k^2 + \varphi_k(\theta)}e^{i\theta} - y)dy,$$

where $E_{(0,\varphi_k)} = \{we^{i\theta} : \theta \in S^1, w \in [0, \sqrt{\rho_k^2 + \varphi_k(\theta)})\}$ and also for $l \neq k$ we have

$$c_{k,l}(\varphi_k, \varphi_l)(\theta) = \gamma \int_{E_{\varphi_l}} G(\xi_k + \sqrt{\rho_k^2 + \varphi_k(\theta)}e^{i\theta} - y)dy.$$

Also, the operator W_k is defined in Equation (1.12). Here, we note that the operators I_k, H_k, A_k and B_k do not depend on the centers $\xi_1, ..., \xi_K$. Also, these operators depend only on φ_k. The term $c_{k,l}$ is the interaction term. Also, by the first problem we get

$$W_k'(0)(u_k) = -\frac{u_k'' + u_k}{2\rho_k^3} \tag{4.2}$$

and also calculations show

$$I_k'(0)(u_k)(\theta) = -\gamma\frac{\log \rho_k}{4\pi}\int_0^{2\pi} u_k(\omega)d\omega \tag{4.3}$$

$$A_k'(0)(u_k)(\theta) = -\frac{\gamma}{4\pi}\int_0^{2\pi} u_k(\omega)\log|e^{i\theta} - e^{i\omega}|d\omega - \frac{\gamma u_k(\theta)}{4} \tag{4.4}$$

$$B_k'(0)(u_k)(\theta) = \frac{\gamma}{2}\int_0^{2\pi} u_k(\omega)R(\rho_k e^{i\theta} - \rho_k e^{i\omega})d\omega +$$

$$+\frac{\gamma u_k(\theta)}{2\rho_k}\int_{B(0,\rho_k)} \nabla R(\rho_k e^{i\theta} - y)\cdot e^{i\theta}dy. \tag{4.5}$$

Also, we have

$$c_{k,l}'(0,0)(u_k, u_l)(\theta) = \frac{\gamma}{2}\int_0^{2\pi} u_l(\omega)G(\xi_k - \xi_l + \rho_k e^{i\theta} - \rho_l e^{i\omega})d\omega +$$

$$+\frac{\gamma u_k(\theta)}{2\rho_k}\int_{B(\xi_l,\rho_l)}\nabla G(\xi_k+\rho_k e^{i\theta}-y)\cdot e^{i\theta}dy.$$

Let

$$X* = \{\varphi = (\varphi_1, ..., \varphi_K) : \varphi_k \in H^2(S^1), \sum_{i=1}^{K}\bar{\varphi}_i = 0\}$$

and

$$Y* = \{\varphi = (\varphi_1, ..., \varphi_K) : \varphi_k \in L^2(S^1), \sum_{i=1}^{K}\bar{\varphi}_i = 0\},$$

where $\bar{\varphi_k} = \frac{1}{2\pi}\int_0^{2\pi}\varphi_k(\theta)d\theta$. Now, let $L_{1,k}(u)(\theta) = W_k'(0)(u_k)(\theta)+A_k'(0)(u_k)(\theta)+I_k'(0)(u_k)(\theta)+l_1(u)$. The operator $l_1(u)$ is independent of k and θ. It is chosen in order for L_1 to map $X*$ to $Y*$. Let Π be the orthogonal projection from $L^2(S^1, \mathbb{R}^K)$ to Y. We will prove the following lemma.

Lemma 4.1. *The operator* $\Pi L_1 : X \to Y$ *has eigenvalues* $\lambda_{k,n} = \frac{n^2-1}{2\rho_k^3} + \frac{\gamma}{4|n|} - \frac{\gamma}{4}$ *with eigenfunctions* $\cos n\theta e_k$, $\sin n\theta e_k$*, where* $e = (e_1, .., e_K)$ *is the standard basis for* $\mathbb{R}^K$ *and* $n \in \mathbb{Z}$ *with* $n \neq 0$.

Proof. We will prove the result for $n \geq 1$. The other case can be proved similarly. We have, by following [2] that $\Pi L_1(u) = L_1(u) - l_1(u) - I_k'(0)(u_k)$ on X. By considering Fourier series we get $\widehat{u_k}(n) = \frac{1}{2\pi}\int_0^{2\pi}u_k(\theta)e^{-in\theta}d\theta$. We claim that $\Pi\widehat{L_{1,k}(u_k)}(n) = \lambda_{k,n}\widehat{u_k}(n)$. To prove this claim we consider

$$\Pi\widehat{L_{1,k}(u_k)}(n) = \frac{1}{2\pi}\int_0^{2\pi}-\frac{u_k''(\theta)+u_k(\theta)}{2\rho_k^3}e^{-in\theta}d\theta + \frac{1}{2\pi}\int_0^{2\pi}-\frac{\gamma}{4}u_k(\theta)e^{-in\theta}d\theta +$$

$$+\frac{1}{2\pi}\int_0^{2\pi}-\frac{\gamma}{4\pi}\int_0^{2\pi}u_k(\omega)\log|e^{i\theta}-e^{i\omega}|e^{-in\theta}d\omega d\theta.$$

Consider the first term in the sum. Using integration by parts we can show that

$$\frac{1}{2\pi}\int_0^{2\pi}-\frac{u_k''(\theta)+u_k(\theta)}{2\rho_k^3}e^{-in\theta}d\theta = \frac{n^2-1}{2\rho_k^3}\widehat{u_k}(n).$$

Also, we can see that

$$\frac{1}{2\pi}\int_0^{2\pi}-\frac{\gamma}{4}u_k(\theta)e^{-in\theta}d\theta = -\frac{\gamma}{4}\widehat{u_k}(n).$$

Now, we consider the last term on the sum. We get

$$-\frac{\gamma}{4\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u_k(\omega)\log|e^{i\theta}-e^{i\omega}|e^{-in\theta}d\theta d\omega =$$
$$-\frac{\gamma}{4\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u_k(\omega)\log|1-e^{i(\omega-\theta)}|e^{-in\theta}d\theta d\omega.$$

Here, we will use that

$$\log|1-e^{i\theta}| = -\sum_{m=1}^{\infty}\frac{\cos(m\theta)}{m}$$

and also that $\cos(a-b)=\cos a\cos b+\sin a\sin b$. Thus, we can prove that

$$-\frac{\gamma}{4\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u_k(\omega)\log|1-e^{i(\omega-\theta)}|e^{-in\theta}d\omega d\theta =$$
$$=\frac{\gamma}{4\pi}\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\int_0^{2\pi}\cos(m\theta)e^{-in\theta}d\theta\int_0^{2\pi}u_k(\omega)\cos(m\omega)d\omega$$
$$+\frac{\gamma}{4\pi}\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_0^{2\pi}\frac{1}{m}\sin(m\theta)e^{-in\theta}d\theta\int_0^{2\pi}u_k(\omega)\sin(m\omega)d\omega.$$

Thus by orthogonality of $\sin$ and $\cos$ we get

$$-\frac{\gamma}{4\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u_k(\omega)\log|1-e^{i(\omega-\theta)}|e^{-in\theta}d\omega d\theta = \frac{\gamma}{4}\frac{1}{n}\widehat{u_k}(n)$$

and the proof of the Claim is complete. Thus, we have proved that

$$\Pi\widehat{L_{1,k}(u_k)}(n) = \lambda_{k,n}\widehat{u_k}(n).$$

Thus,

$$\Pi L_{1,k}(\cos n\theta) = \sum_{m=-\infty}^{\infty}\Pi\widehat{L_{1,k}(v)}(m)e^{im\theta} = \sum_{m=-\infty}^{\infty}\lambda_{k,m}\widehat{v}(m)e^{im\theta},$$

where $v=\cos n\theta$. Also, we have $\widehat{v}(m)=\frac{1}{2}$ when $m=\pm n$ and 0 otherwise. Therefore, we can see that

$$\Pi L_{1,k}(\cos n\theta) = \lambda_{k,n}\cos n\theta.$$

By arguing similarly we can prove the Lemma. □

Now, we define an operator L_2 as $L_2(u) = T'(0)(u) - L_1(u) + l_1(u) + l_2(u)$.

$l_2(u)$ is included so that L_2 maps $X*$ to $Y*$ and it is independent of θ.

Lemma 4.2. *There exists $C > 0$ independent of ξ_i, ρ_i and γ such that*

$||L_2(u)||_{L^2} \leq \frac{C}{\rho^2}||u||_{L^2}$ *for every $u \in X$.*

Proof. Let $L_{2,k}$ be the k-th component of L_2. By following [2] and by Equation (4.5) we have

$$L_{2,k}(u)(\theta) = \frac{\gamma}{2}\int_0^{2\pi} u_k(\omega)(R(\rho_k e^{i\theta} - \rho_k e^{i\omega}) - R(0))d\omega +$$

$$+\frac{\gamma u_k(\theta)}{2\rho_k}\int_{B(\xi_k,\rho_k)} \nabla R(\xi_k + \rho_k e^{i\theta} - y)\cdot e^{i\theta}dy +$$

$$+\sum_{l\neq k}\frac{\gamma}{2}\int_0^{2\pi} u_l(\omega)(G(\xi_k + \rho_k e^{i\theta} - \xi_l - \rho_l e^{i\omega}) - G(\xi_k - \xi_l))d\omega +$$

$$+\sum_{l\neq k}\frac{\gamma u_k(\theta)}{2\rho_k}\int_{B(\xi_l,\rho_l)} \nabla G(\xi_k + \rho_k e^{i\theta} - y)\cdot e^{i\theta}dy + l_2(u).$$

We observe that by Taylor's expansion we have

$$R(\rho_k e^{i\theta} - \rho_k e^{i\omega}) - R(0) = O(\rho)$$

and also

$$G(\xi_k - \xi_l + \rho_k e^{i\theta} - \rho_l e^{i\omega}) - G(\xi_k - \xi_l) = O(\rho).$$

Thus,

$$||\frac{\gamma}{2}\int_0^{2\pi} u_l(\omega)[G(\xi_k - \xi_l + \rho_k e^{i\theta} - \rho_l e^{i\omega}) - G(\xi_k - \xi_l)]d\omega||_{L^2} \leq$$

$$\leq C\gamma||O(\rho)||_{L^\infty}||\int_0^{2\pi} u_l(\omega)d\omega||_{L^2} \leq C\gamma\rho||u||_{L^2}.$$

Also,

$$||\frac{\gamma}{2}\int_0^{2\pi} u_k(\omega)(R(\rho_k e^{i\theta} - \rho_k e^{i\omega}) - R(0))d\omega||_{L^2} \leq C\gamma\rho||u||_{L^2}$$

$$||\frac{\gamma u_k(\theta)}{2\rho_k}\int_{B(\xi_k,\rho_k)} \nabla R(\xi_k + \rho_k e^{i\theta} - y)\cdot e^{i\theta}dy||_{L^2} \leq C\gamma\rho||u_k||_{L^2}$$

since R is smooth near 0 and

$$||\frac{\gamma u_k(\theta)}{2\rho_k}\int_{B(\xi_l,\rho_l)}\nabla G(\xi_k+\rho_k e^{i\theta}-y)\cdot e^{i\theta}dy||_{L^2}\leq C\gamma\rho||u||_{L^2}$$

by the smoothness of G, because $\overline{B(\xi_k,\rho_k)},\overline{B(\xi_l,\rho_l)}$ are disjoint. To finish the proof we can see that the condition $\sum_{k=1}^{K}\bar{L_{2,k}}=0$ implies that $|l_2(u)|\leq C\gamma\rho$. Thus, we have proved that

$$||L_2(u)||_{L^2}\leq C\gamma\rho||u||_{L^2}\leq\frac{C}{\rho^2}||u||_{L^2}$$

because $\gamma\rho^3=O(1)$. □

Lemma 4.3. *There exists a constant $C>0$ such that $||u||_{H^2}\leq C\rho^3||\Pi T'(0)(u)||_{L^2}$ for all $u\in X$.*

Proof. By Lemma 4.1 and Equations (4.1), (4.2), (4.3) and (4.4), we have that $\lambda_{k,n}$ are eigenvalues of ΠL_1, where $\lambda_{k,n}=\frac{n^2-1}{2\rho_k^3}+\gamma(\frac{1}{4|n|}-\frac{1}{4})$. Thus, since $|\gamma\rho^3-2n(n+1)|>\epsilon n^2$ for $n\in\mathbb{N}$,$n\geq 2$ we can see that

$$\frac{|\lambda_{k,n}|}{n^2}\geq\frac{\epsilon(n-1)}{4\rho_k^3 n}\geq\frac{C}{\rho^3}.$$

Now, using Fourier series we can write

$$u_k(\theta)=\sum_{n=2}^{\infty}A_{k,n}\cos(n\theta)+B_{k,n}\sin(n\theta).$$

Therefore, we get

$$||u_k||_{L^2}^2=\pi\sum_{n=2}^{\infty}A_{k,n}^2+B_{k,n}^2$$

$$||u_k'||_{L^2}^2=\pi\sum_{n=2}^{\infty}n^2A_{k,n}+n^2B_{k,n}$$

and

$$||u_k''||_{L^2}^2=\pi\sum_{n=2}^{\infty}n^4(A_{k,n}^2+B_{k,n}^2).$$

Thus,

$$||u_k||_{H^2}^2=\pi\sum_{n=2}^{\infty}A_{k,n}^2(1+n^2+n^4)+B_{k,n}^2(1+n^2+n^4)\leq C\sum_{n=2}^{\infty}n^4(A_{k,n}^2+B_{k,n}^2).$$

Also,

$$L_{1,k}[u_k] = L_{1,k}\big(\sum_{n=2}^{\infty}\big(A_{k,n}\cos n\theta + B_{k,n}\sin n\theta\big)\big) =$$

$$= \sum_{n=2}^{\infty} A_{k,n}L_{1,k}(\cos n\theta) + B_{k,n}L_{1,k}(\sin n\theta) = \sum_{n=2}^{\infty} A_{k,n}\lambda_{k,n}\cos n\theta + B_{k,n}\lambda_{k,n}\sin n\theta.$$

Thus,

$$||\Pi L_{1,k}(u_k)||_{L^2}^2 = \pi\sum_{n=2}^{\infty}\lambda_{k,n}^2(A_{k,n}^2 + B_{k,n}^2)$$

and

$$\lambda_{k,n}^2 \geq C\frac{n^4}{\rho^6}.$$

Thus,

$$||\Pi L_{1,k}(u_k)||_{L^2}^2 \geq \frac{C}{\rho^6}\sum_{n=2}^{\infty} n^4(A_{k,n}^2 + B_{k,n}^2) \geq \frac{C}{\rho^6}||u_k||_{H^2}^2.$$

Thus,

$$||u_k||_{H^2} \leq C\rho^3||\Pi L_{1,k}(u_k)||_{L^2}$$

which proves that

$$||u||_{H^2} \leq C\rho^3||\Pi L_1(u)||_{L^2}.$$

Now, to finish the proof we have

$$||\Pi T'(0)(u)||_{L^2} \geq ||\Pi L_1(u)||_{L^2} - ||\Pi L_2(u)||_{L^2} \geq \frac{C}{\rho^3}||u||_{H^2} - ||L_2(u)||_{L^2} \geq$$

$$\geq \frac{C}{\rho^3}||u||_{H^2} - \frac{C}{\rho^2}||u||_{H^2} = \frac{C_1 - C_2\rho}{\rho^3}||u||_{H^2}.$$

By taking ρ small enough we have $C_1 - C_2\rho \geq C > 0$. It is easy to see now that

$$||u||_{H^2} \leq C\rho^3||\Pi T'(0)(u)||_{L^2},$$

which proves the Lemma. □

Remark 4.4. *By Lemma 4.3 it is evident that $\Pi T'(0)$ is one to one from X to Y. To prove Proposition 2.8 it suffices to prove that $\Pi T'(0)$ is onto Y. It is known that $\Pi T'(0)$ is self-adjoint and also self-adjoint operators have closed graph. We will use the self-adjointness of $\Pi T'(0)$ to prove that it is onto Y.*

Lemma 4.5. *Consider the operator $\Pi T'(0) : X \to Y$.Then the range $\Pi T'(0)(X)$ is closed in Y.*

Proof. We argue as in [2]. Suppose that we have a sequence u_n such that $\Pi T'(0)(u_n) \to \omega$ in Y. Then $\Pi T'(0)(u_n)$ is a Cauchy sequence. Thus,

$$||\Pi T'(0)(u_n) - \Pi T'(0)(u_m)||_{L^2} \to 0,$$

when $n, m \to \infty$. Thus, by Lemma 4.3 we get

$$||u_n - u_m||_{H^2} \to 0$$

when $n, m \to \infty$. Thus, u_n is a Cauchy sequence in H^2 and since H^2 is a Banach space we have that there exists u such that $u_n \to u$ in H^2. Then we have

$$(u_n, \Pi T'(0)(u_n)) \to (u, \omega).$$

Therefore, since $\Pi T'(0)$ has closed graph we get

$$\omega = \Pi T'(0)(u)$$

This shows that the range of $\Pi T'(0)$ is closed and proves the Lemma. □

4.2 Proof of the main results of the second problem

In this section we will prove the main results of the second problem as presented in chapter 2.

Proof of Proposition 2.8

Let $z \in Y$ with $z \in R(\Pi T'(0))^{\perp}$, where $R(\Pi T'(0))$ is the range of the operator $\Pi T'(0)$. We have $< \Pi T'(0)(x), z >= 0$ for every $x \in X$. Then, we have that z is in the domain of the adjoint of $\Pi T'(0)$ and since $\Pi T'(0)$ is self adjoint we have $z \in X$.

Thus, $< x, \Pi T'(0)(z) >= 0$ for every $x \in X$. Also X is dense in Y which gives that $\Pi T'(0)(z) = 0$. Also, we have already proved that $\Pi T'(0)$ is one to one, which gives $z = 0$. Thus, $Y \cap R(T'(0))^{\perp} = \{0\}$. Now, we have $Y = R(\Pi T'(0)) \oplus R(\Pi T'(0))^{\perp}$, because $R(\Pi T'(0))$ is closed in Y by Lemma 4.5. Now it is easy to see that $Y = R(\Pi T'(0))$ and by Remark 4.4 the proof is complete. □

Lemma 4.6. *(1) There exists a constant $C > 0$ independent of ρ and γ such that for $\varphi \in X$ with $||\varphi||_{H^2} \leq c\rho^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have*

$$||W_k''(\varphi_k)(u_k, v_k)||_{L^2} \leq \frac{C}{\rho^5}||u_k||_{H^2}||v_k||_{H^2}.$$

(2) There exists a constant $C > 0$ independent of ρ and γ such that for $\varphi \in X$ with $||\varphi||_{H^2} \leq c\rho^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have

$$||A_k''(\varphi_k)(u_k, v_k)||_{L^2} \leq \frac{C\gamma}{\rho^2}||u_k||_{H^2}||v_k||_{H^2}.$$

(3) There exists a constant $C > 0$ independent of ρ and γ such that for $\varphi \in X$ with $||\varphi||_{H^2} \leq c\rho^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have

$$||B_k''(\varphi_k)(u_k, v_k)||_{L^2} \leq \frac{C\gamma}{\rho}||u_k||_{H^2}||v_k||_{H^2}.$$

(4) There exists a constant $C > 0$ independent of ρ and γ such that for $\varphi \in X$ with $||\varphi||_{H^2} \leq c\rho^2$ where c is sufficiently small and $u, v \in H^2(S^1)$ we have

$$||c_{k,l}''(\varphi_k, \varphi_l)(u_k, u_l)(v_k, v_l)||_{L^2} \leq \frac{C}{\rho^4}(||u_k||_{H^2} + ||u_l||_{H^2})(||v_k||_{H^2} + ||v_l||_{H^2}).$$

Note that $I'' = 0$.

Proof. The formulas (1) to (3) have been proved in the first problem (Lemma 3.7). Also, calculations show that

$$c''_{k,l}(\varphi_k,\varphi_l)(u_k,u_l)(v_k,v_l)(\theta) =$$

$$= \frac{\gamma v_k(\theta)}{4\sqrt{\rho_k^2+\varphi_k(\theta)}}\int_0^{2\pi} u_l(\omega)\nabla G(\xi_k-\xi_l+\sqrt{\rho_k^2+\varphi_k(\theta)}e^{i\theta}-\sqrt{\rho_l^2+\varphi_l(\omega)}e^{i\omega})\cdot e^{i\theta}d\omega+$$

$$+\frac{\gamma u_k(\theta)}{4\sqrt{\rho_k^2+\varphi_k(\theta)}}\int_0^{2\pi} v_l(\omega)\nabla G(\xi_k-\xi_l+\sqrt{\rho_k^2+\varphi_k(\theta)}e^{i\theta}-\sqrt{\rho_l^2+\varphi_l(\omega)}e^{i\omega})\cdot e^{i\theta}d\omega$$

$$-\frac{\gamma}{4}\int_0^{2\pi}\frac{u_l(\omega)v_l(\omega)}{\sqrt{\rho_l^2+\varphi_l(\omega)}}\nabla G(\xi_k-\xi_l+\sqrt{\rho_k^2+\varphi_k(\theta)}e^{i\theta}-\sqrt{\rho_l^2+\varphi_l(\omega)}e^{i\omega})\cdot e^{i\omega}d\omega+$$

$$+\frac{\gamma u_k(\theta)v_k(\theta)}{4(\rho_k^2+\varphi_k(\theta))}\int_{E_{\varphi_l}}[(\nabla\nabla_1 G)(\xi_k+\sqrt{\rho_k^2+\varphi_k(\theta)}e^{i\theta}-y)\cos\theta+$$

$$+(\nabla\nabla_2 G)(\xi_k+\sqrt{\rho_k^2}+\varphi_k(\theta)e^{i\theta}-y)\sin\theta]\cdot e^{i\theta}dy$$

$$-\frac{\gamma u_k(\theta)v_k(\theta)}{4(\rho_k^2+\varphi_k(\theta))^{\frac{3}{2}}}\int_{E_{\varphi_l}}\nabla G(\xi_k+\sqrt{\rho_k^2+\varphi_k(\theta)}e^{i\theta}-y)\cdot e^{i\theta}dy.$$

Now, observe that since $\overline{E_{\varphi_k}}$ and $\overline{E_{\varphi_l}}$ have positive distance and also they are compactly contained in D.Thus, we have that $G(x-y)$ is smooth for $x\in\overline{E_{\varphi_k}}$ and $y\in\overline{E_{\varphi_l}}$. Therefore ∇G is bounded. Thus,

$$|c''_{k,l}(\varphi_k,\varphi_l)(u_k,u_l)(v_k,v_l)(\theta)|\le C\gamma||v_k||_\infty||||u_l||_\infty||\frac{1}{\sqrt{\rho_k^2+\varphi_k(\theta)}}||_\infty+$$

$$+C\gamma||u_k||_\infty||v_l|_\infty||\frac{1}{\sqrt{\rho_k^2+\varphi_k(\theta)}}||_\infty+C\gamma||u_l||_\infty||v_l||_\infty||\frac{1}{\sqrt{\rho_l^2+\varphi_l}}||_\infty+$$

$$+C\gamma\rho_l^2||u_k||_\infty||v_k||_\infty||\frac{1}{\rho_k^2+\varphi_k(\theta)}||_\infty+C\gamma\rho_l^2||u_k||_\infty||v_k||_\infty||\frac{1}{(\rho_k^2+\varphi_k(\theta))^{\frac{3}{2}}}||_\infty$$

But, $||\frac{1}{\sqrt{\rho_i^2+\varphi_i}}||_\infty\le\frac{C}{\rho}$ since $||\varphi||_{H^2}\le c\rho^2$ and c is sufficiently small. Also,

$$\rho_l^2||\frac{1}{\rho_k^2+\varphi_k(\theta)}||_\infty=O(1)$$

and

$$\rho_l^2 || \frac{1}{(\rho_k^2 + \varphi_k(\theta))^{\frac{3}{2}}} ||_\infty = O(\frac{1}{\rho}).$$

By combining the previous results and using that $\gamma\rho^3 = O(1)$, $||\psi||_\infty \leq C||\psi||_{H^2}$ for every ψ and $||c''_{k,l}(\varphi_k, \varphi_l)(u_k, u_l)(v_k, v_l)(\theta)||_{L^2} \leq C||c''_{k,l}(\varphi_k, \varphi_l)(u_k, u_l)(v_k, v_l)(\theta)||_\infty$ we proved the result.

□

Lemma 4.7. *It holds that* $||\Pi T(0)||_{L^2} = O(\gamma\rho^4)$.

Proof. We have as in [2]

$$T_k(0)(\theta) = \frac{1}{\rho_k} + \sum_{l=1}^{K} \gamma \int_{B(\xi_l, \rho_l)} G(\xi_k + \rho_k e^{i\theta} - y) dy. \tag{4.6}$$

We consider two cases. Firslty, $l = k$ and secondly $l \neq k$. If $l = k$ we have

$$\gamma \int_{B(\xi_k, \rho_k)} G(\xi_k + \rho_k e^{i\theta} - y) dy = \gamma \int_{B(0, \rho_k)} G(\rho_k e^{i\theta} - y) dy =$$

$$= \gamma \int_{B(0,\rho_k)} -\frac{1}{2\pi} \log \frac{2\pi}{\sqrt{|D|}} |\rho_k e^{i\theta} - y| dy +$$

$$+\gamma \int_{B(0,\rho_k)} \frac{|\rho_k e^{i\theta} - y|^2}{4|D|} + \gamma \int_{B(0,\rho_k)} H(\rho_k e^{i\theta} - y) dy.$$

By [1]

$$\int_{B(0,\rho_k)} \log |\rho_k e^{i\theta} - y| dy = \rho_k^2 \log \rho_k.$$

Also, by [1]

$$\int_{B(0,\rho_k)} \frac{|\rho_k e^{i\theta} - y|^2}{4|D|} = \int_{B(0,\rho_k)} \frac{|\rho_k e^{i\theta}|^2 + |y|^2}{4|D|} = C_1$$

where C_1 is independent of θ. Now,

$$\int_{B(0,\rho_k)} H(\rho_k e^{i\theta} - y) dy = \pi \rho_k^2 H(\rho_k e^{i\theta})$$

because H is harmonic near 0. Thus, finally

$$\gamma \int_{B(0,\rho_k)} G(\rho_k e^{i\theta} - y) dy = C + \gamma\pi\rho_k^2 H(\rho_k e^{i\theta}),$$

where C is independent of θ. Now, by Taylor's expansion we write $H(\rho_k e^{i\theta}) = H(0) + \nabla H(0) \cdot e^{i\theta}\rho_k + O(\rho_k^2)$. Thus, finally

$$\gamma \int_{B(0,\rho_k)} G(\rho_k e^{i\theta} - y)dy = C + \gamma\pi\rho_k^2(H(0) + \nabla H(0) \cdot e^{i\theta}\rho_k + O(\rho_k^2)) =$$

$$= M + (M_1, M_2) \cdot e^{i\theta} + O(\gamma\rho^4),$$

where M, M_1, M_2 are independent of θ.

Now we consider the case $l \neq k$. We have

$$\gamma \int_{B(0,\rho_l)} G(\xi_k - \xi_l + \rho_k e^{i\theta} - y)dy = C - \frac{\gamma}{2\pi} \int_{B(0,\rho_l)} \log|\xi_k - \xi_l + \rho_k e^{i\theta} - y|dy +$$

$$+\gamma \int_{B(0,\rho_l)} |\xi_k - \xi_l + \rho_k e^{i\theta} - y|^2 \frac{1}{4|D|}dy + \gamma \int_{B(0,\rho_l)} H(\xi_k - \xi_l + \rho_k e^{i\theta} - y)dy.$$

Consider the first term in this sum. We have

$$-\frac{\gamma}{2\pi} \int_{B(0,\rho_l)} \log|\xi_k - \xi_l + \rho_k e^{i\theta} - y|dy = -\frac{\gamma}{2\pi}\pi\rho_l^2 \log|\xi_k - \xi_l + \rho_k e^{i\theta}|$$

because the $\log|\xi_k - \xi_l + \rho_k e^{i\theta} - y|$ function is harmonic for y near 0. Now,

$$\int_{B(0,\rho_l)} |\xi_k - \xi_l + \rho_k e^{i\theta} - y|^2 \frac{1}{4|D|}dy = \frac{1}{4|D|} \int_{B(0,\rho_l)} |y|^2 + |\xi_k - \xi_l + \rho_k e^{i\theta}|^2 dy =$$

$$= C_1 + C_2\pi\rho_l^2|\xi_k - \xi_l + \rho_k e^{i\theta}|^2.$$

Also,

$$\int_{B(0,\rho_l)} H(\xi_k - \xi_l + \rho_k e^{i\theta} - y)dy = \pi\rho_l^2 H(\xi_k - \xi_l + \rho_k e^{i\theta}).$$

Thus, finally

$$\gamma \int_{B(0,\rho_l)} G(\xi_k - \xi_l + \rho_k e^{i\theta} - y)dy = C - \frac{\gamma}{2}\rho_l^2 \log|\xi_k - \xi_l + \rho_k e^{i\theta}| + C\gamma\rho_l^2|\xi_k - \xi_l + \rho_k e^{i\theta}|^2 +$$

$$\gamma\pi\rho_l^2 H(\xi_k - \xi_l + \rho_k e^{i\theta}).$$

By the Taylor's expansion of the functions $\log|\xi_k - \xi_l + \rho_k e^{i\theta}|$, $|\xi_k - \xi_l + \rho_k e^{i\theta}|^2$ and $H(\xi_k - \xi_l + \rho_k e^{i\theta})$ near 0 we have

$$\log|\xi_k - \xi_l + \rho_k e^{i\theta}| = \log|\xi_k - \xi_l| + (C_1, C_2) \cdot e^{i\theta} + O(\rho_k^2)$$

$$|\xi_k - \xi_l + \rho_k e^{i\theta}|^2 = |\xi_k - \xi_l| + (C_3, C_4) e^{i\theta} + O(\rho_k^2)$$

and

$$H(\xi_k - \xi_l + \rho_k e^{i\theta}) = H(\xi_k - \xi_l) + \rho_k \nabla H(\xi_k - \xi_l) \cdot e^{i\theta} + O(\rho_k^2),$$

where the C_i's are independent of θ. Thus, we can see that

$$\gamma \int_{B(0,\rho_l)} G(\xi_k - \xi_l + \rho_k e^{i\theta} - y) dy = M + (M_1, M_2) \cdot e^{i\theta} + O(\gamma\rho^4),$$

where the M's are independent of θ. Therefore, by Equation (4.6)

$$T_k(0)(\theta) = W_k + (W_{k,1}, W_{k,2}) \cdot e^{i\theta} + O(\gamma\rho^4),$$

where the W_k's are independent of θ.

$$||\Pi T(0)||_{L^2}^2 = \sum_{k=1}^{K} ||\Pi T_k(0)||_{L^2}^2 = \sum_{k=1}^{K} ||\Pi(T_k(0) - W_k - (W_{k,1}, W_{k,2}) \cdot e^{i\theta})||_{L^2}^2 \leq$$

$$\leq \sum_{k=1}^{K} ||T_k(0) - W_k - (W_{k,1}, W_{k,2}) \cdot e^{i\theta}||_{L^2}^2 = \sum_{k=1}^{K} ||O(\gamma\rho^4)||_{L^2}^2 = O(\gamma^2\rho^8).$$

Therefore, $||\Pi T(0)||_{L^2} = O(\gamma\rho^4)$. □

Proof of Proposition 2.9

We take $M(\varphi) = -(\Pi T'(0))^{-1}(\Pi T(0) + \Pi N(\varphi))$ for $\varphi \in D(M)$, where

$$D(M) = \{\varphi \in X : ||\varphi||_{H^2} \leq \widetilde{C}\rho^4\},$$

for a constant $\widetilde{C}$ suffiiciently large to be determined.

We will use the Contraction Mapping Principle to show that there exists a function φ which solves

Equation (2.3). Equivalently, we will show that the Equation (2.4) holds. By Lemma 4.7, we have that $||\Pi T(0)||_{L^2} = O(\gamma\rho^4) = O(\rho)$. Now, take

$N_{1,k}(\varphi_k) = W_k(\varphi_k) - W_k(0) - W_k'(0)(\varphi_k)$. for $k \in \{1,..,K\}$.

By problem 1 we have that

$$||W_k(\varphi_k) - W_k(0) - W_k'(0)(\varphi_k)||_{L^2} \leq C||W_k''(\tau\varphi_k)(\varphi_k, \varphi_k)||_{L^2}$$

for some $\tau \in (0,1)$.Thus, $||W_k(\varphi_k) - W_k(0) - W_k'(0)(\varphi_k)||_{L^2} \leq \frac{C}{\rho^5}||\varphi_k||^2_{H^2}$. Thus, $||N_{1,k}(\varphi_k)||_{L^2} \leq \frac{C}{\rho^5}||\varphi_k||^2_{H^2}$. Now, take $N_1 = (N_{1,1}, N_{1,2}, ..., N_{1,k})$. We have

$$||N_1||_{L^2} = (\int_0^{2\pi} |N_{1,1}|^2 + ... + |N_{1,k}|^2)^{\frac{1}{2}} \leq ||N_{1,1}||_{L^2} + .. + ||N_{1,k}||_{L^2} \leq$$

$$\leq \frac{C}{\rho^5}||\varphi_1||^2_{H^2} + .. + \frac{C}{\rho^5}||\varphi_k||^2_{H^2} = \frac{C}{\rho^5}||\varphi||^2_{H^2}.$$

Thus, finally $||N_1(\varphi)||_{L^2} \leq \frac{C}{\rho^5}||\varphi||^2_{H^2}$. Let

$N_{2,k}(\varphi) = A_k(\varphi) - A_k(0) - A'(0)_k(\varphi) + B_k(\varphi) - B_k(0) - B_k'(0)(\varphi) + C_k(\varphi) - C_k(0) - C_k'(0)(\varphi)$,

where $C_k(\varphi) = \sum_{l \neq k} c_{k,l}(\varphi_k, \varphi_l)$.

We have

$$||A_k(\varphi_k) - A_k(0) - A_k'(0)(\varphi_k)||_{L^2} \leq C||A_k''(\tau\varphi_k)(\varphi_k, \varphi_k)||_{L^2} \leq \frac{C\gamma}{\rho^2}||\varphi_k||^2_{H^2} \leq \frac{C}{\rho^5}||\varphi_k||^2_{H^2},$$

by problem 1. Also, $||B_k(\varphi_k) - B_k(0) - B_k'(0)(\varphi_k)||_{L^2} \leq \frac{C}{\rho^4}||\varphi_k||^2_{H^2}$ again by problem 1. By arguing similarly as in the proof of Proposition 2.3 we get

$$||C_{k,l}(\varphi)||_{L^2} \leq C||c_{k,l}''(\tau\varphi)(\varphi_k, \varphi_l)||_{L^2}$$

for some $\tau \in (0,1)$. Therefore by 4.6 we have

$||C_{k,l}(\varphi)||_{L^2} \leq \frac{C}{\rho^4}(||\varphi_k||_{H^2} + ||\varphi_l||_{H^2})^2 \leq \frac{C}{\rho^5}||\varphi||^2_{H^2}$. Thus, finally $||N(\varphi)||_{L^2} \leq \frac{C}{\rho^5}||\varphi||^2_{H^2}$.

Therefore,

$$||(\Pi T'(0))^{-1}(\Pi N(\varphi))||_{L^2} \leq \frac{C}{\rho^5}C\rho^3||\varphi||^2_{H^2} = \frac{C}{\rho^2}||\varphi||^2_{H^2}.$$

Therefore, $||M(\varphi)||_{H^2} \leq C\rho^4 + \frac{C\widetilde{C}^2\rho^8}{\rho^2} = C\rho^4 + C\widetilde{C}^2\rho^6 = C\rho^4 + (C\rho^2\widetilde{C})\widetilde{C}\rho^4$. Now, we have $C\rho^2\widetilde{C} < 1$ for ρ small enough. Thus if we take $\widetilde{C} > 2C$ we get $||M(\varphi)||_{H^2} \leq \widetilde{C}\rho^4$. This means that M is a function from $D(M)$ to $D(M)$. Now, let φ_1,$\varphi_2 \in D(M)$. Then $M(\varphi_1) = -(\Pi T'(0))^{-1}(\Pi T(0) + \Pi N(\varphi_1))$ and $M(\varphi_2) = -(\Pi T'(0))^{-1}(\Pi T(0) + \Pi N(\varphi_2))$. We have $M(\varphi_1) - M(\varphi_2) = -(\Pi T'(0))^{-1}(\Pi N(\varphi_1) - \Pi N(\varphi_2))$. Also, we have $N(\varphi_1) = T(\phi_1) - T(0) - T'(0)(\varphi_1)$ and $N(\varphi_2) = T(\phi_2) - T(0) - T'(0)(\varphi_2)$.

Thus, $N(\varphi_1) - N(\varphi_2) = T(\varphi_1) - T(\varphi_2) - T'(0)(\varphi_1 - \varphi_2)$.

By similar proof as in 2.3 we have $||T_k(\varphi_1) - T_k(\varphi_2) - T_k'(\varphi_2)(\varphi_1 - \varphi_2)||_{L^2} \leq$

$\leq ||T_k''(\tau(\varphi_1 - \varphi_2) + \varphi_2)(\varphi_1 - \varphi_2)(\varphi_1 - \varphi_2)||_{L^2}$ for some $\tau \in (0,1)$. But,

$||T_k''(\tau(\varphi_1 - \varphi_2) + \varphi_2)(\varphi_1 - \varphi_2)(\varphi_1 - \varphi_2)||_{L^2} \leq \frac{C}{\rho^5}||\varphi_1 - \varphi_2||^2_{H^2}$ by Lemma 4.6.

Now,

$$N(\varphi_1) - N(\varphi_2) = T(\varphi_1) - T(\varphi_2) - T'(0)(\varphi_1 - \varphi_2) =$$

$$= T(\varphi_1) - T(\varphi_2) - T'(\varphi_2)(\varphi_1 - \varphi_2) + T'(\varphi_2)(\varphi_1 - \varphi_2) - T'(0)(\varphi_1 - \varphi_2).$$

Thus,

$$||N(\varphi_1) - N(\varphi_2)||_{L^2} \leq \frac{C}{\rho^5}||\varphi_1 - \varphi_2||^2_{H^2} + ||T'(\varphi_2)(\varphi_1 - \varphi_2) - T'(0)(\varphi_1 - \varphi_2)||_{L^2}.$$

Now, again by the proof of Proposition 2.3 we get

$$||T'(\varphi_2)(\varphi_1 - \varphi_2) - T'(0)(\varphi_1 - \varphi_2)||_{L^2} \leq ||T''(\tau\varphi_2)(\varphi_2)(\varphi_1 - \varphi_2)||_{L^2} \leq$$

$$\frac{C}{\rho^5}||\varphi_2||_{H^2}||\varphi_1 - \varphi_2||_{H^2}.$$

Thus, finally

$$||N(\varphi_1) - N(\varphi_2)||_{L^2} \leq \frac{C}{\rho^5}||\varphi_1 - \varphi_2||^2_{H^2} + \frac{C}{\rho^5}||\varphi_2||_{H^2}||\varphi_1 - \varphi_2||_{H^2}.$$

We have $||\varphi_2||_{H^2} \leq \widetilde{C}\rho^4$ and $||\varphi_1 - \varphi_2||_{H^2} \leq 2\widetilde{C}\rho^4$.

Thus, finally

$$||N(\varphi_1) - N(\varphi_2)||_{L^2} \leq ||\varphi_1 - \varphi_2||_{H^2}(\frac{C}{\rho^5}||\varphi_1 - \varphi_2||_{H^2} + \frac{C}{\rho^5}||\varphi_2||_{H^2}) \leq$$

$$\leq ||\varphi_1 - \varphi_2||_{H^2}(\frac{C\widetilde{C}}{\rho} + \frac{2C\widetilde{C}}{\rho}) \leq \frac{C}{\rho}||\varphi_1 - \varphi_2||_{H^2}.$$

Therefore,

$$||M(\varphi_1) - M(\varphi_2)||_{H^2} \leq ||(\Pi T'(0))^{-1}||||\Pi N(\varphi_1) - \Pi N(\varphi_2)||_{L^2} \leq C\rho^3\frac{C}{\rho}||\varphi_1 - \varphi_2||_{H^2} =$$

$$C\rho^2||\varphi_1 - \varphi_2||_{H^2}.$$

To finish the proof we have that for ρ small enough $C\rho^2 < 1$. Thus M is a contraction and from the Contraction Mapping Principle (Proposition 3.9) we have that M has a unique fixed point and the proof is complete. □

Lemma 4.8. *For the solution φ in Proposition 2.9 and for the energy functional J, we have $J(\varphi) = J(0, .., 0) + O(\rho^3)$.*

Proof. We have $J(\varphi) = P_D(E_\varphi) + \frac{\gamma}{2}\int_D |\nabla I_\Lambda(E_\varphi)(x)|^2 dx$ by Equation (1.1).

By Equation (1.2) we have

$$-\Delta I_\Lambda(\Omega)(x) = \chi_\Omega(x) - \omega, x \in \mathbb{C}, \int_D I_\Lambda(\Omega)(y)dy = 0.$$

Thus, by using integration by parts and Equation (1.6) we get

$$\int_D |\nabla I_\Lambda(E_\varphi)(x)|^2 dx = \int_{E_\varphi}\int_{E_\varphi} G(x-y)dxdy.$$

Also, we have $P_D(E_\varphi) = \sum_{k=1}^K P_D(E_{\varphi_k})$. Also we can see that

$$\int_{E_\varphi}\int_{E_\varphi} G(x-y)dxdy = \sum_{i=1}^K\sum_{j=1}^K \int_{E_{\varphi_i}}\int_{E_{\varphi_j}} G(x-y)dxdy.$$

We have for $i \neq j$

$$|\int_{E_{\varphi_i}}\int_{E_{\varphi_j}} G(x-y)dxdy| \leq C\rho^4,$$

since the closures of E_{φ_i} and E_{φ_j} are disjoint. Thus,

$$J(\varphi) = \sum_{i=1}^{K} J(\varphi_i) + O(\rho^4)$$

Thus, for $i \in \{1, .., K\}$

$$J(\varphi_i) = J(0, \rho_i) + \frac{\gamma\pi\rho_i^2}{2}\int_0^{2\pi} H(\rho_i e^{i\theta})\varphi_i(\theta)d\theta + \frac{1}{4}\int_0^{2\pi} S'(\tau\varphi_i)(\varphi_i)\varphi_i(\theta)d\theta$$

by Lemma 3.10. Also, $J(0, .., 0) = \sum_{i=1}^{K} J(0, \rho_i) + O(\rho^4)$.

$$J(0, ..., 0) = \sum_{i=1}^{K}[J(\varphi_i) - \frac{\gamma\pi\rho_i^2}{2}\int_0^{2\pi} H(\rho_i e^{i\theta})\varphi_i(\theta)d\theta - \frac{1}{4}\int_0^{2\pi} S'(\tau\varphi_i)(\varphi_i)\varphi_i(\theta)d\theta] + O(\rho^4).$$

Therefore,

$$J(\varphi) = J(0, .., 0) + \sum_{i=1}^{K}\frac{\gamma\pi\rho_i^2}{2}\int_0^{2\pi} H(\rho_i e^{i\theta})\varphi_i(\theta)d\theta + \frac{1}{4}\int_0^{2\pi} S'(\tau\varphi_i)(\varphi_i)\varphi_i(\theta)d\theta + O(\rho^4).$$

Thus, it suffices to show that

$$\frac{\gamma\pi\rho_i^2}{2}\int_0^{2\pi} H(\rho_i e^{i\theta})\varphi_i(\theta)d\theta + \frac{1}{4}\int_0^{2\pi} S'(\tau\varphi_i)(\varphi_i)\varphi_i(\theta)d\theta = O(\rho^3).$$

We have

$$|\frac{\gamma\pi\rho_i^2}{2}\int_0^{2\pi} H(\rho_i e^{i\theta})\varphi_i(\theta)d\theta| \leq C\gamma\rho_i^2\int_0^{2\pi} |\varphi_i(\theta)|d\theta.$$

. Also,

$$\int_0^{2\pi} |\varphi_i(\theta)|d\theta \leq C||\varphi_i||_{L^2} \leq C||\varphi_i||_{H^2} \leq C\gamma\rho^7.$$

Thus, finnaly

$$|\frac{\gamma\pi\rho_i^2}{2}\int_0^{2\pi} H(\rho_i e^{i\theta})\varphi_i(\theta)d\theta| \leq C\gamma^2\rho^9 = O(\rho^3),$$

since $\gamma^2\rho^6 = O(1)$. Now, by the proof of Proposition 2.3 we have for M sufficiently large

$$||S'(w_2)(w_1 - w_2) - S'(0)(w_1 - w_2)||_{L^2} \leq C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})||w_2||_{H^2}||w_1 - w_2||_{H^2}$$

for all $w_1, w_2 \in D(T) = \{w \in H^2(S^1) : ||w||_{H^2} \leq M\gamma\rho_i^7\}$. Now for $w_2 = \varphi_i$ and $w_1 = 0$ we have

$$||S'(\varphi_i)(-\varphi_i) - S'(0)(-\varphi_i)||_{L^2} \leq C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})||\varphi_i||_{H^2}^2.$$

Change φ_i to $\tau\varphi_i$. Then we have

$$||S'(\tau\varphi_i)(\tau\varphi_i) - S'(0)(\tau\varphi_i)||_{L^2} \leq C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})\tau^2||\varphi_i||_{H^2}^2.$$

Thus,

$$||S'(\tau\varphi_i)(\varphi_i) - S'(0)(\varphi_i)||_{L^2} \leq C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})\tau||\varphi_i||_{H^2}^2 \leq$$

$$\leq C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})||\varphi_i||_{H^2}^2 = C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})M^2\gamma^2\rho_i^{14} =$$

$$CM^2\gamma^2\rho^9 + M^2\gamma^3\rho_i^{12} = O(\rho^3).$$

Consider the term $||S'(0)(\varphi_i)||_{L^2}$. We have

$$S'(0)(\varphi_i) = L_1(\varphi_i) + B(\varphi_i)$$

by Equations (3.1), (3.2) and (3.3). Also,

$$||B(\varphi_i)||_{L^2} \leq C\gamma\rho_i||\varphi_i||_{L^2} \leq C\gamma M\gamma\rho_i^8 = O(\rho^2).$$

Also, we have

$$L_1(\varphi_i) = -\frac{\varphi_i'' + \varphi_i}{2\rho_i^3} - \frac{\gamma}{4}\varphi_i - \frac{\gamma}{8\pi}\int_0^{2\pi} \varphi_i(\omega)\log(1 - \cos(\theta - \omega))d\omega$$

Now using that $||\varphi_i||_{H^2} = O(\gamma\rho^7)$ we get $L_1(\varphi_i) = O(\rho)$. Thus we proved that $||S'(0)(\varphi_i)||_{L^2} = O(\rho)$. Also, we have proved already that

$$||S'(\tau\varphi_i)(\varphi_i) - S'(0)(\varphi_i)||_{L^2} \leq C(\frac{1}{\rho_i^5} + \frac{\gamma}{\rho_i^2})\tau||\varphi_i||_{H^2}^2 = O(\rho^3).$$

Therefore, $||S'(\tau\varphi_i)(\varphi_i)||_{L^2} = O(\rho)$. So,

$$|\frac{1}{4}\int_0^{2\pi} S'(\tau\varphi_i)(\varphi_i)\varphi_i(\theta)d\theta| \leq C||S'(\tau\varphi_i)(\varphi_i)||_{L^2}||\varphi_i||_{L^2} = O(\rho^5)$$

and the proof is complete. $\square$

Now, we will calculate the free energy of (0,...,0). This corresponds to the union of K round disks in D.

Lemma 4.9. *We have*

$$J(0,...,0)=\sum_{k=1}^{K}2\pi\rho_k+\frac{\gamma}{2}\sum_{i=1}^{K}[\frac{\pi^2\rho_i^6}{4|D|}+\pi^2\rho_i^4H(0)-\pi\frac{\rho_i^4}{2}(\log\rho_i-\frac{1}{4})-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi\rho_i^4]+$$

$$+\frac{\gamma}{2}\sum_{i\neq j}\pi^2\rho_i^2\rho_j^2G(\xi_j-\xi_i)+\frac{1}{8|D|}\pi^2\rho_i^2\rho_j^2(\rho_i^2+\rho_j^2).$$

Proof. We have

$$G(z)=-\frac{1}{2\pi}\log(\frac{2\pi|z|}{\sqrt{|D|}})+\frac{|z|^2}{4|D|}+H(z).$$

Then, we have

$$J(0,...,0)=\sum_{k=1}^{K}2\pi\rho_k+\frac{\gamma}{2}\int_{\cup B(\xi_i,\rho_i)}\int_{\cup B(\xi_i,\rho_i)}G(x-y)dxdy=$$

$$=\sum_{k=1}^{K}2\pi\rho_k+\frac{\gamma}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}G(x-y)dxdy$$

$$=\sum_{k=1}^{K}2\pi\rho_k+\frac{\gamma}{2}\sum_{i=1}^{K}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_i,\rho_i)}G(x-y)dxdy+\frac{\gamma}{2}\sum_{i\neq j}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}G(x-y)dxdy.$$

We have

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_i,\rho_i)}H(x-y)dxdy=\int_{B(0_i,\rho_i)}\int_{B(0,\rho_i)}H(x-y)dxdy=\pi^2\rho_i^4H(0)$$

by the mean value theorem for harmonic funactions, because H is harmonic near 0. Also,

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_i,\rho_i)}\log|x-y|dxdy=\int_{B(0,\rho_i)}\int_{B(0,\rho_i)}\log|x-y|dxdy=\pi^2\rho_i^4(\log\rho_i-\frac{1}{4}).$$

Therefore,

$$-\frac{1}{2\pi}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_i,\rho_i)}\log\frac{2\pi|x-y|}{\sqrt{|D|}}dxdy=-\frac{1}{2\pi}\log(\frac{2\pi}{\sqrt{|D|}})\pi^2\rho_i^4-\frac{\pi\rho_i^4}{2}(\log\rho_i-\frac{1}{4}).$$

Now, we have

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_i,\rho_i)}\frac{|x-y|^2}{4|D|}=\int_{B(0,\rho_i)}\int_{B(0,\rho_i)}\frac{|x-y|^2}{4|D|}=\frac{\pi^2\rho_i^6}{4|D|}$$

by using polar coordinates.

Thus, we have proved that

$$\sum_{i=1}^{K}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_i,\rho_i)}G(x-y)dxdy =$$

$$=\sum_{i=1}^{K}[\frac{\pi^2\rho_i^6}{4|D|}+\pi^2\rho_i^4H(0)-\pi\frac{\rho_i^4}{2}(\log\rho_i-\frac{1}{4})-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi\rho_i^4].$$

Consider now $i\neq j$. Then,

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}H(x-y)dxdy=\pi^2\rho_i^2\rho_j^2H(\xi_j-\xi_i)$$

by the mean value theorem for harmonic functions. Now,

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}\log|x-y|dxdy=\pi^2\rho_i^2\rho_j^2\log|\xi_j-\xi_i|$$

again my the mean value theorem for harmonic functions. Also, we have

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}\frac{|x-y|^2}{4|D|}dxdy=\frac{1}{4|D|}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}|x|^2+|y|^2-2x\cdot ydxdy.$$

Also, we can prove that

$$-\frac{1}{4|D|}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}2x\cdot ydxdy=-\frac{1}{2|D|}\int_{B(0,\rho_i)}\int_{B(0,\rho_j)}(x+\xi_i)(y+\xi_j)dxdy=$$

$$=-\frac{\pi^2\rho_i^2\rho_j^2}{2|D|}\xi_i\cdot\xi_j.$$

Also, we have to calculate

$$\frac{1}{4|D|}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}|x|^2+|y|^2dxdy.$$

We have

$$\frac{1}{4|D|}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}|x|^2+|y|^2dxdy=\frac{1}{4|D|}\pi\rho_i^2\int_{B(\xi_j,\rho_j)}|y|^2dy+\frac{1}{4|D|}\pi^2\rho_j^2\int_{B(\xi_i,\rho_i)}|x|^2dx.$$

Also,

$$\int_{B(\xi_j,\rho_j)}|y|^2dy=\int_{B(0,\rho_j)}|y+\xi_j|^2dy=\int_{B(0,\rho_j)}|y|^2+|\xi_j|^2+2y\cdot\xi_jdy=|\xi_j|^2\pi\rho_j^2+$$

$$+\int_{B(0,\rho_j)}|y|^2dy.$$

and also we can see that

$$\int_{B(0,\rho_j)}|y|^2dy=\int_0^{\rho_j}s^2 2\pi s ds=\frac{\pi\rho_j^4}{2}.$$

Thus, finally

$$\int_{B(\xi_j,\rho_j)}|y|^2dy=\pi\rho_j^2|\xi_j|^2+\frac{\pi\rho_j^4}{2}.$$

Similarly, we can see that

$$\int_{B(\xi_i,\rho_i)}|x|^2dx=\pi\rho_i^2|\xi_i|^2+\frac{\pi\rho_i^4}{2}.$$

Thus,

$$\frac{1}{4|D|}\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}|x|^2+|y|^2dxdy=\frac{1}{4|D|}\pi\rho_i^2(\pi\rho_j^2|\xi_j|^2+\frac{\pi\rho_j^4}{2})+\frac{1}{4|D|}\pi\rho_j^2(\pi\rho_i^2|\xi_i|^2+\frac{\pi\rho_i^4}{2}).$$

Therefore, we can see that

$$\int_{B(\xi_i,\rho_i)}\int_{B(\xi_j,\rho_j)}\frac{|x-y|^2}{4|D|}dxdy=\frac{1}{4|D|}\pi^2\rho_i^2\rho_j^2|\xi_j|^2+\frac{1}{4|D|}\pi^2\rho_i^2\rho_j^2|\xi_i|^2+\frac{1}{8|D|}\pi^2\rho_i^2\rho_j^4+$$

$$+\frac{1}{8|D|}\pi^2\rho_j^2\rho_i^4-\frac{\pi^2\rho_i^2\rho_j^2}{2|D|}\xi_i\cdot\xi_j=$$

$$\frac{\pi^2\rho_i^2\rho_j^2}{4|D|}|\xi_i-\xi_j|^2+\frac{1}{8|D|}(\pi^2\rho_i^2\rho_j^2(\rho_i^2+\rho_j^2)).$$

Finally, we get

$$J(0,..,0)=\sum_{k=1}^K 2\pi\rho_k+\frac{\gamma}{2}\sum_{i=1}^K[\frac{\pi^2\rho_i^6}{4|D|}+\pi^2\rho_i^4H(0)-\pi\frac{\rho_i^4}{2}(\log\rho_i-\frac{1}{4})-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi\rho_i^4]+$$

$$+\frac{\gamma}{2}\sum_{i\neq j}\pi^2\rho_i^2\rho_j^2H(\xi_j-\xi_i)-\frac{1}{2\pi}\pi^2\rho_i^2\rho_j^2\log|\xi_j-\xi_i|+\frac{\pi^2\rho_i^2\rho_j^2}{4|D|}|\xi_j-\xi_i|^2+\frac{1}{8|D|}\pi^2\rho_i^2\rho_j^2(\rho_i^2+\rho_j^2)$$

$$-\frac{1}{2\pi}\log\frac{2\pi}{\sqrt{|D|}}\pi^2\rho_i^2\rho_j^2.$$

Also, $G(\xi_j-\xi_i)=-\frac{1}{2\pi}\log\frac{2\pi|\xi_j-\xi_i|}{\sqrt{|D|}}+\frac{1}{4|D|}|\xi_j-\xi_i|^2+H(\xi_j-\xi_i)$. Thus,

$$J(0,..,0)=\sum_{k=1}^K 2\pi\rho_k+\frac{\gamma}{2}\sum_{i=1}^K[\frac{\pi^2\rho_i^6}{4|D|}+\pi^2\rho_i^4H(0)-\pi\frac{\rho_i^4}{2}(\log\rho_i-\frac{1}{4})-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi\rho_i^4]+$$

$$+\frac{\gamma}{2}\sum_{i\neq j}\pi^2\rho_i^2\rho_j^2G(\xi_j-\xi_i)+\frac{1}{8|D|}\pi^2\rho_i^2\rho_j^2(\rho_i^2+\rho_j^2).$$

□

In the next lemma we will use the condition $\gamma\rho^3\log\frac{1}{\rho} > 1+\delta$. Also, we will consider the sets $G=\{\eta : F(\eta)=\min_{\xi\in D^K}F(\xi)\}$ and $U_2=\{(\rho_1,...,\rho_K) : \rho_i\in((1-\delta_1)\rho,(1+\delta_1)\rho),\sum_{k=1}^K\rho_k^2=K\rho^2\}$, where δ_1 is sufficiently small. Also, we consider U_1 to be a small open neighborhood of G. We assume that the radii of E_φ belong to the set U_2 and the centers belong to U_1.

Lemma 4.10. *When ρ is sufficiently small $J(E_{\varphi(\xi,r)})$ is minimized at some $(\xi,r)=(\zeta,s)\in U_1\times U_2$. Also, as $\rho\to 0$ we have $\frac{s}{\rho}\to(1,..,1)$ and $\zeta\to\zeta_0$ along a subsequence, where ζ_0 is a global minimum of F, where $F(\xi_1,..,\xi_K)=\sum_{i\neq j}G(\xi_j-\xi_i)$.*

Proof. Following [2] we let $R=\frac{r}{\rho}$, where $r=(\rho_1,..,\rho_K)$. Also, let
$\widetilde{J}(\xi,R)=\frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}J(E_{\varphi(\xi,r)})$, where $(\xi,R)\in U_1\times\widetilde{U_2}$ and
$\widetilde{U_2}=\{(R_1,..,R_K) : 1-\delta_1<R_i<1+\delta_1,\sum_k R_k^2=K\}$. We have

$$\frac{4}{\pi\gamma\rho^3\log\frac{1}{\rho}}\leq\frac{4}{(1+\delta)\pi}.$$

Thus, $\frac{4}{\pi\gamma\rho^3\log\frac{1}{\rho}}$ is bounded. Thus, by Bolzano-Weierstrass Theorem we have that it has a convergent subsequence. Therefore, we have

$$\frac{4}{\pi\gamma\rho_n^3\log\frac{1}{\rho_n}}\to b_0,$$

where $\rho_n\to 0$. Let (ζ,S) be the global minimum of $\widetilde{J}$ in the closure of $U_1\times\widetilde{U_2}$, where $S=\frac{s}{\rho}$. Then we have by Bolzano-Weierstrass Theorem that $(\zeta,S)\to(\zeta_0,S_0)$ along a subsequence as $\rho\to 0$. We claim that $S_0=(1,...,1)$. To prove that we suppose that $S_0\neq(1,..,1)$ and we will derive a contradiction.

We have as $\rho \to 0$ $\widetilde{J}(\zeta,(1,..,1)) = \frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}} J(\zeta,(1,..,1)) =$

$$\frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}\sum_{k=1}^{K}2\pi s_k+\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i=1}^{K}[\frac{\pi^2 s_i^6}{4|D|}+\pi^2 s_i^4 H(0)-\pi\frac{s_i^4}{2}(\log s_i-\frac{1}{4})-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi s_i^4]+$$

$$+\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i\neq j}[\pi^2 s_i^2 s_j^2 G(\xi_j-\xi_i)+\frac{1}{8|D|}\pi^2 s_i^2 s_j^2(s_i^2+s_j^2)]+\frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}O(\rho^3),$$

by Lemmas 4.8 and 4.9. Note that $(1,...,1)$ corresponds to $s_i = \rho$ for every $i \in \{1,.,,K\}$.

Therefore,

$$\widetilde{J}(\zeta,(1,..,1)) =$$

$$= \frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}\sum_{k=1}^{K}2\pi\rho+\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i=1}^{K}[\frac{\pi^2\rho^6}{4|D|}+\pi^2\rho^4 H(0)-\pi\frac{\rho^4}{2}(\log\rho-\frac{1}{4})-$$

$$\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi\rho^4]+\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i\neq j}[\pi^2\rho^4 G(\zeta_j-\zeta_i)+\frac{1}{4|D|}\pi^2\rho^6]+$$

$$\frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}O(\rho^3) = \frac{4K}{\pi\gamma\rho^3\log\frac{1}{\rho}}+\frac{K\rho^2}{4|D|\log\frac{1}{\rho}}+\frac{KH(0)}{\log\frac{1}{\rho}}+\frac{K}{2\pi}-\frac{K}{8\pi\log\rho}-$$

$$\frac{K}{2\pi\log\frac{1}{\rho}}\log(\frac{2\pi}{\sqrt{|D|}})+$$

$$(K^2-K)\frac{G(\zeta_j-\zeta_i)}{\log\frac{1}{\rho}}+(K^2-K)\frac{\rho^2}{4|D|\log\frac{1}{\rho}}+O(\frac{1}{\gamma\rho\log\frac{1}{\rho}}).$$

Now, we can see that $\frac{4K}{\pi\gamma\rho^3\log\frac{1}{\rho}} \to Kb_0$. Also, $\frac{KH(0)}{\log\frac{1}{\rho}} \to 0$ and also

$(K^2-K)\frac{G(\zeta_j-\zeta_i)}{\log\frac{1}{\rho}} \to 0$, as $\rho \to 0$. Also, we can see that $\frac{K}{8\pi\log\rho} \to 0$, as $\rho \to 0$ and $\frac{K}{2\pi\log\frac{1}{\rho}}\log(\frac{2\pi}{\sqrt{|D|}}) \to 0$.

Now, it is easy to see that $\frac{K\rho^2}{4|D|\log\frac{1}{\rho}} \to 0$ and $(K^2-K)\frac{\rho^2}{4|D|\log\frac{1}{\rho}} \to 0$. Now, we will use our assumption $\gamma\rho^3\log\frac{1}{\rho} > 1+\delta$ and get $\frac{1}{\gamma\rho\log\frac{1}{\rho}} < \frac{\rho^2}{1+\delta}$.

Thus, we can see now that $\widetilde{J}(\zeta,(1,...,1)) \to Kb_0+\frac{K}{2\pi}$, as $\rho \to 0$.

Now, consider $\widetilde{J}(\zeta,S)$. We have

$$\widetilde{J}(\zeta,S) = \frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}\sum_{k=1}^{K}2\pi s_k+\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i=1}^{K}[\frac{\pi^2 s_i^6}{4|D|}+\pi^2 s_i^4 H(0)-\pi\frac{s_i^4}{2}(\log s_i-\frac{1}{4})$$

$$-\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi s_i^4]+\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i\neq j}[\pi^2 s_i^2 s_j^2 G(\zeta_j-\zeta_i)+\frac{1}{8|D|}\pi^2 s_i^2 s_j^2(s_i^2+s_j^2)]+$$

$$+\frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}O(\rho^3).$$

Now, we get

$$\frac{2}{\pi^2\gamma\rho^4\log\frac{1}{\rho}}\sum_{k=1}^{K}2\pi s_k=\frac{4}{\pi\gamma\rho^4\log\frac{1}{\rho}}\sum_{k=1}^{K}s_k=\frac{4}{\pi\gamma\rho^3\log\frac{1}{\rho}}\sum_{k=1}^{K}S_k\to b_0\sum_{k=1}^{K}S_{0,k},$$

as $\rho\to 0$. Now, we have

$$\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\sum_{i=1}^{K}\frac{\pi^2 s_i^6}{4|D|}\to 0.$$

Now, as $\rho\to 0$ we get

$$\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\pi^2 s_i^4 H(0)\to 0$$

$$\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}(-\pi\frac{s_i^4}{2}(\log s_i-\frac{1}{4}))=-\frac{1}{2\pi}S_i^4(\frac{\log s_i-\frac{1}{4}}{\log\frac{1}{\rho}})\to\frac{1}{2\pi}S_{0,i}^4$$

, because $s_i=S_i\rho$. Also, we have

$$\frac{2}{\pi^2\rho^4\log\frac{1}{\rho}}\frac{1}{2}\log(\frac{2\pi}{\sqrt{|D|}})\pi s_i^4\to 0.$$

Now, we get

$$\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\pi^2 s_i^2 s_j^2 G(\zeta_j-\zeta_i)\to 0$$

as $\rho\to 0$.

$$\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}\frac{\pi^2 s_i^2 s_j^2(s_i^2+s_i^2)}{8|D|}=\frac{1}{\pi^2\rho^4\log\frac{1}{\rho}}O(\rho^6)=O(\frac{\rho^2}{\log\frac{1}{\rho}}).$$

Therefore, as $\rho\to 0$ we get

$$\widetilde{J}(\zeta,(1,...,1))-\widetilde{J}(\zeta,S)\to Kb_0+\frac{K}{2\pi}-b_0\sum_{k=1}^{K}S_{0,k}-\sum_{k=1}^{K}\frac{S_{0,k}^4}{2\pi}=$$

$$=(K-\sum_{k=1}^{K}S_{0,k})b_0+\frac{1}{\pi}(\frac{K}{2}-\sum_{k=1}^{K}\frac{S_{0,k}^4}{2}).$$

Now we will prove that

$$(K-\sum_{k=1}^{K}S_{0,k})b_0+\frac{1}{\pi}(\frac{K}{2}-\sum_{k=1}^{K}\frac{S_{0,k}^4}{2})<0$$

and this will be the desired contradiction since (ζ, S) is the global minimum of $\widetilde{J}$. To prove that we let $f(x) = \frac{x^2}{2\pi} + b_0\sqrt{x}$, for $x > (\frac{1}{1+\delta})^{\frac{2}{3}}$. We have $f'(x) = \frac{x}{\pi} + \frac{b_0}{2\sqrt{x}}$. Thus, $f''(x) = \frac{1}{\pi} - \frac{b_0}{4}x^{-\frac{3}{2}}$. Now, since $x > (\frac{1}{1+\delta})^{\frac{2}{3}}$, we get $f''(x) > \frac{1}{\pi} - \frac{b_0}{4}[(\frac{1}{1+\delta})^{\frac{2}{3}}]^{-\frac{3}{2}} \geq \frac{1}{\pi} - \frac{1}{\pi} = 0$. Here we used that $b_0 \leq \frac{4}{(1+\delta)\pi}$. Thus, f is strictly convex. Let $\eta_0 = 1 - \frac{1}{(1+\delta)^{\frac{1}{3}}}$. Then for $\eta < \eta_0$ we have $(1-\eta)^2 > (\frac{1}{1+\delta})^{\frac{2}{3}}$. Also, for η such that $1-\eta \leq S_{0,k} \leq 1+\eta$, we get $S_{0,k}^2 > (\frac{1}{1+\epsilon})^{\frac{2}{3}}$. Now since f is strictly convex we have $f(\frac{x_1+..+x_K}{K}) \leq \sum_{k=1}^{K} \frac{1}{K} f(x_k)$, with equality only if $x_1 = .. = x_K$. Now, set $x_k = S_{0,k}^2$. Note that we do not have $x_1 = .. = x_K$ since $\sum_k S_{0,k} = K$ and $S_0 \neq (1, ..., 1)$. Now,

$$f(\frac{x_1 + .. + x_K}{K}) = \frac{1}{2\pi} + b_0$$

and

$$\sum_{k=1}^{K} \frac{1}{K} f(x_k) = \frac{1}{K} \sum_{k=1}^{K} \frac{S_{0,k}^4}{2\pi} + b_0 S_{0,k}.$$

. Thus, we get

$$\frac{1}{2\pi} + b_0 < \frac{1}{K} \sum_{k=1}^{K} \frac{S_{0,k}^4}{2\pi} + b_0 S_{0,k}$$

which gives

$$(K - \sum_{k=1}^{K} S_{0,k}) b_0 + \frac{1}{\pi}(\frac{K}{2} - \sum_{k=1}^{K} \frac{S_{0,k}^4}{2}) < 0.$$

Thus, we proved that $S_0 = (1, .., 1)$.

Nextly, we claim that ζ_0 minimizes F in U_1. Let η be a global minimum of F in U_1. Assume that $F(\eta) < F(\zeta_0)$. Then we have

$$\log \frac{1}{\rho}(\widetilde{J}(\eta, S) - \widetilde{J}(\zeta, S)) = \frac{2}{\pi^2 \gamma \rho^4}(J(\eta, S) - J(\zeta, S)) =$$

$$\frac{2}{\pi^2 \gamma \rho^4} \sum_{i \neq j} \gamma \pi^2 s_1^2 s_2^2 G(\eta_j - \eta_i) - \gamma \pi^2 s_1^2 s_2^2 G(\zeta_j - \zeta_i) + O(\rho^2 \log \frac{1}{\rho}) \to F(\eta) - F(\zeta_0) < 0,$$

as $\rho \to 0$. Therefore, $\widetilde{J}(\eta, S) - \widetilde{J}(\zeta, S) < 0$ for ρ small enough, which is a contradiction. Therefore, ζ_0 is a global minimizer of F. Thus, we have $\zeta_0 \in U_1$ which proves $\zeta \in U_1$ for ρ small enough. Also, $S_0 \in \widetilde{U_2}$ which gives that $s \in U_2$ for ρ small enough and the proof is complete. $\square$

Now we will prove a helpfull Lemma.

Lemma 4.11. *At $\xi = \zeta$ and $r = s$ there exists a constant λ such that $\lambda_k = \lambda$ for every $k \in \{1, .., K\}$, where λ_k are as in Proposition 2.9.*

Proof. Let $p_k = \rho_k^2$.

We will show that $\frac{d}{dp_k} J(E_\varphi) = \frac{1}{2} \int_0^{2\pi} T(\varphi)(\theta) \cdot \frac{d}{dp_k}(p + \varphi)(\theta) d\theta$, where φ is the solution as in Proposition 2.9. To prove this we have

$$J(E_\varphi) = P_D(E_\varphi) + \frac{\gamma}{2} \int_{E_\varphi} \int_{E_\varphi} G(x - y) dx dy.$$

The term $P_D(E_\varphi)$ is the length of the boundary of E_φ. Thus, we get

$$P_D(E_\varphi) = \sum_{k=1}^{K} \int_0^{2\pi} \sqrt{\rho_k^2 + \varphi_k(\theta) + \frac{(\varphi_k'(\theta))^2}{4(\rho_k^2 + \varphi_k(\theta))}} d\theta.$$

Let

$$A(u) = A(u_1, .., u_K) = \sum_{k=1}^{K} \int_0^{2\pi} \sqrt{u_k(\theta) + \frac{(u_k'(\theta))^2}{4 u_k(\theta)}} d\theta.$$

Then we have $P_D(E_\varphi) = A(p + \varphi)$.

Let $B(u) = B(u_1, .., u_K) = \frac{\gamma}{2} \int_{Q(u)} \int_{Q(u)} G(x - y) dx dy$, where $Q(u) = Q(u_1, .., u_K) = \cup_{i=1}^{K} \{\xi_i + a e^{i\theta} : a \in [0, \sqrt{u_i(\theta)})\}$. Then we have

$$J(E_\varphi) = A(p + \varphi) + B(p + \varphi).$$

. We want to differentiate $J(E_\varphi)$ with respect to p_k. Let $C(u) = A(u) + B(u)$. Then,

$$\frac{d}{dp_k} J(E_\varphi) = \frac{d}{dp_k} C(p + \varphi).$$

Also, $\varphi_k(\epsilon + p_1) = \varphi_k(p_1) + \epsilon \frac{d}{dp_1} \varphi_k + O(\epsilon^2)$. Then, using the Chain Rule and the fact that

$$\frac{d}{d\eta}|_{\eta=0} J(\varphi + \eta\psi) = \frac{1}{2} \int_0^{2\pi} T(\varphi)(\theta) \cdot \psi(\theta) d\theta$$

we can see that $\frac{d}{dp_k}J(E_\varphi) = \frac{1}{2}\int_0^{2\pi} T(\varphi)(\theta)\cdot\frac{d}{dp_k}(p+\varphi)(\theta)d\theta$. Now, using this fact we get

$$\frac{d}{dp_k}J(E_\varphi) = \sum_{l=1}^{K}\frac{1}{2}\int_0^{2\pi} T_l(\varphi)(\theta)\frac{d}{dp_k}(p_l+\varphi_l)(\theta)d\theta =$$

$$= \sum_{l\neq k}\frac{1}{2}\int_0^{2\pi} T_l(\varphi)(\theta)\frac{d}{dp_k}\varphi_l(\theta)d\theta + \frac{1}{2}\int_0^{2\pi} T_k(\varphi)(\theta)(1+\frac{d}{dp_k}\varphi_k(\theta))d\theta.$$

Now we will use the Proposition 2.9 to write $T_k(\varphi) = \lambda_k + \mu_k\cos\theta + \nu_k\sin\theta$. Therefore, by using that $\frac{d}{dp_k}\varphi_l \perp \cos\theta, \sin\theta, 1$, we get that

$$\frac{d}{dp_k}J(E_\varphi) = \pi\lambda_k.$$

Now, at $\xi = \zeta$ and $r = s$ we have by Lemma 4.10 that $J(E_\varphi)$ is minimized under the constraint $\sum_{k=1}^{K} p_k = K\rho^2$. Thus,

$$\nabla_{p_k}J(\varphi) = \omega\nabla_{p_k}(\sum_{k=1}^{K} p_k - K\rho^2))$$

for some constant ω, by Lagrange multiplier method, at $\xi = \zeta$ and $r = s$. Therefore $\nabla_{p_k}J(\varphi) = (\omega, ..., \omega)$. Therefore $\lambda_1 = ... = \lambda_K = \lambda$ for some constant λ and the proof is complete. $\square$

We will now prove an important estimate that will be used later.

Lemma 4.12. *There exists a constant $C > 0$ such that for all $u \in X$ it holds*

$$||u||_{H^2} \leq C\rho^3||\Pi T'(\varphi)(u)||_{L^2},$$

where φ is the solution to the fixed point argument of Proposition 2.9.

Proof. We have as in [2],

$$||\Pi T'(\varphi)(u)||_{L^2} \geq ||\Pi T'(0)(u)||_{L^2} - ||\Pi T'(\varphi)(u) - \Pi T'(0)(u)||_{L^2}.$$

Also, by Lemma 4.3 we get

$$||\Pi T'(0)(u)||_{L^2} \geq \frac{C}{\rho^3}||u||_{H^2}.$$

Now, we claim that

$$||\Pi(T'(\varphi) - T'(0))(u)||_{L^2} \leq \frac{C}{\rho}||u||_{H^2}.$$

Let $\psi \in L^2$. Then by Taylor's expansion we have

$$< T'(t\varphi)(u), \psi >=< T'(0)(u), \psi > +t\frac{d}{d\eta}|_{\eta=\tau} < T'(\eta\varphi)(u), \psi > .$$

Then, by setting $t = 1$ we have

$$< T'(\varphi)(u), \psi > - < T'(0)(u), \psi >= \frac{d}{d\eta}|_{\eta=\tau} < T'(\eta\varphi)(u), \psi >,$$

where $\tau \in (0, 1)$. Also we can see that

$$\frac{d}{d\eta}|_{\eta=\tau}T'(\eta\varphi)(u) = T''(\tau\varphi)(\varphi, u).$$

Now, by taking the supremum for all $\psi \in L^2$, where $||\psi||_{L^2} \leq 1$, we get

$$||\Pi(T'(\varphi) - T'(0))(u)||_{L^2} = ||T''(\tau\varphi)(\varphi, u)||_{L^2} \leq \frac{C}{\rho^5}||\varphi||_{H^2}||u||_{H^2}$$

by 4.6 and the solution φ satisfies $||\varphi||_{H^2} = O(\rho^4)$. Therefore,

$$||\Pi T'(\varphi)(u) - \Pi T'(0)(u)||_{L^2} \leq \frac{C}{\rho}||u||_{H^2}$$

and the proof of the claim is complete. Now, we can see that

$$||\Pi T'(\varphi)(u)||_{L^2} \geq \frac{C_1}{\rho^3}||u||_{H^2} - \frac{C_2}{\rho}||u||_{H^2} = \frac{(C_1 - \rho^2 C_2)}{\rho^3}||u||_{H^2}.$$

Now, by taking ρ small enough we have that

$$C_1 - \rho^2 C_2 > C > 0,$$

where C is a constant. Thus,

$$||u||_{H^2} \leq C\rho^3||\Pi T'(\varphi)(u)||_{L^2}$$

and the proof is complete. □

Now, we need to get an estimate of the solution function φ of $\Pi T(\varphi) = 0$.

Lemma 4.13. *The fixed point φ satisfies $||\frac{\partial \varphi}{\partial \xi_{i,j}}||_{H^2} = O(\rho^2)$.*

Proof. We have already that $\Pi T(\varphi(\xi), \xi) = 0$. Here the solution φ and the operator T depend on ξ. Therefore by differentiating we can see that $\Pi \frac{d}{d\xi_{i,j}} T(\varphi(\xi), \xi) = 0$.For simplicity we write ξ for $\xi_{i,j}$. We will now use the Chain Rule and we define a function A as follows

$$A(u, v) = T(\varphi(u), v).$$

Let also functions $u(\xi), v(\xi)$. By letting $u(\xi) = v(\xi) = \xi$ and using the Chain Rule we have

$$\frac{d}{d\xi} T(\varphi(\xi), \xi) = \frac{dA}{du}|_{u=\xi} + \frac{dA}{dv}|_{v=\xi}.$$

Also,

$$\frac{dA}{du}|_{u=\xi} = \frac{d}{d\eta}|_{\eta=0} T(\varphi(\eta + \xi), v).$$

By Taylor's expansion we get

$$\varphi(\eta + \xi) = \varphi(\xi) + \eta \frac{d}{dx}|_{x=0} \varphi(x + \xi) + O(\eta^2).$$

By using the Chain Rule again we get

$$\frac{dA}{du}|_{u=\xi} = T'(\varphi)(\frac{d\varphi}{d\xi}).$$

Now, we will calculate the term $\frac{dA}{dv}|_{v=\xi}$. We have that this term is about the dependence of the operator T on ξ. It is easy to see that

$$\frac{d}{dv}|_{v=\xi} T(\varphi(u), v) = \frac{d}{dv}|_{v=\xi} \sum_{i \neq k} c_{k,l} e_k,$$

where e_k is the standard basis for $\mathbb{R}^K$, since T depends on ξ only in the term $c_{k,l}$. Also, we have that

$$c_{k,l} = \gamma \int_{E_{\varphi_l}} G(\xi_k + \sqrt{\rho_k^2 + \varphi_k(\theta)} e^{i\theta} - y) dy.$$

Now, since the closures of E_{φ_k} and E_{φ_l} are disjoint we get that the gradient of G exists and it is bounded, since G is smooth. Therefore, we get

$$\frac{d}{dv}|_{v=\xi} c_{k,l} = O(\gamma\rho^2).$$

Thus, we have

$$\Pi(T'(\varphi)(\frac{d\varphi}{d\xi}) + O(\gamma\rho^2)) = 0$$

So, we get

$$||\Pi T'(\varphi)(\frac{d\varphi}{d\xi})||_{L^2} = O(\gamma\rho^2).$$

Therefore, by Lemma 4.12 we get

$$||\frac{d\varphi}{d\xi}||_{H^2} \leq C\rho^3 ||\Pi T'(\varphi)(\frac{d\varphi}{d\xi})||_{L^2} \leq \rho^3 O(\gamma\rho^2) = O(\rho^2)$$

and the proof is complete. $\square$

Now we are in position to prove Proposition 2.10.

<u>Proof of Proposition 2.10</u>

Arguing as in [2], we set

$$\zeta_k + \sqrt{\rho_k^2 + \psi_k} e^{i\widetilde{\eta_k}} = \xi_k + \sqrt{\rho_k^2 + \varphi_k} e^{i\theta}, \tag{4.7}$$

where $\widetilde{\eta_k} = \widetilde{\eta_k}(\theta, \xi)$ and $\psi_k = \psi_k(\widetilde{\eta_k}, \xi)$. Here ζ_k is the minimum of the energy functional J, like in Lemma 4.10.

We set $E_{\psi_k} = E_{\varphi_k}$ and $\widetilde{\eta_k}(\theta, \xi) : (0, 2\pi) \to (0, 2\pi)$ is one to one and onto.

Let the function $f_k : A \to \mathbb{R}^2$, $A \subseteq \mathbb{R}^5$, with

$$A = \{(x_1, .., x_5) : x_2 > -\rho_k^2, x_3 \in (0, 2\pi)\}$$

and

$$f_k(x_1, .., x_5) = \zeta_k + \sqrt{\rho_k^2 + x_2} e^{ix_1} - (x_4, x_5) - \sqrt{\rho_k^2 + \varphi(x_3, x_4, x_5)} e^{ix_3}.$$

Then, we have by Equation (4.7)

$$f_k(\widetilde{\eta_k}(\theta,\xi),\psi_k(\widetilde{\eta_k}(\theta,\xi),\xi),\theta,\xi_{k,1},\xi_{k,2})=0.$$

Let

$$g_k(\theta,\xi_{k,1},\xi_{k,2})=(\widetilde{\eta_k}(\theta,\xi),\psi_k(\widetilde{\eta_k}(\theta,\xi),\xi)).$$

Then we get

$$f_k(g_k(\theta,\xi_{k,1},\xi_{k,2}),\theta,\xi_{k,1},\xi_{k,2})=0. \tag{4.8}$$

Now, since ζ is a minimum, we have that

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta}J(E_\varphi)=0 \tag{4.9}$$

for every k,j. We will prove in the end of the proof that

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta}J(E_\psi)=\frac{1}{2}\int_0^{2\pi}T(\psi)(\eta)\frac{d\psi}{d\xi_{k,j}}(\eta)d\eta. \tag{4.10}$$

We assume this result for the moment. Let $\lambda(\psi)=\frac{1}{2K\pi}\sum_{k=1}^K\int_0^{2\pi}T_k(\psi)(\widetilde{\eta}(\theta,\xi))d\theta$. Also, we have $\int_0^{2\pi}\psi_k(\eta)d\eta=0$, since the area of E_{φ_k} is $\pi\rho_k^2$, which gives that $\int_0^{2\pi}\frac{d\psi_k}{d\xi_{k,j}}(\eta)d\eta=0$. Thus, we get

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta}J(E_\psi)=\frac{1}{2}\sum_{k=1}^K\int_0^{2\pi}(T_k(\psi)(\eta)-\lambda(\psi))\frac{d\psi_k}{d\xi_{k,j}}(\eta)d\eta.$$

We claim that

$$T_l(\varphi)(\theta)=T_l(\psi)(\widetilde{\eta}(\theta,\xi)).$$

To prove this result we note that

$$W_l(\varphi)(\theta)=W_l(\psi)(\widetilde{\eta}(\theta,\xi))$$

since E_φ and E_ψ have the same boundary. Now,

$$\gamma\int_{E_{\psi_k}}G(\zeta_k+\sqrt{\rho_k^2+\psi(\widetilde{\eta}(\theta,\xi),\xi)}e^{i\widetilde{\eta}(\theta,\xi)}-y)dy=$$

$$\gamma \int_{E_{\varphi_k}} G(\xi_k + \sqrt{\rho_k^2 + \varphi_k(\theta)} e^{i\theta} - y) dy,$$

by the definition of E_ψ. Therefore, we can see easily that

$$T_l(\varphi)(\theta) = T_l(\psi)(\widetilde{\eta}(\theta, \xi)).$$

Now, we claim that $\lambda(\psi) = \lambda$, where λ is the constant of the Lemma 4.11. To show this, we use that

$$T_l(\psi)(\widetilde{\eta}(\theta, \xi)) - \lambda(\psi) = T_l(\varphi)(\theta) - \lambda(\psi).$$

Also,

$$\int_0^{2\pi} \sum_{k=1}^{K} T_k(\psi)(\widetilde{\eta}(\theta, \xi)) - K\lambda(\psi) d\theta = 0.$$

Therefore,

$$\int_0^{2\pi} \sum_{k=1}^{K} T_k(\varphi) - K\lambda(\psi) = 0.$$

Now, we use that

$$T_k(\varphi) = \lambda + \mu_k \cos\theta + \nu_k \sin\theta$$

to conclude that $\lambda(\psi) = \lambda$. Now, we have

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta} J(E_\psi) = \frac{1}{2} \sum_{k=1}^{K} \int_0^{2\pi} (T_k(\psi)(\eta) - \lambda(\psi)) \frac{d\psi_k}{d\xi_{k,j}}(\eta) d\eta.$$

Thus, by change of variables and setting $\eta = \widetilde{\eta}(\theta, \xi)$

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta} J(E_\psi) = \frac{1}{2} \sum_{k=1}^{K} \int_0^{2\pi} (T_k(\varphi)(\theta) - \lambda) \frac{d\psi_k}{d\xi_{k,j}}(\widetilde{\eta}) \frac{\partial \widetilde{\eta}}{\partial \theta} d\theta.$$

Now, we use the impicit function theorem to the Equation (4.8). Also, we define $\Psi_k(\theta, \xi) = \psi_k(\widetilde{\eta}_k(\theta, \xi), \xi)$. By implicit differentiation we get

$$\begin{bmatrix} \frac{\partial \widetilde{\eta}_k}{\partial \theta} & \frac{\partial \widetilde{\eta}_k}{\partial \xi_{k,1}} & \frac{\partial \widetilde{\eta}_k}{\partial \xi_{k,2}} \\ \\ \frac{\partial \Psi_k}{\partial \theta} & \frac{\partial \Psi_k}{\partial \xi_{k,1}} & \frac{\partial \Psi_k}{\partial \xi_{k,2}} \end{bmatrix} = - \begin{bmatrix} -\sqrt{\rho_k^2 + \Psi_k} \sin \widetilde{\eta}_k & \frac{\cos \widetilde{\eta}_k}{2\sqrt{\rho_k^2 + \Psi_k}} \\ \\ \sqrt{\rho_k^2 + \Psi_k} \cos \widetilde{\eta}_k & \frac{\sin \widetilde{\eta}_k}{2\sqrt{\rho_k^2 + \Psi_k}} \end{bmatrix}^{-1} \times$$

$$\times$$

$$\begin{bmatrix} -\frac{\cos\theta}{2\sqrt{\rho_k^2+\varphi_k}}\frac{\partial\varphi_k}{\partial\theta}+\sqrt{\rho_k^2+\varphi_k}\sin\theta, & -1-\frac{\cos\theta}{2\sqrt{\rho_k^2+\varphi_k}}\frac{\partial\varphi_k}{\partial\xi_{k,1}}, & -\frac{\cos\theta}{2\sqrt{\rho_k^2+\varphi_k}}\frac{\partial\varphi_k}{\partial\xi_{k,2}} \\ \\ -\frac{\sin\theta}{2\sqrt{\rho_k^2+\varphi_k}}\frac{\partial\varphi_k}{\partial\theta}-\sqrt{\rho_k^2+\varphi_k}\cos\theta, & -\frac{\sin\theta}{2\sqrt{\rho_k^2+\varphi_k}}\frac{\partial\varphi_k}{\partial\xi_{k,1}}, & -1-\frac{\sin\theta}{2\sqrt{\rho_k^2+\varphi_k}}\frac{\partial\varphi_k}{\partial\xi_{k,2}} \end{bmatrix} =$$

$$= 2\begin{bmatrix} \frac{\sin\widetilde{\eta}_k}{2\sqrt{\rho_k^2+\Psi_k}} & -\frac{\cos\widetilde{\eta}_k}{2\sqrt{\rho_k^2+\Psi_k}} \\ \\ -\sqrt{\rho_k^2+\Psi_k}\cos\widetilde{\eta}_k & -\sqrt{\rho_k^2+\Psi_k}\sin\widetilde{\eta}_k \end{bmatrix}\times$$

$$\times\begin{bmatrix} \sqrt{\rho_k^2+\varphi_k}\sin\theta+O(\rho^3), & -1+O(\rho) & O(\rho) \\ \\ -\sqrt{\rho_k^2+\varphi_k}\cos\theta+O(\rho^3), & O(\rho) & -1+O(\rho) \end{bmatrix}.$$

Here, we used that $||\varphi||_{H^2}=O(\rho^4)$ and also Lemma 4.13. Therefore, evaluating at $\xi=\zeta$, $\widetilde{\eta}=\theta$ and $\Psi=\varphi$ we get

$$\begin{bmatrix} \frac{\partial\widetilde{\eta}_k}{\partial\theta} & \frac{\partial\widetilde{\eta}_k}{\partial\xi_{k,1}} & \frac{\partial\widetilde{\eta}_k}{\partial\xi_{k,2}} \\ \\ \frac{\partial\Psi_k}{\partial\theta} & \frac{\partial\Psi_k}{\partial\xi_{k,1}} & \frac{\partial\Psi_k}{\partial\xi_{k,2}} \end{bmatrix}_{\xi=\zeta} =$$

$$\begin{bmatrix} 1+O(\rho^2) & -\frac{\sin\theta}{2\sqrt{\rho_k^2+\varphi_k}}+O(1) & \frac{\cos\theta}{\sqrt{\rho_k^2+\varphi_k}}+O(1) \\ \\ O(\rho^4) & 2\sqrt{\rho_k^2+\varphi_k}\cos\theta+O(\rho^2) & 2\sqrt{\rho_k^2+\varphi_k}\sin\theta+O(\rho^2) \end{bmatrix}.$$

Also, by using that $||\varphi||_{H^2}=O(\rho^4)$ we get $2\sqrt{\rho_k^2+\varphi_k}\cos\theta=2\rho_k\cos\theta+O(\rho^3)$ and $2\sqrt{\rho_k^2+\varphi_k}\sin\theta=2\rho_k\sin\theta+O(\rho^3)$. Thus, we have found that

$$\frac{\partial\Psi_k}{\partial\xi_{k,1}}|_{\xi=\zeta}=2\rho_k\cos\theta+O(\rho^2)$$

and

$$\frac{\partial\Psi_k}{\partial\xi_{k,2}}|_{\xi=\zeta}=2\rho_k\sin\theta+O(\rho^2).$$

Our goal is to compute $\frac{\partial \psi_k}{d\xi_{k,j}}|_{\xi=\zeta}$. For this purpose, we invert the function $\widetilde{\eta}_k$ and we consider the function $\widetilde{\theta}_k$ such that $\widetilde{\eta}_k(\widetilde{\theta}_k(\eta,\xi),\xi) = \eta$, for $\eta \in S^1$. Thus, we have

$$\frac{d}{d\xi_{k,j}}\widetilde{\eta}_k(\widetilde{\theta}_k(\eta,\xi),\xi) = 0.$$

Using this relation and using the Chain Rule, we can prove that

$$\frac{d\widetilde{\eta}_k}{d\theta}\frac{d\widetilde{\theta}_k}{d\xi_{k,j}} + \frac{d\widetilde{\eta}_k}{d\xi_{k,j}} = 0$$

and also we have already proved that $\frac{d\widetilde{\eta}_k}{d\theta} = 1 + O(\rho^2)$ and also

$$\frac{d\widetilde{\eta}_k}{d\xi_{k,j}} = \begin{cases} -\frac{\sin\theta}{\sqrt{\rho_k^2+\varphi_k}} + O(1), & j = 1 \\ \frac{\cos\theta}{\sqrt{\rho_k^2+\varphi_k}} + O(1), & j = 2 \end{cases}$$

Therefore,

$$\frac{d\widetilde{\theta}_k}{d\xi_{k,j}} = \begin{cases} \frac{-\frac{\sin\theta}{\sqrt{\rho_k^2+\varphi_k}}+O(1)}{1+O(\rho)}, & j = 1 \\ \frac{\frac{\cos\theta}{\sqrt{\rho_k^2+\varphi_k}}+O(1)}{1+O(\rho)}, & j = 2 \end{cases}$$

Now, using the Chain Rule again and the relation

$$\psi_k(\eta,\xi) = \Psi_k(\widetilde{\theta}_k(\eta,\xi),\xi)$$

we get

$$\frac{d\psi_k}{d\xi_{k,1}} = O(\rho^3)(\frac{-\frac{\sin\theta}{\sqrt{\rho_k^2+\varphi_k}} + O(1)}{1+O(\rho)}) + 2\rho_k\cos\eta + O(\rho^2) =$$

$$= \frac{O(\rho^3)(O(\frac{1}{\rho}) + O(1))}{1+O(\rho)} + 2\rho_k\cos\eta + O(\rho^2) =$$

$$= 2\rho_k\cos\eta + O(\rho^2)$$

and similarly we get

$$\frac{d\psi_k}{d\xi_{k,2}} = 2\rho_k\sin\eta + O(\rho^2).$$

When $l \neq k$, by implicit differentiation we get

$$\begin{bmatrix} \frac{\partial \widetilde{\eta}_l}{\partial \theta} & \frac{\partial \widetilde{\eta}_l}{\partial \xi_{k,1}} & \frac{\partial \widetilde{\eta}_l}{\partial \xi_{k,2}} \\ \\ \frac{\partial \Psi_l}{\partial \theta} & \frac{\partial \Psi_l}{\partial \xi_{k,1}} & \frac{\partial \Psi_l}{\partial \xi_{k,2}} \end{bmatrix}_{\xi=\zeta} =$$

$$= - \begin{bmatrix} -\sqrt{\rho_l^2 + \Psi_l} \sin \widetilde{\eta}_l & \frac{\cos \widetilde{\eta}_l}{2\sqrt{\rho_l^2+\Psi_l}} \\ \\ \sqrt{\rho_l^2 + \Psi_l} \cos \widetilde{\eta}_l & \frac{\sin \widetilde{\eta}_l}{2\sqrt{\rho_l^2+\Psi_l}} \end{bmatrix}_{\xi=\zeta}^{-1} \times$$

$$\times$$

$$\begin{bmatrix} -\frac{\cos\theta}{2\sqrt{\rho_l^2+\varphi_l}} \frac{\partial \varphi_l}{\partial \theta} + \sqrt{\rho_l^2+\varphi_l} \sin\theta, & -\frac{\cos\theta}{2\sqrt{\rho_l^2+\varphi_l}} \frac{\partial \varphi_l}{\partial \xi_{k,1}}, & -\frac{\cos\theta}{2\sqrt{\rho_l^2+\varphi_l}} \frac{\partial \varphi_l}{\partial \xi_{k,2}} \\ \\ -\frac{\sin\theta}{2\sqrt{\rho_l^2+\varphi_l}} \frac{\partial \varphi_l}{\partial \theta} - \sqrt{\rho_l^2+\varphi_l} \cos\theta, & -\frac{\sin\theta}{2\sqrt{\rho_l^2+\varphi_l}} \frac{\partial \varphi_l}{\partial \xi_{k,1}}, & -\frac{\sin\theta}{2\sqrt{\rho_l^2+\varphi_l}} \frac{\partial \varphi_l}{\partial \xi_{k,2}} \end{bmatrix}_{\xi=\zeta}$$

=

$$= 2 \begin{bmatrix} \frac{\sin\theta}{2\sqrt{\rho_l^2+\varphi_l}} & -\frac{\cos\theta}{2\sqrt{\rho_l^2+\varphi_l}} \\ \\ -\sqrt{\rho_l^2+\varphi_l}\cos\theta & -\sqrt{\rho_l^2+\varphi_l}\sin\theta \end{bmatrix} \times$$

$$\begin{bmatrix} \sqrt{\rho_l^2+\varphi_l}\sin\theta + O(\rho^3) & O(\rho) & O(\rho) \\ \\ -\sqrt{\rho_l^2+\varphi_l}\cos\theta + O(\rho^3) & O(\rho) & O(\rho) \end{bmatrix}.$$

Here, we used that $||\varphi||_{H^2} = O(\rho^4)$ and $||\frac{d\varphi}{d\xi_{k,j}}||_{H^2} = O(\rho^2)$. Thus, finally we get

$$\begin{bmatrix} \frac{\partial \widetilde{\eta}_l}{\partial \theta} & \frac{\partial \widetilde{\eta}_l}{\partial \xi_{k,1}} & \frac{\partial \widetilde{\eta}_l}{\partial \xi_{k,2}} \\ \\ \frac{\partial \Psi_l}{\partial \theta} & \frac{\partial \Psi_l}{\partial \xi_{k,1}} & \frac{\partial \Psi_l}{\partial \xi_{k,2}} \end{bmatrix}_{\xi=\zeta} = \begin{bmatrix} 1 + O(\rho^2) & O(1) & O(1) \\ \\ O(\rho^4) & O(\rho^2) & O(\rho^2) \end{bmatrix}.$$

Therefore, we proved that $\frac{d\Psi_l}{d\xi_{k,j}}|_{\xi=\zeta} = O(\rho^2)$.

Now, we invert the function $\widetilde{\eta}_l$ and we have

$$\widetilde{\eta}_l(\widetilde{\theta}_l(\xi,\theta),\xi) = \theta.$$

Thus, by using the Chain Rule we get

$$\frac{d\psi_l}{d\xi_{k,j}} = \frac{d\Psi_l}{d\xi_{k,j}} + \frac{d\Psi_l}{d\theta}\frac{d\widetilde{\theta}_l}{d\xi_{k,j}}.$$

At $\xi = \zeta$, we have

$$\frac{d\Psi_l}{d\theta} = O(\rho^4)$$

and also

$$\frac{d\widetilde{\theta}_l}{d\xi_{k,j}} = -\frac{\frac{d\widetilde{\eta}_l}{d\xi_{k,j}}}{\frac{d\widetilde{\eta}_l}{d\theta}} = O(1)$$

Therefore, finally

$$\frac{d\psi_l}{d\xi_{k,j}} = O(\rho^2).$$

Then we have by equation (4.9)

$$2\frac{dJ(E_\psi)}{d\xi_{k,1}}|_{\xi=\zeta} = \int_0^{2\pi}(T_k(\varphi)(\theta)-\lambda)(2\rho_k\cos\theta+O(\rho^2))d\theta+\sum_{l\neq k}\int_0^{2\pi}(T_l(\varphi)(\theta)-\lambda)O(\rho^2)d\theta = 0$$

$$2\frac{dJ(E_\psi)}{d\xi_{k,2}}|_{\xi=\zeta} = \int_0^{2\pi}(T_k(\varphi)(\theta)-\lambda)(2\rho_k\sin\theta+O(\rho^2))d\theta+\sum_{l\neq k}\int_0^{2\pi}(T_l(\varphi)(\theta)-\lambda)O(\rho^2)d\theta = 0.$$

Here, we used that $\frac{d\psi_l}{d\xi_{k,j}}|_{\xi=\zeta} = O(\rho^2)$ and also $\frac{d\widetilde{\eta}_l}{d\theta}|_{\xi=\zeta} = 1+O(\rho^2)$. Now, we use that $T_k(\varphi)(\theta) = \lambda + \mu_k\cos\theta + \nu_k\sin\theta$ and by taking approximations we get that $\mu_k = \nu_k = 0$ for every k.

The only thing it remains to prove is that

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta}J(E_\psi) = \frac{1}{2}\int_0^{2\pi}T(\psi)(\eta)\cdot\frac{d\psi}{d\xi_{k,j}}(\eta)d\eta.$$

To prove that we use the Taylor's expansion and have

$$\psi_k(\theta,\xi_{k,j}) = \psi_k(\theta,\zeta_{k,j}) + (\xi_{k,j}-\zeta_{k,j})\frac{d\psi_k(\theta,\epsilon)}{d\epsilon}|_{\epsilon=\zeta_{k,j}} + O((\xi_{k,j}-\zeta_{k,j})^2).$$

Also, we write

$$\psi = (\psi_1, .., \psi_K).$$

Therefore, $\psi(\theta, \xi_{k,j}) = \psi(\theta, \zeta_{k,j}) + (\xi_{k,j} - \zeta_{k,j}) \frac{d\psi}{d\epsilon}|_{\epsilon=\zeta_{k,j}} + O(|\xi_{k,j} - \zeta_{k,j}|^2)$. Let

$$A(\epsilon_1, \epsilon_2) = J(\psi(\theta, \zeta_{k,j}) + \epsilon_1 \frac{d\psi}{dx}|_{x=\zeta_{k,j}} + O(\epsilon_2^2)).$$

Thus, by using the Chain Rule and setting $\epsilon_1 = \epsilon_2 = \xi_{k,j} - \zeta_{k,j}$, we have

$$\frac{dA}{d\xi_{k,j}}|_{\xi=\zeta} = \frac{dA}{d\epsilon_1}|_{\epsilon_1=0} + \frac{dA}{d\epsilon_2}|_{\epsilon_2=0}.$$

Also,

$$\frac{dA}{d\epsilon_1}|_{\epsilon_1=0} = \frac{1}{2} \int_0^{2\pi} T_\zeta(\psi) \cdot \frac{d\psi}{dx}|_{x=\zeta_{k,j}} d\theta.$$

Also, we can see that

$$\frac{dA}{d\epsilon_2}|_{\epsilon_2=0} = 0.$$

Thus, finally

$$\frac{d}{d\xi_{k,j}}|_{\xi=\zeta} J(E_\psi) = \frac{1}{2} \int_0^{2\pi} T(\psi)(\eta) \cdot \frac{d\psi}{d\xi_{k,j}}(\eta) d\eta.$$

and Equation (4.10) holds. Therefore, we have a non-singular linear system, which implies that $\mu_1 = ... = \mu_K = \nu_1 = ... = \nu_K = 0$ and the proof is complete. □

Chapter 5

Finding the centers of the disks

In this Chapter we will prove the remaining results of the Second Problem. For the next propositions we assume that $K = 2$ and we will turn our attention on the global minima of the function $F(\xi_1, \xi_2) = G(\xi_2 - \xi_1)$. This is because we are interested in locating the centers (ζ_1, ζ_2) of the stationary set E_φ, since by Theorem 2.11 the centers of the stationary set E_φ are close to a global minimum of the function F.

Lemma 5.1. *Let $\omega_1 = 1$ and $\omega_2 = \zeta$ and consider the lattice generated by ω_1 and ω_2, where $\zeta = \frac{b}{a}$. Also, consider the lattice generated by a and b, with $\frac{b}{a} = \zeta$. Let G_1 and G_2 be the Green's functions corresponding to the first and second lattice respectively. Then, we have $G_2(az) = G_1(z)$ for every z.*

Proof. We have

$$G_2(az) = -\frac{1}{2\pi} \log \frac{2\pi|az|}{\sqrt{|D_2|}} + \frac{|az|^2}{4|D_2|} + H_2(az).$$

We can see that $|D_2| = Im(\bar{a}b) = |a|^2 Im(\zeta)$. Also, we can see easily that $|D_1| = Im(\zeta)$.

Therefore,

$$G_2(az) = -\frac{1}{2\pi}\log\frac{2\pi|z|}{\sqrt{|D_1|}} + \frac{|z|^2}{4|D_1|} + H_2(az).$$

Therefore, to prove the result it suffices to prove that $H_2(az) = H_1(z)$. Now,

$$H_2(az) = -\frac{1}{2\pi}\log|e(\frac{\bar{a}az^2}{4|D_2|i}| - \frac{z}{2} + \frac{\zeta}{12}) - \frac{1}{2\pi}\log|\frac{(1-e(z))\sqrt{|D_2|}}{2\pi az}|$$
$$-\frac{1}{2\pi}\log|\prod_{n=1}^{\infty}(1-e(n\zeta+z))(1-e(n\zeta-z))|.$$

Now, use that $|D_2| = |a|^2 Im(\zeta) = |a|^2|D_1|$ to get $H_2(az) = H_1(z)$ and the proof is complete. $\square$

By the Lemma 5.1 we have that G_1 attains a global minimum at z if and only if G_2 attains a global minimum at az.

Remark 5.2. *Now, we take the case $\zeta = \frac{b}{a} = e^{\frac{\pi i}{3}}$. Following [7] we have that for the lattice generated by $\omega_1 = 1$ and $\omega_2 = e^{\frac{\pi i}{3}}$ the global minimums of the Green's function G are attained for $\pm\frac{1+e^{\frac{\pi i}{3}}}{3}$. Therefore, by the Lemma 5.1 and for the lattice generated by a and b with $\zeta = \frac{a}{b} = e^{\frac{\pi i}{3}}$ the global minimums of G in $\overline{D}$ are attained in the points $\frac{a+b}{3}$ and $\frac{2(a+b)}{3}$.*

Consider now the lattice generated by $\omega_1 = 1$ and $\omega_2 = ti$, for $t > \frac{4\log 2}{\pi}$. By [7], we have that the only critical points of the Green's function associated with this lattice are the points $\frac{1}{2}$, $\frac{ti}{2}$ and $\frac{1+ti}{2}$. Therefore, since the global minimum of G exists, we have that the global minimum is attained in the one of these three points which has the smaller value.

Let's first introduce the Dedekind eta function from [6], which is defined as $\eta : \{z \in \mathbb{C} : Im(z) > 0\} \to \mathbb{C}$ with

$$\eta(z) = e^{\frac{\pi iz}{12}}\prod_{n=1}^{\infty}(1-e(nz)). \tag{5.1}$$

This function is central in the theory of elliptic modular functions in Number Theory.

Lemma 5.3. *Assume $\omega_1 = 1$ and $\omega_2 = ti$, where $t > 0$. Then for the critical points of the Green's function $\frac{1}{2}$, $\frac{ti}{2}$ and $\frac{1+ti}{2}$, we have*

$$G(\frac{1}{2}) = \frac{t}{12} - \frac{1}{2\pi}\log 2 - \frac{1}{\pi}\log\prod_{n=1}^{\infty}(1 + exp(-2\pi nt)) \tag{5.2}$$

$$G(\frac{ti}{2}) = -\frac{1}{\pi}\log|\frac{\eta(\frac{ti}{2})}{\eta(ti)}| \tag{5.3}$$

and

$$G(\frac{1+ti}{2}) = -\frac{1}{\pi}\log|\frac{\eta(\frac{1+ti}{2})}{\eta(ti)}|. \tag{5.4}$$

Proof. We have

$$G(\frac{1}{2}) = -\frac{1}{2\pi}\log\frac{\pi}{\sqrt{t}} + \frac{1}{16t} + H(\frac{1}{2}).$$

Also, $H(\frac{1}{2}) = H_1(\frac{1}{2}) + H_2(\frac{1}{2}) + H_3(\frac{1}{2})$. We have

$$H_1(\frac{1}{2}) = -\frac{1}{2\pi}\log|e(\frac{1}{16ti} - \frac{1}{4} + \frac{ti}{12})| = -\frac{1}{2\pi}\log|exp(2\pi i(-\frac{1}{16t}i + \frac{ti}{12} - \frac{1}{4})| =$$

$$= -\frac{1}{2\pi}Re(\frac{1}{8t}\pi - \frac{\pi t}{6} - \frac{\pi i}{2}) = -\frac{1}{16t} + \frac{t}{12}.$$

Also,

$$H_2(\frac{1}{2}) = -\frac{1}{2\pi}\log|\frac{\sqrt{t}(1-e(\frac{1}{2}))}{\pi}| = -\frac{1}{2\pi}\log 2 - \frac{1}{2\pi}\log\frac{\sqrt{t}}{\pi}.$$

Now,

$$H_3(\frac{1}{2}) = -\frac{1}{2\pi}\log|\prod_{n=1}^{\infty}(1 - e(nti + \frac{1}{2}))(1 - e(nti - \frac{1}{2})) = -\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1 + exp(-2\pi nt)).$$

Finally, if we put it all together we proved that

$$G(\frac{1}{2}) = \frac{t}{12} - \frac{1}{2\pi}\log 2 - \frac{1}{\pi}\log\prod_{n=1}^{\infty}(1 + exp(-2\pi nt)).$$

and Equation (5.2) holds. Now,

$$G(\frac{ti}{2}) = -\frac{1}{2\pi}\log\frac{\pi t}{\sqrt{t}} + \frac{t}{16} + H(\frac{ti}{2})$$

and also

$$H(\frac{ti}{2}) = H_1(\frac{ti}{2}) + H_2(\frac{ti}{2}) + H_3(\frac{ti}{2}).$$

Now,

$$H_1(\frac{ti}{2}) = -\frac{1}{2\pi}\log|e(\frac{(\frac{ti}{2})^2}{4ti} - \frac{ti}{4} + \frac{ti}{12})| = -\frac{1}{2\pi}\log|e(-\frac{5ti}{48})| = -\frac{5t}{48}.$$

Now,

$$H_2(\frac{ti}{2}) + H_3(\frac{ti}{2}) = -\frac{1}{2\pi}\log\frac{1}{\pi\sqrt{t}} - \frac{1}{2\pi}\log|\prod_{n=0}^{\infty}(1 - e(nti + \frac{ti}{2}))\prod_{n=1}^{\infty}(1 - e(nti - \frac{ti}{2}))| =$$

$$= -\frac{1}{2\pi}\log\frac{1}{\pi\sqrt{t}} - \frac{1}{2\pi}\log|\prod_{n=0}^{\infty}(1 - e(\frac{ti}{2}(2n+1)))\prod_{n=1}^{\infty}(1 - e(\frac{ti}{2}(2n-1)))| =$$

$$= -\frac{1}{2\pi}\log\frac{1}{\pi\sqrt{t}} - \frac{1}{\pi}\log|\prod_{n=1}^{\infty}(1 - e(\frac{ti}{2}(2n-1)))| =$$

$$= -\frac{1}{2\pi}\log\frac{1}{\pi\sqrt{t}} - \frac{1}{\pi}\log|\prod_{n=1}^{\infty}(1 - exp(-\pi t(2n-1)))|.$$

Also, we have

$$|\prod_{n=1}^{\infty}(1 - exp(-\pi t(2n-1)))| = |\frac{\prod_{n=1}^{\infty}(1 - exp(-n\pi t))}{\prod_{n=1}^{\infty}(1 - exp(-2n\pi t))}| = |\frac{\eta(\frac{ti}{2})}{\eta(ti)}exp(-\frac{\pi t}{24})|.$$

Now, if we put it all together we get

$$G(\frac{ti}{2}) = \frac{t}{16} - \frac{5t}{48} + \frac{t}{24} - \frac{1}{\pi}\log|\frac{\eta(\frac{ti}{2})}{\eta(ti)}| = -\frac{1}{\pi}\log|\frac{\eta(\frac{ti}{2})}{\eta(ti)}|,$$

where the function η is defined in Equation (5.1). Therefore, Equation (5.3) holds. Now, we will calculate $G(\frac{1+ti}{2})$. We have,

$$G(\frac{1+ti}{2}) = -\frac{1}{2\pi}\log\pi\frac{\sqrt{1+t^2}}{\sqrt{t}} + \frac{1+t^2}{16t} + H(\frac{1+ti}{2}).$$

Then,

$$H(\frac{1+ti}{2}) = H_1(\frac{1+ti}{2}) + H_2(\frac{1+ti}{2}) + H_3(\frac{1+ti}{2}).$$

$$H_1(\frac{1+ti}{2}) = -\frac{1}{2\pi}\log|e(\frac{(\frac{1+ti}{2})^2}{4ti} - \frac{1+ti}{4} + \frac{ti}{12}| =$$

$$= -\frac{1}{2\pi}\log|e(\frac{t^2-1}{16t}i - \frac{ti}{6} - \frac{1}{8})| = \frac{t}{16} - \frac{t}{6} - \frac{1}{16t}.$$

Now,

$$H_2(\frac{1+ti}{2}) + H_3(\frac{1+ti}{2}) = -\frac{1}{2\pi}\log\frac{\sqrt{t}}{\pi\sqrt{t^2+1}}$$

$$-\frac{1}{2\pi}\log|\prod_{n=0}^{\infty}(1-e(nti+\frac{1+ti}{2}))\prod_{n=1}^{\infty}(1-e(nti-\frac{1+ti}{2}))|$$

$$= -\frac{1}{2\pi}\log\frac{\sqrt{t}}{\pi\sqrt{t^2+1}} - \frac{1}{2\pi}\log|\prod_{n=0}^{\infty}(1+exp(-2\pi nt-\pi t)\prod_{n=1}^{\infty}(1+exp(-2\pi nt+\pi t)| =$$

$$= -\frac{1}{2\pi}\log\frac{\sqrt{t}}{\pi\sqrt{t^2+1}} - \frac{1}{2\pi}\log\prod_{n=1}^{\infty}(1+exp(-\pi t(2n-1)))^2 =$$

$$= -\frac{1}{2\pi}\log\frac{\sqrt{t}}{\pi\sqrt{t^2+1}} - \frac{1}{\pi}\log\prod_{n=1}^{\infty}(1+exp(-\pi t(2n-1))).$$

Now,

$$|\prod_{n=1}^{\infty}(1+exp(-\pi t(2n-1)))| = |\prod_{n=1}^{\infty}(1-exp(-\pi t(2n-1)+(2n-1)\pi i))| =$$

$$= |\frac{\prod_{n=1}^{\infty}(1-exp(2\pi ni(\frac{i-t}{2i})))}{\prod_{n=1}^{\infty}(1-exp(2\pi ni(\frac{i-t}{i})))}| = |\frac{\prod_{n=1}^{\infty}(1-exp(2\pi ni(\frac{1+ti}{2})))}{\prod_{n=1}^{\infty}(1-exp(2\pi ni(1+ti)))}| =$$

$$= |\frac{\eta(\frac{1+ti}{2})}{\eta(1+ti)}|\frac{exp(\frac{\pi t}{24})}{exp(\frac{\pi t}{12})}.$$

Therefore, finally we have

$$H_2(\frac{1+ti}{2}) + H_3(\frac{1+ti}{2}) = \frac{t}{24} - \frac{1}{\pi}\log|\frac{\eta(\frac{1+ti}{2})}{\eta(1+ti)}|.$$

If we put it all together we have

$$G(\frac{1+ti}{2}) = \frac{t}{8} - \frac{t}{6} + \frac{t}{24} - \frac{1}{\pi}\log|\frac{\eta(\frac{1+ti}{2})}{\eta(1+ti)}| = -\frac{1}{\pi}\log|\frac{\eta(\frac{1+ti}{2})}{\eta(ti)}|$$

and Equation (5.4) holds. □

Lemma 5.4. *Consider the lattice generated by $\omega_1 = 1$ and $\omega_2 = ti$, where $t > 0$. Then for the critical points $\frac{1}{2}$, $\frac{ti}{2}$ and $\frac{1+ti}{2}$ of the Green's function G associated with this lattice we have*

$$G(\frac{1+ti}{2}) < G(\frac{ti}{2})$$

for every $t > 0$ and also

$$G(\frac{1+ti}{2}) < G(\frac{1}{2})$$

for $t > \frac{4\log 2}{\pi} = 0.8825....$ Therefore for $t > \frac{4\log 2}{\pi}$ G has a unique global minimum in $\overline{D}$ attained at the point $\frac{1+ti}{2}$, with value

$$G(\frac{1+ti}{2}) = -\frac{1}{\pi}\log|\frac{\eta(\frac{1+ti}{2})}{\eta(ti)}|.$$

Proof. We have

$$G(\frac{1}{2}) = \frac{t}{12} - \frac{1}{2\pi}\log 2 - \frac{1}{\pi}\log\prod_{n=1}^{\infty}(1 + exp(-2\pi nt))$$

$$G(\frac{ti}{2}) = -\frac{1}{\pi}\log|\frac{\eta(\frac{ti}{2})}{\eta(ti)}|$$

and

$$G(\frac{1+ti}{2}) = -\frac{1}{\pi}\log|\frac{\eta(\frac{1+ti}{2})}{\eta(ti)}|$$

by Lemma 5.3. We have

$$G(\frac{ti}{2}) - G(\frac{1+ti}{2}) = -\frac{1}{\pi}\log|\frac{\eta(\frac{ti}{2})}{\eta(\frac{1+ti}{2})}|.$$

$$|\frac{\eta(\frac{ti}{2})}{\eta(\frac{1+ti}{2})}| = |exp(\frac{\pi i}{24})||\frac{\prod_{n=1}^{\infty}(1 - e(\frac{nti}{2}))}{\prod_{n=1}^{\infty}(1 - (-1)^n e(\frac{nti}{2}))} =$$

$$= \prod_{n=1}^{\infty}\frac{1 - exp(-\pi nt)}{1 - (-1)^n exp(-\pi nt)}.$$

We have

$$1 - exp(-\pi nt) \le 1 - (-1)^n exp(-\pi nt),$$

where we have strict inequality for n odd. Thus,

$$\prod_{n=1}^{\infty} \frac{1 - exp(-\pi nt)}{1 - (-1)^n exp(-\pi nt)} < 1,$$

which gives $G(\frac{ti}{2}) > G(\frac{1+ti}{2})$ for every $t > 0$. For the second part, we have

$$G(\frac{1}{2}) = \frac{t}{12} - \frac{1}{2\pi} \log 2 - \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 + exp(-2\pi nt)).$$

Also,

$$G(\frac{1+ti}{2}) = -\frac{1}{\pi} \log |exp(\frac{\pi t}{24}) \prod_{n=1}^{\infty} \frac{1 - (-1)^n e(\frac{nti}{2})}{1 - e(nti)}| = -\frac{t}{24} - \frac{1}{\pi} \log | \prod_{n=1}^{\infty} \frac{1 - (-1)^n e(\frac{nti}{2})}{1 - e(nti)}|.$$

Thus,

$$G(\frac{1}{2}) - G(\frac{1+ti}{2}) = \frac{t}{12} + \frac{t}{24} - \frac{1}{2\pi} \log 2 - \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 + exp(-2\pi nt)) +$$

$$+ \frac{1}{\pi} \log | \prod_{n=1}^{\infty} \frac{1 - (-1)^n e(\frac{nti}{2})}{1 - e(nti)}| =$$

$$= \frac{t}{8} - \frac{\log 2}{2\pi} - \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 + exp(-2\pi nt)) - \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 - exp(-2\pi nt)) +$$

$$+ \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 - (-1)^n exp(-\pi nt)).$$

Now, we use that

$$(1 - exp(-2\pi nt))(1 + exp(-2\pi nt)) = 1 - exp(-4\pi nt).$$

Thus,

$$G(\frac{1}{2}) - G(\frac{1+ti}{2}) = \frac{t}{8} - \frac{\log 2}{2\pi} - \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 - exp(-4\pi nt)) +$$

$$+ \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 - (-1)^n exp(-\pi nt)).$$

Now we have

$$\frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 - (-1)^n exp(-\pi nt)) = \frac{1}{\pi} \log \prod_{n=1}^{\infty} (1 - exp(-2\pi nt)) +$$

$$+\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1+exp(-\pi(2n-1)t)).$$

Thus,

$$G(\frac{1}{2})-G(\frac{1+ti}{2})=\frac{t}{8}-\frac{\log 2}{2\pi}-\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1-exp(-4\pi nt))+\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1-exp(-2\pi nt))+$$

$$+\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1+exp(-\pi(2n-1)t)).$$

Also,

$$-\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1-exp(-4\pi nt))+\frac{1}{\pi}\log\prod_{n=1}^{\infty}(1-exp(-2\pi nt))=-\frac{1}{\pi}\log\prod_{n=1}^{\infty}\frac{1-exp(-4\pi nt)}{1-exp(-2\pi nt))}=$$

$$=-\frac{1}{\pi}\log\prod_{n=1}^{\infty}1+exp(-2\pi nt).$$

Therefore,

$$G(\frac{1}{2})-G(\frac{1+ti}{2})=\frac{t}{8}-\frac{\log 2}{2\pi}-\frac{1}{\pi}\log\prod_{n=1}^{\infty}1+exp(-2\pi nt)+\frac{1}{\pi}\log\prod_{n=1}^{\infty}1+exp(-2\pi nt+\pi t)$$

and also we can see that $exp(-2\pi nt) < exp(-2\pi nt+\pi t)$, for $t > 0$. Thus,

$$G(\frac{1}{2})-G(\frac{1+ti}{2}) > \frac{t}{8}-\frac{\log 2}{2\pi} > 0,$$

for $t > \frac{4\log 2}{\pi}$. □

Proof of Proposition 2.12

By the Lemmas 5.4 and 5.1 and also by Remark 5.2 the proof is complete. □

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