How to create a universe

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Abstract

Creating a universe out of nothing seems an impossible task. A closer investigation shows that some difficult steps pose problems, but most steps are feasible and realistic. This document explains the realistic steps and analyses the steps that pose problems.

The analysis intensively applies the concept of vector space and the extensions of this concept, such as number systems, the Hilbert space, and the Hilbert repository. Set theory plays an important role in the exploration of the behavior of the universe. The paper shows that it is impossible to comprehend physical reality without comprehending these parts of mathematics. The paper also shows that the mentioned mathematical concepts are not yet fully explored and might reveal more of the secrets of physical reality.

For comprehending physical reality, it is crucial to be able to grasp the change in behavior of a set of numbers when irrational numbers are added to the rational numbers. This occurs in the one-dimensional real numbers and the one-dimensional or three-dimensional spatial numbers. It appears that the change in the behavior of spatial numbers explains the origin of gravity. The typical behavior also explains the existence of dark energy objects and dark matter objects.

1 Introduction

1.1 Abridgment

This paper will be kept compact and sufficiently comprehensive. More details and citations were published in a previous paper. They will not be repeated here, but the referred paper offers all information that cannot easily be found online in free accessible publications. Further, the paper refers to most applied concepts that are accessible free online. The author advises getting familiar with the results of great mathematicians that contributed to set theory, number theory, vector spaces, and Hilbert spaces. This paper builds on their results. The paper will not offer a foundation to currently accepted theoretical physical theories. For example, the paper does not explain the least action principle that leads to the Lagrangian equations and Hamiltonian equations. Instead, the paper explains the field equations that describe the behavior of the objects that exist in the universe.

1.2 Humans, science, and physical reality

Humans cannot perform science without extensive linguistic tools such as identifications and descriptions of structures and behaviors. This document uses number systems to navigate space in a huge number of locations. Humans use both text and formulas to be able to describe behavior. We need to realize that physical reality can find its way without these tools. Humans can take an intelligent approach. Physical reality probably uses an approach of trial and error that is supported by stochastic mechanisms. That way of working must be sufficiently efficient because the universe has been running for billions of years without it coming to a standstill.

1.3 Plan

We start from complete nothingness. A candidate for complete nothingness is empty space. Empty space contains nothing to which can be referred. On the other hand, if it is possible to consider space as a container from which everything that it contains can be stripped, then is empty space a realistic concept. So, if space exists, then we turn it into empty space and after that, we start by turning the empty space into a vector space by adding two point-like objects that are connected by a direction line. One of the point-like objects is the base point of the vector. The other point-like object is the pointer of the vector. If the vector is shifted such that it does not change its length and its direction, then the integrity of the vector is conserved. With the help of such vectors, every location in the vector space can be reached.

2 Number systems

We intend to apply the vector space to cover the empty space with one or more associated number systems that become the base of a coordinate system that helps us to navigate in the vector space. The coordinate markers will be point-like objects that will be identified with the corresponding element of the number system. After finishing the construction of the number system, the point-like coordinate markers will be detached from the numbers, but these markers will keep their identification with the number. In this way, the life story of the coordinate marker can be followed via this identifier. For multidimensional numbers, the real part of the number will act as progression.

With the currently available ingredients, it is possible to shift the vector along the direction line until the base point locates at the old location of the pointer. This action creates two new vectors. One of them is the vector that consists of the old base point and the new pointer. Now three point-like objects, two lengths of vectors, and the direction line are involved. Repeating the shift adds another point-like object. The shifts introduce the procedures of counting and addition. The point-like objects are members of an ordered set. An ongoing shift generates the set of the natural numbers. It is obvious that this set is countable. We shall call a set countable when every member of the set can be labeled with a natural number. Reversing the shift introduces the subtraction procedure. If counting is reversed, then the point will be reached that the space is empty again. This is the reason to give the first base point a special identifier. It will be called point zero. If subtraction proceeds past point zero, then the negative integer numbers are introduced. Together with zero and the natural numbers, the negative numbers form the set of the integer numbers.

We can add multiple shifts in one action or shifts that involve longer vectors. This does not introduce new integer numbers. It introduces a procedure that in arithmetic is called multiplication. The reverse procedure is called division. Division can introduce new numbers that can be interpreted as ratios. Numbers that can be interpreted as ratios are called rational numbers. Rational numbers form classes of numbers that in addition and multiplication feature the same value. Each class corresponds with a location in space. Scientists have proven that every rational number class can be labeled with a natural number. Thus, the set of rational number classes is countable. Without the addition of other numbers, all rational numbers appear to be surrounded by empty space. On the applied direction line still, abundant empty space is left to insert other numbers. It is possible to add a coordinate marker to each rational number class. The coordinate marker links the identity of the rational number class to an actual point-like object. Via their values, the rational number classes form an ordered set. Chapter 3 will show the arithmetic in formulas. We are interested in the behavior of the set of the point-shaped objects that we call coordinate markers above.

2.1 Superseding countability

Rational numbers can be squared. The result is again a rational number that results when the number is multiplied by itself. The reverse procedure is called square root. The square root need not result in a rational number. However, a converging sequence of rational numbers can approach the result arbitrarily close. If the converging series does not result in an existing rational number, then the result is called an irrational number. For many situations, a converging series of rational numbers does not result in a rational number. The missing number belongs to the irrational numbers. The set of the irrational numbers cannot be counted. This set is uncountable. If all irrational numbers are added to the selected direction line, then none of the numbers on this line is surrounded by empty space. This fact causes a significant change in several aspects of the coverage of the direction line. The combination of the rational numbers and the irrational numbers forms the set of what will be called the real numbers. Like the rational numbers, the real numbers form classes that in arithmetic feature the same value. The real number classes form an ordered set. The direction line together with the real numbers acts as a mathematical continuum. In this continuum each converging series of real numbers owns a corresponding limit. Therefore, the continuum enables differentiation. Differentiation shows that the continuum can change. To be able to show this change we apply the possibility to detach the locations of the coordinate markers from the virtual location of the original real number class. This means that the coordinate markers can describe deformations, vibrations, and expansions of the coverage of the direction line. The coordinate markers act as the target values of a function that uses the original coverage of the direction line as its parameter space. Differential calculus uses this function to describe in detail the deformation, vibration, and expansion of the current state of the coverage of the direction line. The function relates to its parameter space in the same way as the coordinate system relates to the generated real number system. The coordinate markers will react to disturbances that cause the dynamic behavior. The generated number system will only be used for providing the identifiers for the coordinate markers. Before detachment the coordinate markers will not be disturbed. By detaching the coordinate markers from the corresponding numbers, we are only considering the behavior of the actual point-like coordinate markers. The detached numbers only act as identifiers.

The description of the dynamic behavior applies an independent progression indicator that runs monotone with the undeformed natural numbers. This indicator is represented in the vector space by the generated natural numbers.

2.2 Spatial numbers

The construction of the universe now proceeds from the condition that the selected direction line is fully covered by real numbers. This situation does not allow the insertion of large sets of new numbers to that same direction line.

In the example, the square root of a positive rational number was added. If the square roots of negative rational numbers must be represented, then this extra set of numbers must be placed on an independent direction line. The new numbers show different arithmetic because their squares result in a negative real number, while the square of a real number always results in zero or a positive real number. We call these new numbers spatial numbers. The two involved direction lines may cross at point zero. This choice enables the introduction of the addition of a real number and a spatial number. The mixed numbers then span a plane in a two-dimensional vector space. Mathematicians call these mixed numbers complex numbers.

It is possible to add an extra independent direction line that is also covered with spatial numbers. The spatial arithmetic immediately introduces via the spatial product rule a fourth independent direction line that also is covered with spatial numbers. The resulting number system is known as the quaternionic number system. It consists of a one-dimensional real number subspace and an isotropic three-dimensional spatial subspace. Squaring a spatial number still results in a negative real number. Multiplication of a spatial number by a real number results in a spatial number with the same or opposite direction. Multiplication of two spatial numbers results in a combination of a real number and a new spatial number. The direction of the new spatial number is independent of the directions of the two factors. The real part of the result is called the inner product and the spatial part of the result of the multiplication is called the outer product.

Like real numbers, spatial numbers can be integer, rational or irrational. In spatial numbers, the insertion of all irrational numbers has drastic effects on the behavior of the coverage of space. Like vibrations will deformations and expansions still be possible, but deformations will be removed as quickly as possible by sending them in the form of shock fronts away into all available directions until they vanish at infinity. Because the deformation vanishes at infinity, sudden deformations will be removed quickly and finally turn into the expansion of the spatial part of the space coverage. This means that expansion evolves slowly and gradually. If the causes of deformations are spatially randomly distributed, then expansion is an isotropic effect.

By inspecting the differential equations, it becomes clear that the real parts of the mixed numbers act as progression indicators. The introduction of all irrational numbers not only turns the space coverage into a continuum. It also makes that continuum differentiable and attributes to the space coverage a very special behavior. This very special behavior will explain the origin of gravity. It explains the existence of dark matter and the existence of dark energy.

2.3 Freedom of choice

In creating the number systems. we made several choices. This started with the location and direction of the first vector. Also, on the direction line, we could have shifted up or down. This means that number systems exist in several versions that distinguish in their symmetry. We will use coordinate systems to establish the symmetry of the version. The selected coordinate system removes all selection freedom.

2.4 Dynamics

It is possible to interpret multidimensional numbers as a combination of a scalar timestamp and a one-dimensional or three-dimensional spatial location. The progression indicator runs monotonic with the natural numbers on the real number direction line. In this way, the deformation, vibration, and expansion of the corresponding coordinate system become dynamic behaviors where the timestamps play the role of the progression indicator.

3 Mixed arithmetic

The real number arithmetic and the spatial number arithmetic can be mixed. Spatial numbers that reside on different spatial direction lines can be added and multiplied. This will make the spatial number space of the quaternions isotropic. The coordinate markers will capture the geometric symmetry and the location of the geometric center. This means that the coordinate system removes all selection freedom from the chosen number system. In this way, the coordinate system substantiates the version of the number system until the number system and the coordinate markers are detached. Real numbers can be added and multiplied by spatial numbers.

We will use a vector cap to indicate the spatial part and we will indicate the scalar part with suffix $_r$.

Thus, the number a will be represented by the sum $a = a_r + \vec{a}$. This means that the product $c = ab$ of two numbers a and b will be split into several terms

$$\begin{aligned} c &= c_r + \vec{c} = ab = \left(a_r + \vec{a}\right)\left(b_r + \vec{b}\right) \\ &= a_r b_r + a_r \vec{b} + \vec{a} b_r + \vec{a}\vec{b} \end{aligned} \tag{3.1.1}$$

For real numbers the addition and multiplication are commutative and associative.

$$\begin{aligned} b_r + a_r &= a_r + b_r \\ \left(a_r + b_r\right) + c_r &= a_r + \left(b_r + c_r\right) \end{aligned} \tag{3.1.2}$$

$$\begin{aligned} b_r a_r &= a_r b_r \\ \left(a_r b_r\right) c_r &= a_r \left(b_r c_r\right) \end{aligned} \tag{3.1.3}$$

The addition and multiplication of real numbers with spatial numbers is commutative.

$$a_r + \vec{b} = \vec{b} + a_r$$
$$a_r \vec{b} = \vec{b} a_r \tag{3.1.4}$$

The product d of two spatial numbers $\vec{a}$ and $\vec{b}$ results in a real scalar part d_r and a new spatial part $\vec{d}$

$$d = d_r + \vec{d} = \vec{a}\vec{b} \tag{3.1.5}$$

$d_r = -\langle \vec{a}, \vec{b} \rangle$ is the inner product of $\vec{a}$ and $\vec{b}$

For the inner product and the norm $\|\vec{a}\|$ holds $\langle \vec{a}, \vec{a} \rangle = \|\vec{a}\|^2$

$$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos(\alpha) \tag{3.1.6}$$

$\vec{d} = \vec{a} \times \vec{b}$ is the outer product of $\vec{a}$ and $\vec{b}$

The spatial vector $\vec{d}$ is independent of $\vec{a}$ and independent of $\vec{b}$. This means that $\langle \vec{a}, \vec{d} \rangle = 0$ and $\langle \vec{b}, \vec{d} \rangle = 0$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| |\sin(\alpha)|$$
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \tag{3.1.7}$$

Mathematics often treats spatial numbers as vectors because their behavior corresponds to a large degree to other types of vectors. Also, large differences exist between types of vectors. Mathematics defines the inner product of vectors that represent spatial numbers as the above geometric scalar vector product (3.1.6). It is also called the dot product of two vectors. Hilbert spaces define a different kind of inner product. It is important to distinguish between the inner product in spatial number systems and the inner product in Hilbert spaces. Hilbert spaces will be treated in a later chapter.

Only three mutually independent spatial number parts can be involved in the outer product.

These formulas still do not determine the sign of the outer product. Apart from that sign, the outer product is fixed.

Quaternionic multiplication obeys the equation

$$\begin{aligned} c = c_r + \vec{c} = ab = \left(a_r + \vec{a}\right)\left(b_r + \vec{b}\right) \\ = a_r b_r - \left\langle \vec{a}, \vec{b} \right\rangle + a_r \vec{b} + \vec{a} b_r \pm \vec{a} \times \vec{b} \end{aligned} \tag{3.1.8}$$

The $\pm$ sign indicates the freedom of choice of the handedness of the product rule that exists when selecting a version of the quaternionic number system. In this way the handedness of the product rule is treated as a special kind of symmetry. The version must be selected before it can be used in calculations.

Two quaternions that are each other's inverse can rotate the spatial part of another quaternion.

$$c = ab / a \tag{3.1.9}$$

The construct rotates the spatial part of b that is perpendicular to $\vec{a}$ over an angle that is twice the angular phase θ of $a = \|a\| e^{\vec{i}\theta}$ where $\vec{i} = \vec{a} / \|\vec{a}\|$.

Cartesian quaternionic functions apply a quaternionic parameter space that is sequenced by a Cartesian coordinate system. In the parameter space, the real scalar parts of quaternions are often interpreted as instances of (proper) time, and the spatial parts are often interpreted as spatial locations. The real scalar parts of quaternionic functions represent dynamic scalar fields. The spatial parts of quaternionic functions represent dynamic vector fields.

The vectors that occur in the number systems as spatial numbers differ at essential points from the vectors defined in the underlying vector space. In particular, the product of vectors differs between the two types of vectors. So, there is a difference between the arithmetic of these two types of vectors. This is exploited by the bra-ket combination that is introduced by Paul Dirac.

4 Symmetries

During the generation of each of the number systems, the selected coordinate system substantiated several choices that beforehand were free. An important choice is the location of point zero. The coordinate system takes this location as its geometric center. The first direction line has no spatial direction but the shift on this direction line can be upwards or downwards. The third choice concerns the spatial direction of the first spatial direction line. That choice is free. A related selection concerns the direction of the shift inside this direction line. The direction of the second direction line must be spatially perpendicular to the first spatial direction line and reduces the angular choice to two pi radians. Inside this direction line, the direction of the shift has two choices. In the resulting spatial dimension, the choice of a direction line leaves a range of pi radians. The independent direction line can be oriented right-handed or left-handed. Inside that independent direction line, the shift has again two choices. The handedness of the multiplication of spatial numbers is a special kind of symmetry.

In the Hilbert repository, the symmetry of the number systems used plays a crucial role. In this system of Hilbert spaces, only a limited number of symmetries are allowed to participate.

5 Hilbert spaces

Hilbert spaces are ideally suited for modeling multidimensional functions that apply number systems as their natural parameter space. Hilbert spaces can only cope with number systems that are associative division rings. This limits the choice to the real numbers, the complex numbers, and the quaternions. Number systems exist in multiple versions. The version is captured by a selected coordinate system. Each Hilbert space chooses a suitable number system and selects a single version of that number system. This restriction is stronger than is applied in most textbooks about Hilbert spaces. The restriction attributes some special properties to Hilbert spaces and their operators.

Hilbert spaces support operators that manage eigenspaces that are constituted of the members of the selected version of the number system. Separate Hilbert spaces own operators that have countable eigenspaces. Non-separable Hilbert spaces also own operators that have uncountable eigenspaces. These eigenspaces are continuous and differentiable.

A dedicated operator, called the reference operator, manages the selected version of the chosen number system in its eigenspace. A category of operators shares the eigenvectors of the reference operator and uses the target values of a selected function that belong to the parameter value that equals the eigenvalue of the reference operator as the new eigenvalue of the new operator. This turns the Hilbert space into a function space and turns the eigenspace of the reference operator into the natural parameter space of the Hilbert space. The members of the new category are called natural operators. The reference operator is a normal operator, and its eigenvectors span the whole Hilbert space.

A quaternionic Hilbert space includes a series of complex-number-based Hilbert spaces as its subspaces. This can be seen by selecting all eigenvectors of the reference operator that belong to a given direction line of the selected version of the chosen number system and construct a new Hilbert space from these vectors and the vectors that belong to the real numbers. This also shows that a real-number-based Hilbert space is a subspace of a complex-number-based Hilbert space and a quaternionic Hilbert space.

5.1 Inner product space

In mathematics, a Hilbert space is known as an inner product space. This inner product concerns the inner product of vectors from the underlying vector space. At the same time, that inner product is used to calculate the eigenvalue of an eigenvector from the original vector and the value of the vector image on its own direction line. This is conceivable if the values of inner products were limited to the geometric scalar product. After all, this product only produces real values. A corresponding extension, shortening, and/or reversal is conceivable for a map onto itself. However, the eigenvalues can also be complex numbers or quaternions. That's less imaginable for a map of a vector onto its own direction line. The calculation of eigenvalues from the inner product has not been used in the foregoing and is not necessary. We have applied a trick that in the initial phase links generated numbers to vectors of the underlying vector space associated with point-like objects that we called coordinate markers. After the formation of the number systems, the coordinate markers are detached from the numbers while the reference to the number is retained as identification. In this way, the disconnected coordinate markers can be followed for the rest of their lives. The notions of eigenvalues and eigenvectors stay coupled to the operators that manage the eigenspaces.

5.2 Bra's and ket's

Paul Dirac introduced a handy notation for the relationship that exists between operators, its eigenvectors, its eigenvalues, and between functions that define operators. The bra-ket combination acts as a replacement for the inner product of the inner product space. The bra-ket combination implements an equivalent of the functionality of the inner product in an inner product space. This is not the dot product $\left\langle \vec{f}, \vec{g} \right\rangle$ that exists in spatial number systems and that also is called inner product. In contrast, the bra-ket combination provides the opportunity to use complex numbers and quaternions as eigenvalues. By selecting a version of the number system, the corresponding symmetry is fixed. This section treats the case that the inner product space applies quaternions to specify the values of bra-ket combinations. The bra-ket combination will be used in linear combinations of vectors and as eigenvalues of operators.

To make this possible, the bra-ket method distinguishes the vectors from the underlying vector space into vectors with different arithmetic. The ket $\left\langle \vec{f} \right|$ is a covariant vector, and the bra $\left| \vec{g} \right\rangle$ is a contravariant vector. The vectors $\vec{f}$ and $\vec{g}$ reside in the underlying vector space. The arithmetic of the ket vectors differs from the arithmetic of the bra vectors. The bra-ket combination $\left\langle \vec{f} \mid \vec{g} \right\rangle$ acts as a metric. It has a quaternionic value. Since the product of quaternions is not commutative, care must be taken with the format of the formulas.

5.2.1 Ket vectors

The addition of ket vectors is commutative and associative.

$$\left| \vec{f} \right\rangle + \left| \vec{g} \right\rangle = \left| \vec{g} \right\rangle + \left| \vec{f} \right\rangle = \left| \vec{f} + \vec{g} \right\rangle \tag{5.2.1}$$

$$\left(\left| \vec{f} + \vec{g} \right\rangle\right) + \left| \vec{h} \right\rangle = \left| \vec{f} \right\rangle + \left(\left| \vec{g} + \vec{h} \right\rangle\right) = \left| \vec{f} + \vec{g} + \vec{h} \right\rangle \tag{5.2.2}$$

Together with quaternions, a set of ket vectors forms a ket vector space. Ket vectors are covariant vectors.

A quaternion α can be used to construct a covariant linear combination with the ket vector $\left| \vec{f} \right\rangle$

$$\left| \alpha \vec{f} \right\rangle = \left| \vec{f} \right\rangle \alpha \tag{5.2.3}$$

5.2.2 Bra vectors

For bra vectors hold

$$\left\langle \vec{f} \right| + \left\langle \vec{g} \right| = \left\langle \vec{g} \right| + \left\langle \vec{f} \right| = \left\langle \vec{f} + \vec{g} \right| \tag{5.2.4}$$

$$\left(\left\langle \vec{f} + \vec{g} \right|\right) + \left\langle \vec{h} \right| = \left\langle \vec{f} \right| + \left(\left\langle \vec{g} + \vec{h} \right|\right) = \left\langle \vec{f} + \vec{g} + \vec{h} \right| \tag{5.2.5}$$

Bra vectors are contravariant vectors.

$$\left\langle \alpha \vec{f} \right| = \alpha^* \left\langle \vec{f} \right| \tag{5.2.6}$$

Quaternions can constitute linear combinations with bra vectors.

A set of bra vectors form the vector space that is adjunct to the vector space of ket vectors that are the origins of these maps. If the map images the adjunct space onto the original vector space, then the bra vectors may be mapped onto the same ket vector.

5.2.3 Bra-ket combination

For the bra-ket combination holds

$$\left\langle \vec{f} \mid \vec{g} \right\rangle = \left\langle \vec{g} \mid \vec{f} \right\rangle^* \tag{5.2.7}$$

For quaternionic numbers α and β hold

$$\left\langle \alpha \vec{f} \mid \vec{g} \right\rangle = \left\langle \vec{g} \mid \alpha \vec{f} \right\rangle^* = \left(\left\langle \vec{g} \mid \vec{f} \right\rangle \alpha \right)^* = \alpha^* \left\langle \vec{f} \mid \vec{g} \right\rangle \tag{5.2.8}$$

$$\left\langle \vec{f} \mid \beta \vec{g} \right\rangle = \left\langle \vec{f} \mid \vec{g} \right\rangle \beta \tag{5.2.9}$$

$$\begin{aligned} \left\langle (\alpha + \beta) \vec{f} \mid \vec{g} \right\rangle &= \alpha^* \left\langle \vec{f} \mid \vec{g} \right\rangle + \beta^* \left\langle \vec{f} \mid \vec{g} \right\rangle \\ &= (\alpha + \beta)^* \left\langle \vec{f} \mid \vec{g} \right\rangle \end{aligned} \tag{5.2.10}$$

This corresponds with (5.2.3) and (5.2.6)

$$\left\langle \alpha \vec{f} \right| = \alpha^* \left\langle \vec{f} \right| \tag{5.2.11}$$

$$\left| \alpha \vec{g} \right\rangle = \left| \vec{g} \right\rangle \alpha \tag{5.2.12}$$

We made a choice. Another possibility would be $\left\langle \alpha \vec{f} \right| = \alpha \left\langle \vec{f} \right|$ and $\left| \alpha \vec{g} \right\rangle = \left| \vec{g} \right\rangle \alpha^*$

5.2.4 Operator construction

$\left| \vec{f} \right\rangle \left\langle \vec{g} \right|$ is a constructed operator.

$$\left| \vec{g} \right\rangle \left\langle \vec{f} \right| = \left(\left| \vec{f} \right\rangle \left\langle \vec{g} \right| \right)^\dagger \tag{5.2.13}$$

The superfix † indicates the adjoint version of the operator.

For the orthonormal base $\{ |\vec{q}_i\rangle \}$ consisting of eigenvectors of the reference operator, holds

$$\left\langle \vec{q}_n \mid \vec{q}_m \right\rangle = \delta_{nm} \tag{5.2.14}$$

The ***bra-ket method*** enables the definition of new operators that are defined by quaternionic functions.

$$\left\langle \vec{g} \mid F \mid \vec{h} \right\rangle = \sum_{i=1}^{N} \left\{ \left\langle \vec{g} | \vec{q}_i \right\rangle F(q_i) \left\langle \vec{q}_i | \vec{h} \right\rangle \right\} \tag{5.2.15}$$

The symbol F is used both for the operator F and the quaternionic function $F(q)$. This enables the shorthand

$$F \equiv |\vec{q}_i\rangle F(q_i)\langle\vec{q}_i| \tag{5.2.16}$$

for operator F. It is evident that for the adjoint operator

$$F^{\dagger} \equiv |\vec{q}_i\rangle F^{*}(q_i)\langle\vec{q}_i| \tag{5.2.17}$$

For ***reference operator*** $\mathfrak{R}$ holds

$$\mathfrak{R} = |\vec{q}_i\rangle q_i \langle\vec{q}_i| \tag{5.2.18}$$

If $\{q_i\}$ consists of all rational values of the version of the quaternionic number system that Hilbert space $\mathfrak{H}$ applies then the eigenspace of $\mathfrak{R}$ represents the natural parameter space of the separable Hilbert space $\mathfrak{H}$. It is also the parameter space of the function $F(q)$ that defines the natural operator F in the formula (5.2.16).

5.2.5 Expected value

Any ket vector can be written as a linear combination of the bra base vectors.

$$\langle\vec{g}| = \sum_{i=1}^{N}\{\langle\vec{g}|\vec{q}_i\rangle\langle\vec{q}_i|\} \tag{5.2.19}$$

Any bra vector can be written as a linear combination of the ket base vectors.

$$|\vec{g}\rangle = \sum_{i=1}^{N}\{|\vec{q}_i\rangle\langle\vec{q}_i|\vec{g}\rangle\} \tag{5.2.20}$$

This means that $|\langle g|q_i\rangle|^2 = \langle g|q_i\rangle\langle q_i|g\rangle$ can take the role of a location density distribution. Because the vectors represent point-shaped objects, the location density distribution is also a probability density distribution and can $\langle g | q_i \rangle$ be seen as a quaternionic probability amplitude.

The expected value for operator $\mathfrak{R}$ and (bra) vector $| \vec{g} \rangle$ is

$$\langle \mathfrak{R} \rangle_{\vec{g}} = \langle \vec{g} | \mathfrak{R} | \vec{g} \rangle = \sum_{i=1}^{N} \left\{ \langle \vec{g} | \vec{q}_i \rangle q_i \langle \vec{q}_i | \vec{g} \rangle \right\} \tag{5.2.21}$$

The expected value plays its role in a series of subsequent observations or events. This gets a special interpretation when the eigenvalues $\{q_i\}$ are restricted to spatial eigenvalues. In that case, a real eigenvalue can be attributed as a timestamp to every sample value. After sequencing the timestamps of the samples, the string of samples represents a hopping path. The mechanism that generates the ongoing hopping path recurrently regenerates a hop landing location swarm that is described by the location density distribution. The density distributions and the probability amplitude are continuous functions. What they describe approximately are discrete sets. The approach fits better if the number of elements in the set is larger and there is a requirement that the coherence of the set is large.

To give the location density distribution a statistical sense, a stochastic selection process must be or have been active. That selection process is then represented by a footprint vector $| \vec{g} \rangle$ that varies over time. How $| \vec{g} \rangle$ varies over time is checked by the characteristic function of the selection process. The footprint vector is represented by a vector in the underlying vector space. The Hilbert space can archive the life history of the footprint vector in the form of a cord of quaternionic eigenvalues from a dedicated footprint operator.

This does not say anything about the essence of the required underlying stochastic selection mechanism.

5.2.6 Operator types

I is used to indicate the identity operator.

For normal operator N holds $NN^{\dagger} = NN^{\dagger}$.

The normed eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

For unitary operator U holds $UU^{\dagger} = U^{\dagger}U = I$

For Hermitian operator H holds $H = H^{\dagger}$

A normal operator N has a Hermitian part $\frac{N + N^{\dagger}}{2}$ and an anti-Hermitian part $\frac{N - N^{\dagger}}{2}$

For anti-Hermitian operator A holds $A = -A^{\dagger}$

5.3 Non-separable Hilbert space

Every infinite-dimensional separable Hilbert space owns a unique non-separable companion Hilbert space that embeds its separable partner. The non-separable Hilbert space allows operators that maintain eigenspaces that in every dimension and every spatial direction contain closed sets of rational and irrational eigenvalues. These eigenspaces behave as dynamic sticky continuums.

Gelfand triple and ***Rigged Hilbert space*** are other names for the general non-separable Hilbert spaces.

In the non-separable Hilbert space, for operators with continuum eigenspaces, the bra-ket method turns from a summation into an integration.

$$\langle \vec{g} \,|\, F \,|\, \vec{h} \rangle \equiv \int\iiint \left\{ \langle \vec{g} | \vec{q} \rangle F(q) \langle \vec{q} | \vec{h} \rangle \right\} dV d\tau \qquad (5.3.1)$$

Here we omitted the enumerating subscripts that were used in the countable base of the separable Hilbert space.

The shorthand for the operator F is now

$$F \equiv |\vec{q}\rangle F(q) \langle \vec{q}| \tag{5.3.2}$$

For eigenvectors $|q\rangle$, the function $F(q)$ defines as

$$F(q) = \langle \vec{q} \mid F\vec{q} \rangle = \int\iiint \{ \langle \vec{q} \mid \vec{q}\,' \rangle F(q') \langle \vec{q}\,' \mid \vec{q} \rangle \} dV' d\tau' \tag{5.3.3}$$

The reference operator $\mathcal{R}$ that provides the continuum natural parameter space as its eigenspace follows from

$$\langle \vec{g} \mid \mathcal{R}\vec{h} \rangle \equiv \int\iiint \{ \langle \vec{g} | \vec{q} \rangle q \langle \vec{q} | \vec{h} \rangle \} dV d\tau \tag{5.3.4}$$

The corresponding shorthand is

$$\mathcal{R} \equiv |\vec{q}\rangle q \langle \vec{q}| \tag{5.3.5}$$

The reference operator is a special kind of defined operator. Via the quaternionic functions that specify defined operators, the claim becomes clear that every infinite-dimensional separable Hilbert space owns a unique non-separable companion Hilbert space that can be considered to embed its separable companion.

The reverse bracket method combines Hilbert space operator technology with quaternionic function theory and indirectly with quaternionic differential and integral technology.

6 Change

6.1 Differentiation

Along a direction line, change can be described by a partial differential. If in a region of the space coverage inside this direction line all converging series of coordinate markers result in a limit that is a coordinate marker, then the partial change of the space coverage along the direction of r is defined as the limit

$$\frac{\partial\psi}{\partial r} = \lim_{\delta r\to 0}\frac{\psi\left(r+\delta r\right)-\psi(r)}{\delta r} \qquad (6.1.1)$$

If the region is covered by all irrational numbers, then this limit exists. The existence of the limit is not ensured. If the limit does not exist, then the location represents a singular point. It is also possible that the surrounding region is covered by a discrete set of point-like objects.

If the spatial part of the neighborhood is isotropic and the limit also exists in the real number space, then the total differential change df of field f equals

$$df = \frac{\partial f}{\partial\tau}d\tau + \frac{\partial f}{\partial x}\vec{i}dx + \frac{\partial f}{\partial y}\vec{j}dy + \frac{\partial f}{\partial z}\vec{k}dz \qquad (6.1.2)$$

In this equation, the partial differentials $\frac{\partial f}{\partial\tau},\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial y}$ behave like quaternionic differential operators.

The quaternionic nabla ∇ assumes the ***special condition*** that partial differentials direct along the axes of the Cartesian coordinate system in a natural parameter space of a non-separable Hilbert space. Thus,

$$\nabla = \sum_{i=0}^{4} \vec{e}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tau} + \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \tag{6.1.3}$$

This will be applied in the next section by splitting both the quaternionic nabla and the function in a scalar part and a vector part.

The first-order partial differential equations divide the first-order change of a quaternionic field into five different parts that each represent a new field. We will represent the quaternionic field change operator by a quaternionic nabla operator. This operator behaves like a quaternionic multiplier.

The first order partial differential follows from

$$\nabla = \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} = \nabla_r + \vec{\nabla} \tag{6.1.4}$$

The spatial nabla $\vec{\nabla}$ is well-known as the del operator and is treated in detail in Wikipedia. The partial derivatives in the change operator only use parameters that are taken from the natural parameter space.

$$\begin{aligned} \phi = \nabla \psi &= \left(\frac{\partial}{\partial \tau} + \vec{\nabla} \right) \left(\psi_r + \vec{\psi} \right) \\ &= \nabla_r \psi_r - \left\langle \vec{\nabla}, \vec{\psi} \right\rangle + \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} \end{aligned} \tag{6.1.5}$$

In a selected version of the quaternionic number system, only the corresponding version of the quaternionic nabla is active. In a selected Hilbert space, this version is always and everywhere the same.

The differential $\nabla \psi$ describes the change of field ψ. The five separate terms in the first-order partial differential have a separate physical meaning. All basic fields feature this decomposition. The terms may represent new fields.

$$\phi_r = \nabla_r \psi_r - \left\langle \vec{\nabla}, \vec{\psi} \right\rangle \qquad (6.1.6)$$

ϕ_r is a scalar field.

$$\vec{\phi} = \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} \qquad (6.1.7)$$

$\vec{\phi}$ is a vector field.

$\vec{\nabla} f$ is the gradient of f .

$\left\langle \vec{\nabla}, \vec{f} \right\rangle$ is the divergence of $\vec{f}$.

$\vec{\nabla} \times \vec{f}$ is the curl of $\vec{f}$.

Important properties of the del operator are

$$\left(\vec{\nabla}, \vec{\nabla} \right) \psi = \Delta \psi = \nabla^2 \psi \qquad (6.1.8)$$

$$\left(\vec{\nabla}, \vec{\nabla} \times \vec{\psi} \right) = 0 \qquad (6.1.9)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \psi_r \right) = 0 \qquad (6.1.10)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) = \vec{\nabla} \left(\vec{\nabla}, \vec{\psi} \right) - \left(\vec{\nabla}, \vec{\nabla} \right) \vec{\psi} \qquad (6.1.11)$$

Sometimes parts of the change get new symbols

$$\vec{E} = -\nabla_r \vec{\psi} - \vec{\nabla} \psi_r \qquad (6.1.12)$$

$$\vec{B} = \vec{\nabla} \times \vec{\psi} \qquad (6.1.13)$$

The formula (6.1.5) does not leave room for gauges. In Maxwell equations, the equation (6.1.6) is treated as a gauge.

$$\left(\vec{\nabla}, \vec{B} \right) = 0 \qquad (6.1.14)$$

$$\vec{\nabla} \times \vec{E} = -\nabla_r \vec{\nabla} \times \vec{\psi} - \vec{\nabla} \times \vec{\nabla} \psi_r = -\nabla_r \vec{B} \qquad (6.1.15)$$

$$\left(\vec{\nabla},\vec{E}\right)=-\nabla_r\left(\vec{\nabla},\vec{\psi}\right)-\left(\vec{\nabla},\vec{\nabla}\right)\psi_r \tag{6.1.16}$$

The conjugate of the quaternionic nabla operator defines another type of field change.

$$\nabla^*=\nabla_r-\vec{\nabla} \tag{6.1.17}$$

$$\begin{aligned}\zeta=\nabla^*\phi&=\left(\frac{\partial}{\partial\tau}-\vec{\nabla}\right)\left(\phi_r+\vec{\phi}\right)\\&=\nabla_r\phi_r+\left\langle\vec{\nabla},\vec{\phi}\right\rangle+\nabla_r\vec{\phi}-\vec{\nabla}\phi_r\mp\vec{\nabla}\times\vec{\phi}\end{aligned} \tag{6.1.18}$$

All dynamic quaternionic fields obey the same first-order partial differential equations (6.1.5) and (6.1.18).

$$\nabla^\dagger=\nabla^*=\nabla_r-\vec{\nabla}=\nabla_r+\vec{\nabla}^\dagger=\nabla_r+\vec{\nabla}^* \tag{6.1.19}$$

In the Hilbert space, the quaternionic nabla is a normal operator. The operators

$$\nabla^\dagger\nabla=\nabla\nabla^\dagger=\nabla^*\nabla=\nabla\nabla^*=\nabla_r\nabla_r+\left\langle\vec{\nabla},\vec{\nabla}\right\rangle \tag{6.1.20}$$

are normal operators who are also Hermitian operators.

The separate operators $\nabla_r\nabla_r$ and $\left\langle\vec{\nabla},\vec{\nabla}\right\rangle$ are also Hermitian operators.

$\left\langle\vec{\nabla},\vec{\nabla}\right\rangle$ is known as the Laplace operator.

The two operators can also be combined as $\square=\nabla_r\nabla_r-\left\langle\vec{\nabla},\vec{\nabla}\right\rangle$. This is the d'Alembert operator.

The solutions of $\nabla_r\nabla_r+\left\langle\vec{\nabla},\vec{\nabla}\right\rangle=0$ and $\nabla_r\nabla_r-\left\langle\vec{\nabla},\vec{\nabla}\right\rangle=0$ differ. These two equations offer different solutions and for that reason, they deliver different dynamic behavior of the field. The equations control the behavior of the embedding field that physicists call their universe. This dynamic field exists everywhere in the reach of the parameter space of the function. Both equations also control the behavior of the symmetry-related fields. The homogeneous d'Alembert equation is known as the wave equation and offers waves and wave packages as its solutions. Both equations offer shock fronts as solutions but only the operators in (6.1.20) deliver shock fronts that feature a spin or polarization vector. Integration over the time domain turns both equations in the Poisson equation and removes the spin or polarization vector. Shock fronts require a corresponding actuator and occur only in odd numbers of participating dimensions. Spherical shock fronts require an isotropic actuator. Otherwise, the shock front does not appear.

6.2 Continuity equations

Continuity equations are partial quaternionic differential equations.

The dynamic changes of the field are interpreted as field excitations or as field deformations or field expansions.

The field excitations that will be discussed here are solutions of mentioned second-order partial differential equations. Without a corresponding actuator, the field will not react. It appears that spherical pulses are the only actuators that deform the field. The field reacts to these pulses by quickly removing the deformation by sending the deformation away in all directions in the form of shock fronts until these fronts vanish at infinity.

One of the second-order partial differential equations results from combining the two first-order partial differential equations $\phi = \nabla \psi$ and $\zeta = \nabla^* \phi$.

$$\begin{aligned} \zeta &= \nabla^* \varphi = \nabla^* \nabla \psi = \nabla \nabla^* \psi = \left(\nabla_r + \vec{\nabla}\right)\left(\nabla_r - \vec{\nabla}\right)\left(\psi_r + \vec{\psi}\right) \\ &= \left(\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle\right)\psi \end{aligned} \tag{6.2.1}$$

All other terms vanish. $\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$ is known as the Laplace operator.

Integration over the time domain results in the Poisson equation

$$\rho = \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \psi \tag{6.2.2}$$

Under isotropic conditions, a very special solution of the Poisson equation is the Green's function $\frac{1}{4\pi\left|\vec{q} - \vec{q'}\right|}$ of the affected field. This solution is the spatial Dirac $\delta\left(\vec{q}\right)$ pulse response of the field under strict isotropic conditions.

$$\nabla \frac{1}{\left|\vec{q} - \vec{q'}\right|} = -\frac{\left(\vec{q} - \vec{q'}\right)}{\left|\vec{q} - \vec{q'}\right|^3} \tag{6.2.3}$$

$$\begin{aligned} \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \frac{1}{\left|\vec{q} - \vec{q'}\right|} &\equiv \left\langle \vec{\nabla}, \vec{\nabla} \frac{1}{\left|\vec{q} - \vec{q'}\right|} \right\rangle \\ &= -\left\langle \vec{\nabla}, \frac{\left(\vec{q} - \vec{q'}\right)}{\left|\vec{q} - \vec{q'}\right|^3} \right\rangle = 4\pi\delta\left(\vec{q} - \vec{q'}\right) \end{aligned} \tag{6.2.4}$$

This solution corresponds with an ongoing source or sink that exists in the field.

Change can take place in one spatial dimension or combined in two or three spatial dimensions.

Under the proper conditions, the dynamic pulse response of the field is a solution of a special form of the equation (6.2.1)

$$\left(\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle\right)\psi = 4\pi\delta\left(\vec{q} - \overrightarrow{q'}\right)\theta\left(\tau \pm \tau'\right) \qquad (6.2.5)$$

Here $\theta(\tau)$ is a temporal step function and $\delta(\vec{q})$ is a spatial Dirac pulse response. For the spherical pulse response, the pulse must be isotropic.

After the instant τ', the equation turns into a homogeneous equation.

A remarkably simple solution is the shock front in one dimension along the line $\vec{q} - \overrightarrow{q'}$.

$$\psi = f\left(\left|\vec{q} - \overrightarrow{q'}\right| \pm c\left(\tau - \tau'\right)\vec{n}\right) \qquad (6.2.6)$$

Here $\vec{n}$ is a normed spatial quaternion. This spatial quaternion has an arbitrary direction that does not vary in time. Here, the normalized vector $\vec{n}$ can be interpreted as the polarization of the solution. We intentionally placed the spatial vector $\vec{n}$ close to speed c. The function f can be a primitive shock front, but it can also be a superposition of primitive shock fronts. The single primitive shock front solution represents a ***dark energy object***. It represents a quantum of energy.

In isotropic conditions, we better switch to spherical coordinates. Then the equation gets the form

$$\begin{aligned}&\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial r^2} + 2\frac{\partial}{r\partial r}\right)\psi \\ &= \left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial r^2}\right)(\psi r) = 0\end{aligned} \qquad (6.2.7)$$

The second line describes the second-order change of ψr in one dimension along the radius r. That solution is described above. A solution of this equation is

$$\psi r = f\left(r \pm c\tau\vec{n}\right) \tag{6.2.8}$$

The solution of (6.2.7) is described by

$$\psi = \frac{f\left(\left|\vec{q} - \overrightarrow{q'}\right| \pm c\left(\tau - \tau'\right)\vec{n}\right)}{\left|\vec{q} - \overrightarrow{q'}\right|} \tag{6.2.9}$$

The normalized vector $\vec{n}$ can be interpreted as the spin of the solution. It might be related to the direction that is selected when the quaternion-based Hilbert space is temporary reduced to a subspace that contains a complex-number-based Hilbert space. The spherical pulse response acts either as an expanding or as a contracting spherical shock front. Over time this pulse response integrates into the Green's function. This means that the isotropic pulse injects the volume of the Green's function into the field. Subsequently, the front spreads this volume over the field. The contracting shock front collects the volume of the Green's function and sucks it out of the field. The $\pm$ sign in the equation (6.2.5) selects between injection and subtraction.

Spherical shock fronts are ***dark matter objects***.

Shock fronts only occur in one and three dimensions. A pulse response can also occur in two dimensions, but in that case, the pulse response is a complicated vibration that looks like the result of a throw of a stone in the middle of a pond.

Equations (6.2.1) and (6.2.2) show that the operators $\frac{\partial^2}{\partial \tau^2}$ and $\langle \vec{\nabla}, \vec{\nabla} \rangle$ are valid second-order partial differential operators. These operators combine in the quaternionic equivalent of the wave equation.

$$\varphi = \left(\frac{\partial^2}{\partial \tau^2} - \langle \vec{\nabla}, \vec{\nabla} \rangle \right) \psi = \Box \psi \tag{6.2.10}$$

This equation also offers one-dimensional and three-dimensional shock fronts as its solutions.

$$\psi = \frac{f\left(\left| \vec{q} - \vec{q'} \right| \pm c(\tau - \tau') \right)}{\left| \vec{q} - \vec{q'} \right|} \tag{6.2.11}$$

$$\psi = f\left(\left| \vec{q} - \vec{q'} \right| \pm c(\tau - \tau') \right) \tag{6.2.12}$$

These pulse responses do not contain the normed vector $\vec{n}$. Apart from pulse responses, the wave equation offers waves as its solutions.

If locally the field can be split into a time-dependent part $T(\tau)$ and a location-dependent part $A(\vec{q})$, then the homogeneous version of the wave equation can be transformed into the Helmholtz equation.

$$\frac{\partial^2 \psi}{\partial \tau^2} = \langle \vec{\nabla}, \vec{\nabla} \rangle \psi = -\omega^2 \psi \tag{6.2.13}$$

$$\psi(\vec{q}, \tau) = A(\vec{q}) T(\tau) \tag{6.2.14}$$

$$\frac{1}{T} \frac{\partial^2 T}{\partial \tau^2} = \frac{1}{A} \langle \vec{\nabla}, \vec{\nabla} \rangle A = -\omega^2 \tag{6.2.15}$$

$$\langle \vec{\nabla}, \vec{\nabla} \rangle A + \omega^2 A = 0 \tag{6.2.16}$$

$$\frac{\partial^2 T}{\partial \tau^2} + \omega^2 T = 0 \tag{6.2.17}$$

ω acts as quantum coupling between(6.2.16) and (6.2.17).

The time-dependent part $T(\tau)$ depends on initial conditions, or it indicates the switch of the oscillation mode.

During the switch, the quaternionic Hilbert space temporarily switches to a complex-number-based Hilbert space that is a subspace of the Hilbert space. The switch takes a corresponding interval and during that interval, the subspace emits or absorbs a sequence of equidistant one-dimensional shock fronts. Together, these shock fronts constitute a photon. The one-dimensional shock fronts are discussed above. The switch of the oscillation mode means that temporarily the oscillation is stopped and instead an object is emitted or absorbed that compensates for the difference in potential energy. The location-dependent part of the field $A(\vec{q})$ describes the possible oscillation modes of the field and depends on boundary conditions. The oscillations have a binding effect. They keep moving objects within a bounded region.

For three-dimensional isotropic spherical conditions, the solutions have the form

$$A(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \left(a_{lm} j_l(kr) \right) + b_{lm} Y_l^m(\theta,\varphi) \right\} \tag{6.2.18}$$

Here j_l and y_l are the spherical Bessel functions, and Y_l^m are the spherical harmonics. These solutions play a role in the spectra of atomic modules.

Planar and spherical waves are the simpler wave solutions of the equation (6.2.13)

$$\psi(\vec{q},\tau)=\exp\left\{\vec{n}\left(\left\langle\vec{k},\vec{q}-\vec{q}_0\right\rangle-\omega\tau+\varphi\right)\right\} \tag{6.2.19}$$

$$\psi(\vec{q},\tau)=\frac{\exp\left\{\vec{n}\left(\left\langle\vec{k},\vec{q}-\vec{q}_0\right\rangle-\omega\tau+\varphi\right)\right\}}{|\vec{q}-\vec{q}_0|} \tag{6.2.20}$$

A more general solution is a superposition of these basic types.

Two quite similar homogeneous second-order partial differential equations exist. They are the homogeneous versions of equations (6.2.5) and (6.2.10). The equation (6.2.5) has spherical shock-front solutions with a spin vector that behaves like the spin of elementary particles. Obviously, the field only reacts dynamically when it gets triggered by corresponding actuators. Pulses may cause shock fronts that after the trigger keep traveling. Oscillations of type (6.2.19) and (6.2.20) must be triggered by periodic actuators.

The inhomogeneous pulse activated equations are

$$\left(\nabla_r\nabla_r \pm \left\langle\vec{\nabla},\vec{\nabla}\right\rangle\right)\psi = 4\pi\delta\left(\vec{q}-\vec{q}\,'\right)\theta\left(\tau\pm\tau'\right) \tag{6.2.21}$$

Without the interaction with actuators, all vibrations and deformations of the field keep busy vanishing until the affected field resembles a flat field. Only an ongoing stream of actuators can generate a more persistently deformed field. This is provided by an ongoing embedding of the actuators into the eigenspaces of operators that archive the dynamic fields.

6.3 Isotropic conditions

The two shock-front solutions show an interesting property of the Laplace operator. In isotropic conditions, the Poisson equation can be rewritten as

$$\varphi = \langle \vec{\nabla}, \vec{\nabla} \rangle \psi = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) \qquad (6.3.1)$$

The product $\phi = (r\psi)$ is a solution of a one-dimensional equation in which r plays the variable.

The same thing holds for all differential equations that contain the Laplace operator $\langle \vec{\nabla}, \vec{\nabla} \rangle$

So, spherical solutions of the second-order differential equations ξ / r can be obtained from the solutions ξ of one-dimensional second-order differential equations by dividing ξ with the distance r to the center.

It looks as if in isotropic conditions the quaternionic differential calculus can be scaled down to complex-number-based differential calculus. This already works at local scales. If on larger scales the isotropic condition is violated, then the coordinates of the complex-number-based abstraction must be adapted to the possibly deformed Cartesian coordinates of the quaternionic platform. This makes sense in the presence of moderate deformations of the quaternionic field. After adaptation, the map of each complex-number-based coordinate line becomes a geodesic.

These tricks are possible because complex-number-based Hilbert spaces can be considered subspaces of quaternionic Hilbert spaces.

If the dimension of the quaternionic Hilbert space is reduced to the dimension of a subspace that contains a complex-number-based Hilbert space, then it might become important whether the selected direction involves a polar angle or an azimuth angle. In mathematics, the range of the polar angle is twice the range of the azimuth angle. In physics, the two ranges are exchanged.

7 Transport of change

The del operator has a direction. This suggests that change moves along that direction.

Enclosure balance equations are quaternionic integral equations that describe the balance between the inside, the border, and the outside of an enclosure.

These integral balance equations base on replacing the del operator $\vec{\nabla}$ with a normed vector $\vec{n}$. The vector $\vec{n}$ is oriented outward and perpendicular to a local part of the closed boundary of the enclosed region.

$$\vec{\nabla}\psi \Leftrightarrow \vec{n}\psi \tag{7.1.1}$$

This approach turns part of the differential continuity equation into a corresponding integral balance equation.

$$\iiint \vec{\nabla}\psi dV = \oiint \vec{n}\psi dS \tag{7.1.2}$$

$\vec{n}\ dS$ plays the role of a differential surface. $\vec{n}$ is perpendicular to that surface.

This result separates into three parts

$$\begin{aligned}\vec{\nabla}\psi &= -\left\langle \vec{\nabla},\vec{\psi}\right\rangle + \vec{\nabla}\psi_r \pm \vec{\nabla}\times\vec{\psi} \Leftrightarrow \vec{n}\psi \\ &= -\left\langle \vec{n},\vec{\psi}\right\rangle + \vec{n}\psi_r \pm \vec{n}\times\vec{\psi}\end{aligned} \tag{7.1.3}$$

The first part concerns the gradient of the scalar part of the field

$$\vec{\nabla}\psi_r \Leftrightarrow \vec{n}\psi_r \tag{7.1.4}$$

$$\iiint \vec{\nabla}\psi_r dV = \oiint \vec{n}\psi_r dS \tag{7.1.5}$$

The divergence is treated in an integral balance equation that is known as the Gauss theorem. It is also known as the divergence theorem.

$$\left\langle \vec{\nabla}, \vec{\psi} \right\rangle \Leftrightarrow \left\langle \vec{n}, \vec{\psi} \right\rangle \tag{7.1.6}$$

$$\iiint \left\langle \vec{\nabla}, \vec{\psi} \right\rangle dV = \oiint \left\langle \vec{n}, \vec{\psi} \right\rangle dS \tag{7.1.7}$$

The curl is treated in a corresponding integrated balance equation

$$\vec{\nabla} \times \vec{\psi} \Leftrightarrow \vec{n} \times \vec{\psi} \tag{7.1.8}$$

$$\iiint \vec{\nabla} \times \vec{\psi} dV = \oiint \vec{n} \times \vec{\psi} dS \tag{7.1.9}$$

Equation (7.1.7) and equation (7.1.9) can be combined in the extended theorem

$$\iiint \vec{\nabla} \vec{\psi} dV = \oiint \vec{n} \; \vec{\psi} dS \tag{7.1.10}$$

The method also applies to other partial differential equations. For example

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) = \vec{\nabla} \left\langle \vec{\nabla}, \vec{\psi} \right\rangle - \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \vec{\psi} \Leftrightarrow \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) \\ = \vec{n} \left\langle \vec{n}, \vec{\psi} \right\rangle - \left\langle \vec{n}, \vec{n} \right\rangle \vec{\psi} \end{aligned} \tag{7.1.11}$$

$$\iiint_V \left\{ \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi} \right) \right\} dV = \oiint_S \left\{ \vec{\nabla} \left\langle \vec{\nabla}, \vec{\psi} \right\rangle \right\} dS - \oiint_S \left\{ \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \vec{\psi} \right\} dS$$

(7.1.12)

One dimension less, a similar relation exists.

$$\iint_S \left(\left\langle \vec{\nabla} \times \vec{a}, \vec{n} \right\rangle \right) dS = \oint_C \left\langle \vec{a}, d\vec{l} \right\rangle \tag{7.1.13}$$

This is known as the Stokes theorem.

The curl can be presented as a line integral

$$\langle \vec{\nabla} \times \vec{\psi}, \vec{n} \rangle \equiv \lim_{A \to 0} \left(\frac{1}{A} \oint_C \langle \vec{\psi}, d\vec{r} \rangle \right) \tag{7.1.14}$$

8 Functions and coordinate systems

This elucidates the sense of introducing the coordinate system. Humans can more easily imagine the dynamic life of coordinate markers than that they can visualize what happens to the target values of a multidimensional function that uses a borderless multidimensional parameter space. If the differential equations that describe the behavior of coordinate markers are everywhere the same, then these equations hold at all scales. Even if spatial expansion plays a role, then its effects can easily be separated from spatial deformation and spatial vibration.

Without triggering by an actuator, the space coverage does not deform or vibrate. This does not exclude the possibility that an encapsulated spatially coherent countable subset of coordinate markers statically deforms the space coverage. This happens for a phenomenon that is called a black hole.

9 Other features of Hilbert spaces

It is already indicated that complex-number-based Hilbert spaces appear as subspaces of quaternionic Hilbert spaces and that real-number-based Hilbert spaces appear as subspaces of complex-number-based Hilbert spaces.

9.1 Position space and change space

This will be used to introduce other orthogonal bases than the natural parameter base. First, we separate the subspace that relates to the real numbers. What is left we call the position space. Next, we introduce the change base, which is an alternative orthogonal base of the position space and that is constituted by the eigenvectors that belong to the change operator. To be able to represent this in a formula we first limit to eigenvectors that belong to a selected direction line. This reduces position space to a single direction. For example, we select the direction $\vec{i}$ along the x coordinate. The change p_x of a field ψ along that direction is $p_x\psi = \frac{\partial\psi}{\partial x}$. The suffix $_x$ indicates the relation with coordinate x.

9.2 Fourier transform

x and p_x are related via a Fourier transform. In this section, we do not indicate in the exponentials the spatial direction number i with a vector cap. Instead, we use the convention that is applied in complex number versions of the exponential function.

The Fourier transform in a separable complex-number-based Hilbert space is given by the relation between $\psi(x)$ and $\tilde{\psi}\left(p_{xn}\right)$ in the sum

$$\psi(x) = \sum_{n=-\infty}^{\infty} \left\{ \tilde{\psi}\left(p_{x,n}\right) e^{2\pi i x p_{x,n}} \left(p_{x,n+1} - p_{x,n}\right) \right\} \qquad (8.2.1)$$

In the limit where $\Delta p_x = \left(p_{x,n+1} - p_{x,n}\right) \rightarrow 0$ the sum becomes an integral

$$\psi(x) = \int_{-\infty}^{\infty} \left\{ \tilde{\psi}\left(p_x\right) e^{2\pi i x p_x} \right\} dp_x \qquad (8.2.2)$$

The reverse Fourier transform runs as

$$\tilde{\psi}(p_x) = \int_{-\infty}^{\infty} \left\{ \psi\left(x\right) e^{-2\pi i x p_x} \right\} dx \qquad (8.2.3)$$

In these formulas, the symbol i represents a normalized spatial number part of a complex number. i corresponds to the spatial direction that was selected for constructing the complex-number-based Hilbert space.

The function $e^{2\pi i x p_x}$ is an eigenfunction of the operator $\vec{p}_x = i \frac{\partial}{\partial x}$ which is recognizable as part of the change operator (6.1.4).

$$\vec{i} \frac{\partial}{\partial x} e^{2\pi i x p_x} = 2\pi \vec{p}_x e^{2\pi i x p_x} \qquad (8.2.4)$$

The eigenvalue p_x represents the eigenfunction and the eigenvector $\vec{p}_x$ in the change space. In the same sense, the function $e^{-2\pi i x p_x}$ is an eigenfunction of the position operator $-\vec{i} \frac{\partial}{\partial p_x}$ and corresponds with the eigenvalue x of that operator.

$$-\vec{i} \frac{\partial}{\partial p_x} e^{-2\pi i x p_x} = 2\pi x e^{-2\pi i x p_x} \qquad (8.2.5)$$

The eigenvalue x represents the eigenfunction and the eigenvector x in the position space.

The Fourier transform of a Dirac delta function is

$$\tilde{\delta}(p_x) = \int_{-\infty}^{\infty} \left\{ \delta\left(x\right) e^{-2\pi i x p_x} \right\} dx = 1 \qquad (8.2.6)$$

The inverse transform tells

$$\delta(x) = \int_{-\infty}^{\infty} \left\{1 \cdot e^{2\pi i x p_x}\right\} dp_x \tag{8.2.7}$$

$$\delta(x-a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-a)p_x} dp_x \tag{8.2.8}$$

$$e^{2\pi i p_x a} = \int_{-\infty}^{\infty} \delta(x-a) e^{2\pi i x p_x} dx \tag{8.2.9}$$

The operator $\vec{p}_x = \vec{i}\,\frac{\partial}{\partial x}$ is often called the momentum operator for the spatial direction $\vec{i}$ of the coordinate x. $\vec{p}$ differs from the classical momentum that is defined as the product of velocity $\vec{v}$ and mass m.

9.3 Uncertainty principle

The uncertainty principle states

$$\left(\int_{-\infty}^{\infty} (x - x_0)^2 \left|\psi(x)\right|^2 dx\right)\left(\int_{-\infty}^{\infty} (p_x - p_{x,0})^2 \left|\tilde{\psi}(p_x)\right|^2 dp_x\right) \geq \frac{1}{16\pi^2}$$

(8.3.1)

For a Gaussian distribution, the equality sign holds. The Fourier transform of a Gaussian distribution is again a Gaussian distribution that has a different standard deviation.

If $\psi(x)$ spreads, then $\tilde{\psi}(p_x)$ shrinks and vice versa.

9.4 Stochastic processes

In this way, the characteristic function of a stochastic process that resides in the change space can control the spread of the location density distribution of the produced location swarm that resides in position space.

The stochastic process consists of a Poisson process that regulates the distribution in the real-number-based Hilbert space that is a subspace of the quaternionic Hilbert space and a binomial process that regulates the distribution in position space. This distribution is described by a location density distribution.

The production of the stochastic process is archived in the eigenspace of a dedicated footprint operator that stores its eigenvalues in quaternionic storage bins that consists of a real number valued timestamp and a three-dimensional spatial number value that represents a hop landing location. After sequencing the timestamps, the hop landing locations represent a hopping path of a point-like object. The hopping path regularly regenerates a coherent hop landing location swarm. The location density distribution describes this swarm.

If this location density distribution is a Gaussian distribution, then its Fourier transform determines exactly the location density distribution of the swarm.

The described stochastic process can deliver the actuators that generate the pulse responses that may deform the dynamic universe field. In some way, an ongoing embedding process must map the eigenspace of the footprint operator onto the embedding field. As previously argued, the footprint operator's eigenspace corresponds to a dynamic footprint vector that defines a location density function and a probability amplitude. The footprint vector resides in the underlying vector space and has a representation in Hilbert space via the footprint operator.

The stochastic processes that own a characteristic function which are described here, are in common use in the qualification of imaging quality via the Optical Transfer Function of an imaging process or imaging equipment. The Optical Transfer Function is the Fourier transform of the point spread function. For point-like objects, the PSF acts as a location density distribution.

A system of Hilbert spaces that share the same underlying vector space can perform the job of the imaging platform. In this system, the imaging process will be called the embedding process. This explanation still says nothing about the essence of the necessary underlying stochastic selection process. That remains a mystery

9.4.1 Gravity and the universe

If the hop landings of a particle cause a spherical shock front, then the deformation that is caused by the footprint is roughly defined by the convolution of the location density distribution that describes the footprint and the Green's function of the embedding field. This formulation is not exact because each spherical shock front quickly fades away. The pulses occur in a sequence and not in a single instant. This effect weakens the deformation. Still, due to the huge number of hops that constitute the swarm, the spherical pulse response will blur the hop landing location swarm such that its image becomes a smooth function. This smooth function describes the local gravitational potential of the considered particle. Far from the geometric center at distance r from the particle, the particle looks point-like, and the gravitational potential $V(r)$ can be described by

$$V(r) = MG / r \tag{8.4.1}$$

Here M is the mass of the particle. G is a constant.

The embedding field is a superposition of gravitational potentials. A formula like (8.4.1) does not directly show that gravity leads to the attraction between massive objects. The gravitational potential does not own a point of engagement. Or that point must be given by the geometrical center of local gravity.

10 Hilbert repository

A system of Hilbert spaces that all share the same underlying vector space can act as a modeling platform that not only supports dynamic fields that obey quaternionic differential equations. It also supports the containers of footprints that can map into the quaternionic fields. The vectors that represent the footprint vectors also occur in the underlying vector field. This enables the maps of these vectors into the dynamic universe field.

The Hilbert repository applies the structured storage capacity of the Hilbert spaces that are members of the system. The requirement that all member Hilbert spaces must share the same underlying vector space restricts the types of Hilbert spaces that can be a member of the Hilbert repository. It appears that the coordinate systems that determine the symmetry type of the Hilbert spaces must have the Cartesian coordinate axes in parallel. This enables the determination of differences in symmetry. Only the sequence along the axis can be freely selected up or down. This means that only a small set of symmetry types will be tolerated. One of the Hilbert spaces will act as the background platform and its symmetry will act as background symmetry. Its natural parameter space will act as background parameter space. All other members of the system will float with the geometric center of their parameter space over the background parameter space. This already generates a dynamic system. The symmetry differences generate symmetry-related sources or sinks that will locate at the geometric center of the natural parameter space of the corresponding floating Hilbert space. The sources and sinks correspond to symmetry-related charges that generate symmetry-related fields.

Not the symmetries of the floating Hilbert spaces are important. Instead, the differences between the symmetry of the floating member and the background symmetry are important for establishing the type of the member Hilbert space. The counts of the differences in symmetry restrict to the shortlist -3, -2, -1, 0, +1, +2, +3.

All floating Hilbert spaces are separable. The background Hilbert space is an infinite-dimensional separable Hilbert space. It owns a non-separable companion Hilbert space that embeds its separable partner.

All member Hilbert spaces are quaternionic Hilbert spaces and own a dedicated footprint operator. An ongoing embedding process maps the footprint vectors onto the uncountable eigenspace of a dedicated operator that resides in the background non-separable Hilbert space. In this way, a huge amount of ongoing hopping paths are mapped onto the embedding field. Physicists call this dynamic field the universe. On the floating platforms, the hopping paths are closed. The movement of the floating platforms breaks the closure of the images of the hopping paths.

10.1 Standard Model

The structure and behavior of the purely mathematical Hilbert repository show a striking resemblance with the structure of the Standard Model of the elementary fermions. The Standard Model of the elementary fermions is part of the Standard Model of particle physics that experimental particle physicists treat as their workbook. This does not include the physical theories that are often considered as part of the Standard Model of particle physics. These theories are Quantum Field Theory, Quantum Electro Dynamics, and Quantum Chromo Dynamics. QFT, QED, and QCD seek their foundation in the Lagrangian that is derived from the least action principle. The author considers this principle a high-level concept that follows from the behavior of the coverage of space by an uncountable set of point-like objects.

The first-order change equations (6.1.5) and (6.1.18) already reflect this typical behavior.

The least action principle does not imply the ongoing recurrent regeneration of the elementary fermions. Chapter 12 shows how the Lagrangian relates to the embedded hopping path of the elementary fermion.

The shortlist of counts of the differences in symmetry corresponds to a shortlist of electric charges -1, -2/3, -1/3, 0, +1/3, +2/3, +1 in the Standard Model.

The mathematical model does not predict that each elementary fermion owns a private footprint vector. Each fermion type has a fixed mass. This means that the private stochastic mechanisms very regularly regenerate the same deformation. The mathematical model does not yet predict this regularity.

10.2 Conglomerates

Elementary fermions appear to behave as elementary modules. The conglomerates of these elementary modules populate the dynamic field that we call our universe. All massive objects, except black holes are conglomerates of elementary fermions. All elementary fermions own mass. This means that the universe is covered by massive modular systems.

A private stochastic process determines the complete local life story of each elementary fermion. That stochastic process is controlled in the change space of its private Hilbert space. The private stochastic process produces a continuous hopping path and corresponds to a footprint vector that consists of a dynamically changing superposition of the reference operator's eigenvectors. This is explained in the formula (5.2.21). Each floating platform of the Hilbert repository owns a single private footprint vector. The footprint vector acts as the state vector of the elementary fermion and the probability amplitude corresponds to what physicists call the wavefunction of the particle.

This invites the idea that conglomerates of elementary fermions are defined by stochastic processes whose characteristic functions are defined in the change space of the background platform. In this change space, the characteristic function of a stochastic process that defines a conglomerate is a superposition of the characteristic functions of the components of the conglomerate. The dynamic superposition coefficients act as displacement generators. This means that these displacement generators define the internal oscillations of the components within the conglomerates.

Since in change space, position is not defined, the fact that a component belongs to a conglomerate does not restrict the distance between the components. This way of defining the membership of a conglomerate introduces entanglement. Independent of their mutual distance, components of a conglomerate must still obey the Pauli exclusion principle.

10.3 Hadrons

Hadrons can be mesons or baryons. They are conglomerates of quarks. Quarks can only bind via oscillations and via the attraction that is induced by their electric charges. Since the symmetry of quarks does not differ from the background symmetry in an isotropic way, the footprint of quarks does not deform the embedding field. So, mass does not help to bind the quarks until they reach an isotropic symmetry difference. This phenomenon is called color confinement. Hadrons feature mass. Thus, these conglomerates are sufficiently isotropic to deform the embedding field. Once configured, the mutual binding of baryons is very strong. The nuclei of atoms are constituted by baryons.

10.4 Atoms

Compound modules are composite modules for which the images of the geometric centers of the platforms of the components coincide in the background platform. The charges of the platforms of the elementary modules establish the binding of the corresponding platforms. Physicists and chemists call these compound modules atoms or atomic ions.

In free compound modules, the geometric symmetry-related charges do not take part in the oscillations. The targets of the private stochastic processes of the elementary modules oscillate. This means that the hopping path of the elementary module folds around the oscillation path and the hop landing location swarm gets smeared along the oscillation path. The oscillation path is a solution to the Helmholtz equation. Each fermion must use a different oscillation mode. A change of the oscillation mode goes together with the emission or absorption of a photon. As suggested earlier the emission or absorption of a photon involves a switch from the quaternionic Hilbert space to a subspace that is represented by a complex-number-based Hilbert space. The duration of the switch lasts a full particle regeneration cycle. During that cycle, the stochastic mechanism does not produce a swarm of hop landing locations that produce pulses that generate spherical shock fronts, but instead, it produces a one-dimensional string of equidistant pulse responses that cause one-dimensional shock fronts. The center of emission coincides with the geometrical center of the compound module. This ensures that the emitted photon does not lose its integrity. All photons will share the same emission duration, and that duration will coincide with the regeneration cycle of the hop landing location swarm. This is the reason that photons obey the Planck-

Einstein relation $E = h\nu$. Absorption cannot be interpreted so easily. It can only be comprehended as a time-reversed emission act. Otherwise, the absorption would require an incredible aiming precision for the photon. The number of one-dimensional pulses in the string corresponds to the step in the energy of the Helmholtz oscillation.

The type of stochastic process that controls the binding of components appears to be responsible for the absorption and emission of photons and the change of oscillation modes. If photons arrive with too low energy, then the energy is spent on the kinetic energy of the common platform. If photons arrive with too high energy, then the energy is distributed over the available oscillation modes, and the rest is spent on the kinetic energy of the common platform, or it escapes into free space. The process must somehow archive the modes of the components. It can apply the private platform of the components for that purpose. Most probably, the current value of the dynamic superposition coefficient is stored in the eigenspace of a special superposition operator.

10.5 Molecules

Molecules are conglomerates of compound modules that each keep their private geometrical center. However, electron oscillations are shared among the compound modules. Together with the geometric symmetry-related charges, this binds the compound modules into the molecule.

11 Dynamics in the Hilbert repository

11.1 Embedding in the background platform

The differences in the symmetry between the platforms only become apparent when a floating platform is embedded into the background platform or more specific when eigenvalues of a dedicated selection operator are mapped to corresponding eigenvectors in the background platform. A special operator in the non-separable Hilbert space of the background platform manages in its eigenspace the dynamic field that embeds discrete eigenvalues that originate from the eigenspace of the selection operator that resides in the floating platform. The eigenspace of the selection operator is filled in advance by a stochastic preselection process. The selector of the stochastic preselection process hops around in the eigenspace of the reference operator such that after sequencing the timestamps, an ongoing hopping path results that recurrently regenerates a hop landing location swarm that can be described by a stable location density distribution. The Fourier transform of this location density distribution equals the characteristic function of the stochastic selection mechanism. The hop landing location swarm generates the footprint of the floating platform in the eigenspace of the operator that manages the embedding field in the background platform. The coverage of the embedding field lets the field act as a sticky medium. The sticky medium resists the embedding of objects that break the symmetry of the embedding field. It appears that only isotropic symmetry breaks can deform the embedding field. The sticky medium reacts to the deformation by moving the deformation in all directions away from the embedding location until it vanishes at infinity. Differential calculus shows that the sticky medium reacts with a spherical pulse response that behaves as a spherical shock front that diminishes its amplitude with increasing distance from the location of the pulse. The pulse responses can superpose and join into a more persistent and more smoothed local deformation. This occurs when large amounts of nearby point-like actuators cooperate during a long enough time interval.

Aside from this footprint streaming mechanism, the symmetry-related charges represent sources or sinks that generate streams that embed symmetry-related fields into the embedding field. The charges are not spread over the root geometry of the floating platform. Instead, they locate in the geometric center of the floating platform. Thus, the map of the footprint spreads around the image of the symmetry-related charge.

Without these streaming processes, not many dynamics would occur in the embedding field.

11.2 Footprint

An ongoing embedding of a stream of symmetry-disturbing eigenvalues will cause a persistent deformation of the embedding field. The eigenspace of the footprint operator can archive a cord of quaternionic storage bins that contain the timestamps and the landing locations that will be embedded. After sequencing the timestamps, the archive shows an ongoing hopping path that is used in an ongoing embedding process. This embedding process runs during the running episode of the Hilbert repository and acts as an ***imaging process*** in which the image quality is characterized by an Optical Transfer Function. This function is the Fourier transfer of the Point Spread Function. The Point Spread Function can be interpreted as a hop landing location density distribution. Its Fourier transform is the Optical Transfer Function of the imaging process that embeds the footprint of the considered object.

11.2.1 Footprint mechanism

The mechanism that generates the content of the eigenspace of the footprint operator did its work in the creation episode of the Hilbert repository. The private natural parameter space of the Hilbert space already exists in this creation episode. The timestamps and the hopping locations of the hopping path were taken from this private parameter space. The footprint mechanism owns a characteristic function that ensures that the hopping path recurrently regenerates a hop landing location swarm that features a stable location density distribution which is the Fourier transform of the characteristic function of the footprint mechanism. The location density distribution equals the mentioned Point Spread Function, and the characteristic function equals the corresponding Optical Transfer Function.

The hopping path, the hop landing location swarm, the location density distribution, and the Point Spread Function reside in the position space of the Hilbert space. The continuous location density distribution equals the Point Spread Function and describes the discrete hop landing location swarm.

The Optical Transfer Function equals the characteristic function of the footprint mechanism, and both reside in the change space.

Nothing is said yet about the distribution of the timestamps. In imaging processes, the distribution of discrete objects in the imaging beam can often be characterized as the result of a combination of a Poisson process and a binomial process, where the binomial process is implemented by a spatial point spread function. In that case, the Poisson process handles the distribution of the timestamps.

11.2.2 Footprint characteristics

The footprint generating mechanism recurrently produces a nearly constant stream of potential point-like actuators in the form of a swarm that features a constant location density distribution. The stream takes the form of an ongoing hopping path. The actuators that originate from the same floating separable Hilbert space have a constant symmetry.

Some of these actuator symmetries can disturb the symmetry of the embedding field and therefore they can generate pulse responses that at least temporarily deform this field. A symmetry disturbance that generates a spherical pulse response must represent an isotropic difference between the two symmetries. A sufficiently constant and sufficiently dense and coherent stream of such actuators can generate a persistent deformation.

11.3 Resisting change

The Green's function, the shock fronts, and the oscillations also demonstrate the stickiness of dynamic quaternionic fields. Discrete sets of quaternions do not show this stickiness.

The stickiness of the field tends to flatten the field and it resists permanent deformations of the field.

11.3.1 Potential

In physics, potential energy is the energy held by an object because of its position relative to other objects.

The gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a reference location where the value of the potential equals zero.

The spherical shock fronts integrate over time into the Green's function of the field. Thus, the shock front injects the content of the Green's function into the affected field. All spherical shock fronts spread the contents of the front over the full field.

We consider the gravitational potential to be zero at infinity. Thus, if infinity is selected as the reference location, then the gravitational potential at a considered location is equal to the work (energy transferred) per unit mass that would be needed to move an object from infinity to that location. The potential at a location represents the reverse action of the combined spherical shock fronts that act at that location.

11.3.2 Center of deformation

The deformation potential $V(r)$ describes the effect of a local response to an isotropic point-like actuator and reflects the work that must be done by an agent to bring a unit amount of the injected stuff from infinity back to the considered location.

$$V(r) = m_p G / r \tag{10.3.1}$$

Here m_p represents the mass that corresponds to the full pulse response. G takes care for adaptation to physical units. r is the distance to the location of the pulse.

A stream of footprint actuators recurrently regenerates a coherent swarm of embedding locations in the dynamic universe field. Viewed from sufficient distance r that swarm generates a potential

$$V(r) = MG / r \tag{10.3.2}$$

Here M represents the mass that corresponds to the considered swarm of pulse responses. r is the distance to the center of the deformation. This formula is valid at sufficiently large values of r such that the whole swarm can be considered as a point-like object.

In a coherent swarm of massive objects $p_i, i = 1, 2, 3, \ldots n$, each with static mass m_i at locations r_i, the center of mass $\vec{R}$ follows from

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{R} \right) = \vec{0} \tag{10.3.3}$$

Thus

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \tag{10.3.4}$$

Where

$$M = \sum_{i=1}^{n} m_i \tag{10.3.5}$$

In the following, we will consider an ensemble of massive objects that own a center of mass $\vec{R}$ and a fixed combined mass M as a single massive object that locates at $\vec{R}$. The separate masses m_i may differ because, at the instant of summation, the corresponding deformation might have partly faded away.

$\vec{R}$ can be a dynamic location. In that case, the ensemble must move as one unit. The problem with the treatise in this paragraph is that in physical reality, point-like objects that possess a static mass do not exist. Only pulse responses that temporarily deform the field exist. Except for black holes, these pulse responses constitute all massive objects that exist in the universe.

11.4 Pulse location density distribution

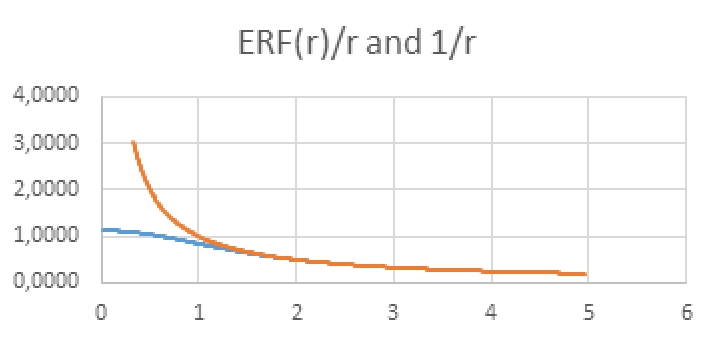
It is false to treat a pulse location density distribution as a set of point-like masses as is done in formulas (10.3.3) and (10.3.4). Instead, the gravitational potential follows from the convolution of the location density distribution and the Green's function. This calculation is still not correct, because the exact result depends on the fact that the deformation that is due to a pulse response quickly fades away and the result also depends on the density of the distribution. If these effects can be ignored, then the resulting gravitational potential of a Gaussian density distribution would be given by

$$g(r) \approx GM \frac{ERF(r)}{r} \tag{10.4.1}$$

Where $ERF(r)$ is the well-known error function. Here the gravitational potential is a perfectly smooth function that at some distance from the center equals the approximated gravitational potential that was described above in the equation (10.3.2). As indicated above, the convolution only offers an approximation because this computation does not account for the influence of the density of the swarm, and it does not compensate for the fact that the deformation by the individual pulse responses quickly fades away. Thus, the exact result depends on the duration of the recurrence cycle of the swarm.

In the example, we apply a normalized location density distribution, but the actual location density distribution might have a higher amplitude.

This might explain why some elementary module types exist in multiple generations. These generations appear to have their own mass. For example, elementary fermions exist in three generations. The two more massive generations usually get the name muon or tau generation.



This might also explain why different first-generation elementary particle types show different masses. Due to the convolution, and the coherence of the location density distribution, the blue curve does not show any sign of the singularity that is contained in the red curve, which shows the Green's function.

In physical reality, no point-like static mass object exists. The most important lesson of this investigation is that far from the gravitational center of the distribution the deformation of the field is characterized by the here shown simplified form of the gravitation potential

$$\phi(r) \approx \frac{GM}{r} \tag{10.4.2}$$

Warning: This simplified form shares its shape with the Green's function of the deformed field. This does not mean that the Green's function owns a mass that equals $M_G = \frac{1}{G}$. The functions only share the form of their tail.

11.5 Rest mass

The weakness in the definition of the gravitation potential is the definition of the unit of mass and the fact that shock fronts move with a fixed finite speed. Thus, the definition of the gravitation potential only works properly if the geometric center location of the swarm of injected spherical pulses is at rest in the affected embedding field. The consequence is that the mass that follows from the definition of the gravitation potential is the ***rest mass*** of the considered swarm. We will call the mass that is corrected for the motion of the observer relative to the observed scene the ***inertial mass***.

11.6 Observer

The inspected location is the location of a hypothetical test object that owns an amount of mass. It can represent an elementary particle or a conglomerate of such particles. This location is the target location in the embedding field. The embedding field is supposed to be deformed by the embedded objects.

Observers can access information that is retrieved from storage locations that for them have a historic timestamp. That information is transferred to them via the dynamic universe field. This dynamic field embeds both the observer and the observed event. The dynamic geometric data of point-like objects are archived in Euclidean format as a combination of a timestamp and a three-dimensional spatial location. The embedding field affects the format of the transferred information. The observers perceive in spacetime format. A hyperbolic Lorentz transform converts the Euclidean coordinates of the background parameter space into the spacetime coordinates that are perceived by the observer.

11.6.1 Lorentz transform

In dynamic fields, shock fronts move with speed c. In the quaternionic setting, this speed is unity.

$$x^2+y^2+z^2=c^2\tau^2 \qquad (10.6.1)$$

In flat dynamic fields, swarms of triggers of spherical pulse responses move with lower speed v.

For the geometric centers of these swarms still holds:

$$x^2+y^2+z^2-c^2\tau^2=x'^2+y'^2+z'^2-c^2\tau'^2 \qquad (10.6.2)$$

If the locations $\{x,y,z\}$ and $\{x',y',z'\}$ move with uniform relative speed v, then

$$ct'=ct\cosh(\omega)-x\sinh(\omega) \qquad (10.6.3)$$

$$x'=x\cosh(\omega)-ct\sinh(\omega) \qquad (10.6.4)$$

$$\cosh(\omega)=\frac{\exp(\omega)+\exp(-\omega)}{2}=\frac{c}{\sqrt{c^2-v^2}} \qquad (10.6.5)$$

$$\sinh(\omega)=\frac{\exp(\omega)-\exp(-\omega)}{2}=\frac{v}{\sqrt{c^2-v^2}} \qquad (10.6.6)$$

$$\cosh(\omega)^2-\sinh(\omega)^2=1 \qquad (10.6.7)$$

This is a hyperbolic transformation that relates two coordinate systems, which is known as a Lorentz boost.

This transformation can concern two platforms P and P' on which swarms reside and that move with uniform relative speed.

However, it can also concern the storage location P that contains a timestamp τ and spatial location $\{x, y, z\}$ and platform P' that has coordinate time t' and location $\{x', y', z'\}$.

In this way, the hyperbolic transform relates two platforms that move with uniform relative speed. One of them may be a floating Hilbert space on which the observer resides. Or it may be a cluster of such platforms that cling together and move as one unit. The other may be the background platform on which the embedding process produces the image of the footprint.

The Lorentz transform converts a Euclidean coordinate system consisting of a location $\{x, y, z\}$ and proper timestamps τ into the perceived coordinate system that consists of the spacetime coordinates $\{x', y', z', ct'\}$ in which t' plays the role of coordinate time. The uniform velocity v causes time dilation $\Delta t' = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}$ and length contraction

$\Delta L' = \Delta L\sqrt{1-\frac{v^2}{c^2}}$

11.6.2 Minkowski metric

Spacetime is ruled by the Minkowski metric.

In flat field conditions, proper time τ is defined by

$$\tau = \pm\frac{\sqrt{c^2t^2 - x^2 - y^2 - z^2}}{c} \tag{10.6.8}$$

And in deformed fields, still

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{10.6.9}$$

Here ds is the spacetime interval and $d\tau$ is the proper time interval. dt is the coordinate time interval

11.6.3 Schwarzschild metric

Polar coordinates convert the Minkowski metric to the Schwarzschild metric. The proper time interval $d\tau$ obeys

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 d\varphi^2\right) \tag{10.6.10}$$

Under pure isotropic conditions, the last term on the right side vanishes.

According to mainstream physics, in the environment of a black hole, the symbol r_s stands for the Schwarzschild radius.

$$r_s = \frac{2GM}{c^2} \tag{10.6.11}$$

The variable r equals the distance to the center of mass of the massive object with mass M .

The Hilbert Book model finds a different value for the boundary of a spherical black hole. That radius is a factor of two smaller.

11.6.4 Event horizon

The gravitational potential energy $U(r)$

$$U(r) = \frac{mMG}{r} \tag{10.6.12}$$

at the event horizon $r = r_{eh}$ of a black hole is supposed to be equal to the mass-energy equivalent of an object that has unit mass $m = 1$ and is brought by an agent from infinity to that event horizon. Dark energy objects are energy packages in the form of one-dimensional shock fronts that are a candidate for this role. Photons are strings of equidistant samples of these energy packages. The energy equivalent of the unit mass objects is

$$E = mc^2 = \frac{mMG}{r_{eh}} \quad (10.6.13)$$

Or

$$r_{eh} = \frac{MG}{c^2} \quad (10.6.14)$$

At the event horizon, all energy of the dark energy object is consumed to compensate for the gravitational potential energy at that location. No field excitation and in particular no shock front can pass the event horizon.

11.7 Inertial mass

The Lorentz transform also gives the transform of the rest mass to the mass that is relevant when the embedding field moves relative to the floating platform of the observed object with uniform speed $\vec{v}$.

In that case, the inertial mass M relates to the test mass M_0 as

$$M = \gamma M_0 = \frac{M_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (10.7.1)$$

This indicates that the formula (10.3.2) for the gravitational potential at distance r must be changed to

$$V(r) = \frac{M_0 G}{r\sqrt{1-\frac{v^2}{c^2}}} \qquad (10.7.2)$$

11.8 Inertia

The relation between inertia and mass is complicated. We apply an artificial field that resists its changing. The condition that for each type of massive object, the gravitational potential is a static function, and the condition that in free space, the massive object moves uniformly, establish that inertia rules the dynamics of the situation. These conditions define an artificial quaternionic field that resists change. The scalar part of the artificial field is represented by the gravitational potential, and the uniform speed of the massive object represents the vector part of the field.

The first-order change of the quaternionic field can be divided into five separate partial changes. Some of these parts can compensate for each other.

Mathematically, the statement that in the first approximation nothing in the field ξ changes indicates that locally, the first-order partial differential $\nabla\xi$ will be equal to zero.

$$\zeta = \nabla\xi = \nabla_r\xi_r - \langle\vec{\nabla},\vec{\xi}\rangle + \vec{\nabla}\xi_r + \nabla_r\vec{\xi} \pm \vec{\nabla}\times\vec{\xi} = 0 \qquad (10.8.1)$$

Thus

$$\zeta_r = \nabla_r\xi_r - \langle\vec{\nabla},\vec{\xi}\rangle = 0 \qquad (10.8.2)$$

$$\vec{\zeta} = \vec{\nabla}\xi_r + \nabla_r\vec{\xi} \pm \vec{\nabla}\times\vec{\xi} = 0 \qquad (10.8.3)$$

These formulas can be interpreted independently. For example, according to the equation (10.8.2), the variation in time of ξ_r can compensate the divergence of $\vec{\xi}$. The terms that are still eligible for change must together be equal to zero. For our purpose, the curl $\vec{\nabla}\times\vec{\xi}$ of the vector field $\vec{\xi}$ is expected to be zero. The resulting terms of the equation (10.8.3) are

$$\nabla_r\vec{\xi}+\vec{\nabla}\xi_r=0 \qquad (10.8.4)$$

In the following text plays $\vec{\xi}$ the role of the vector field and ξ_r plays the role of the scalar gravitational potential of the considered object. For elementary modules, this special field concerns the effect of the hop landing location swarm that resides on the floating platform on its image in the embedding field. It reflects the activity of the stochastic process and the uniform movement of the geometric center of the floating platform over the embedding field in the background platform. It is characterized by a mass value and by the uniform velocity of the floating platform with respect to the background platform. The real (scalar) part conforms to the deformation that the stochastic process causes. The vector part conforms to the speed of movement of the floating platform. The main characteristic of this field is that it tries to keep its overall change zero. The author calls ξ the ***conservation field***.

At a large distance r, we approximate this potential by using the formula

$$\zeta_r(r)\approx\frac{GM}{r} \qquad (10.8.5)$$

Here M is the inertial mass of the object that causes the deformation. The new artificial field $\xi=\left\{\frac{GM}{r},\vec{v}\right\}$ considers a uniformly moving mass as a normal situation. It is a combination of scalar potential $\frac{GM}{r}$ and speed $\vec{v}$. This speed of movement is the relative speed between the floating platform and the background platform. At rest this speed is uniform.

If this object accelerates, then the new field $\left\{\frac{GM}{r},\vec{v}\right\}$ tries to counteract the change of the vector field $\vec{v}$ by compensating this with an equivalent change of the scalar part $\frac{GM}{r}$ of the new field ξ. According to the equation (10.8.4), this equivalent change is the gradient of the real part of the field.

$$\vec{a}=\dot{\vec{v}}=-\vec{\nabla}\left(\frac{GM}{r}\right)=\frac{GM\,\vec{r}}{|\vec{r}|^3} \qquad (10.8.6)$$

This generated vector field acts on masses that appear in its realm.

Thus, if two uniformly moving masses m and M exist in each other's neighborhood, then any disturbance of the situation will cause the gravitational force

$$\vec{F}(\vec{r}_1-\vec{r}_2)=m_0\vec{a}=\frac{Gm_0M(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}=\gamma\frac{Gm_0M_0(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3} \qquad (10.8.7)$$

Here $M=\gamma M_0$ is the inertial mass of the object that causes the deformation. m_0 is the rest mass of the observer.

The inertial mass M relates to its rest mass M_0 as

$$M=\gamma M_0=\frac{M_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (10.8.8)$$

This formula holds for all elementary particles except for quarks.

The problem with quarks is that these particles do not provide an isotropic symmetry difference. They must first combine into hadrons to be able to generate an isotropic symmetry difference. This phenomenon is known as ***color confinement***.

11.9 Momentum

In the formula (10.8.7) that relates mass to force the factor γ that corrects for the relative speed can be attached to m_0 or to M_0

$$\vec{F}(\vec{r}_1 - \vec{r}_2) = \gamma \frac{G m_0 M_0 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} \tag{10.9.1}$$

The force relates to the temporal change of the momentum vector $\vec{P}$ of the observer

$$\vec{F} = \dot{\vec{P}} = \frac{d\vec{P}}{dt} \tag{10.9.2}$$

The momentum vector $\vec{P}$ is part of a quaternionic momentum P. The momentum depends on the relative speed of the moving object that causes the deformation which defines the mass. The speed is determined relative to the field that embeds the object and that gets deformed by the investigated object. For free elementary particles, the speed equals the floating speed of the platform on which the particle resides.

$$P = P_r + \vec{P} \tag{10.9.3}$$

$$\|P\|^2 = P_r^2 + \|\vec{P}\|^2 \tag{10.9.4}$$

$$\vec{P} = \gamma m_0 \vec{v} \tag{10.9.5}$$

$$\|\vec{P}\|^2 = \gamma^2 m_0^2 \|\vec{v}\|^2 \tag{10.9.6}$$

$$\|P\|^2 = \gamma^2 m_0^2 c^2 = P_r^2 + \gamma^2 m_0^2 \|\vec{v}\|^2 \tag{10.9.7}$$

$$\|P\| = \gamma m_0 c = E / c \tag{10.9.8}$$

$$E = \gamma m_0 c^2 \tag{10.9.9}$$

$$\begin{aligned} P_r^2 &= \gamma^2 m_0^2 c^2 - \gamma^2 m_0^2 \|\vec{v}\|^2 \\ &= \gamma^2 m_0^2 \left(c^2 - \|\vec{v}\|^2\right) = \gamma^2 m_0^2 c^2 \left(1 - \left\|\frac{\vec{v}}{c}\right\|^2\right) = m_0^2 c^2 \end{aligned} \tag{10.9.10}$$

$$P_r = m_0 c = \frac{E}{\gamma c} \tag{10.9.11}$$

$$\|\vec{P}\| = \gamma m_0 \|\vec{v}\| \tag{10.9.12}$$

$$P = P_r + \vec{P} = m_0 c + \gamma m_0 \vec{v} = \frac{E}{\gamma c} + \gamma m_0 \vec{v} \tag{10.9.13}$$

If $\vec{v} = \vec{0}$ then $\vec{P} = \vec{0}$ and $\|P\| = P = P_r = m_0 c$

Here Einstein's famous mass-energy equivalence is involved.

$$E = \gamma m_0 c^2 = mc^2 \tag{10.9.14}$$

The disturbance by the ongoing expansion of the embedding field suffices to put the gravitational force into action. The description also holds when the field ξ describes a conglomerate of platforms and M represents the mass of the conglomerate.

The artificial field ξ represents the habits of the underlying model that ensures the constancy of the gravitational potential and the uniform floating of the considered massive objects in free space.

Inertia ensures that the third-order differential (the third-order change) of the deformed field is minimized. It does that by varying the speed of the platforms on which the massive objects reside.

Inertia bases mainly on the definition of mass that applies to the region outside the sphere where the gravitational potential behaves like the Green's function of the field. There, the formula $\xi_r = \frac{GM}{r}$ applies. Further, it bases on the intention of modules to keep the gravitational potential inside the mentioned sphere constant. At least that holds when this potential is averaged over the regeneration period. In that case, the overall change $\nabla\xi$ in the conservation field ξ equals zero. Next, the definition of the conservation field supposes that the swarm which causes the deformation moves as one unit. Further, the fact is used that the solutions of the homogeneous second-order partial differential equation can superpose in new solutions of that same equation.

The popular sketch in which the deformation of our living space is presented by smooth dips is obviously false. The story that is represented in this paper shows the deformations as local extensions of the field, which represents the universe. In both sketches, the deformations elongate the information path, but none of the sketches explain why two masses attract each other. The above explanation founds on the habit of the stochastic process to recurrently regenerate the same time average of the gravitational potential, even when that averaged potential moves uniformly. Without the described habit of the stochastic processes, inertia would not exist.

The applied artificial field also explains the gravitational attraction by black holes.

The artificial field that implements mass inertia also plays a role in other fields. Similar tricks can be used to explain the electrical force from the fact that the electrical field is produced by sources and sinks that can be described with the Green's function.

11.9.1 Forces

In the Hilbert repository, all symmetry-related charges are located at the geometric center of an elementary particle and all these particles own a footprint that for isotropic symmetry differences can deform the embedding field. In that case, the particle features mass and forces might be coupled to acceleration via

$$F = m\vec{a} \tag{10.9.15}$$

Or to momentum via

$$F = \dot{\vec{P}} \tag{10.9.16}$$

12 From hopping path to Lagrangian

We restrict our view to a complex-number-based Hilbert space that orients along the direction $\vec{n}$ of the current displacement generator $\vec{p}$

The hopping path is a series of hop landing locations $\{\vec{a}_i\}$ We divide each hop into steps.

1. Change to Fourier space. This involves inner product $\langle \vec{a}_i | \vec{p} \rangle$
2. Evolve during an infinitesimal progression step into the future.
   a. Multiply with the corresponding displacement generator $\vec{p}$
   b. The generated step in configuration space is $(\vec{a}_{i+1} - \vec{a}_i)$.
   c. The action contribution in Fourier space is $\langle \vec{p}, \vec{a}_{i+1} - \vec{a}_i \rangle$.
   d. This combines in a unitary factor $\exp(\vec{n}\langle \vec{p}, \vec{a}_{i+1} - \vec{a}_i \rangle)$
3. Change back to configuration space. This involves inner product $\langle \vec{p} | \vec{a}_{i+1} \rangle$
   a. The combined term contributes a factor $\langle \vec{a}_i | \vec{p} \rangle \exp(\vec{n}\langle \vec{p}, \vec{a}_{i+1} - \vec{a}_i \rangle)\langle \vec{p} | \vec{a}_{i+1} \rangle$.

Two subsequent steps give:

$$\langle \vec{a}_i | \vec{p} \rangle \exp(\vec{n}\langle \vec{p}, \vec{a}_{i+1} - \vec{a}_i \rangle)\langle \vec{p} | \vec{a}_{i+1} \rangle \langle \vec{a}_{i+1} | \vec{p} \rangle \exp(\vec{n}\langle \vec{p}, \vec{a}_{i+2} - \vec{a}_i \rangle)\langle \vec{p} | \vec{a}_{i+2} \rangle$$

(11.1.1)

The terms in the middle turn into unity. The other terms also join.

$$\begin{aligned}&\langle \vec{a}_i | \vec{p} \rangle \exp(\vec{n}\langle \vec{p}, \vec{a}_{i+1} - \vec{a}_i \rangle)\exp(\vec{n}\langle \vec{p}, \vec{a}_{i+2} - \vec{a}_{i+1} \rangle)\langle \vec{p} | \vec{a}_{i+2} \rangle \\ &= \langle \vec{a}_i | \vec{p} \rangle \exp(\vec{n}\langle \vec{p}, \vec{a}_{i+2} - \vec{a}_i \rangle)\langle \vec{p} | \vec{a}_{i+2} \rangle\end{aligned} \qquad (11.1.2)$$

Over a full particle generation cycle with N steps this results in:

$$\prod_{i=1}^{N-1}\langle\vec{a}_i|\vec{p}\rangle\exp\left(\vec{n}\langle\vec{p},\vec{a}_{i+1}-\vec{a}_i\rangle\right)\langle\vec{p}|\vec{a}_{i+1}\rangle$$
$$=\langle\vec{a}_1|\vec{p}\rangle\exp\left(\vec{n}\langle\vec{p},\vec{a}_N-\vec{a}_1\rangle\right)\langle\vec{p}|\vec{a}_N\rangle$$
$$=\langle\vec{a}_1|\vec{p}\rangle\exp\left(\vec{n}\sum_{i=2}^{N}\langle\vec{p},\vec{a}_{i+1}-\vec{a}_i\rangle\right)\langle\vec{p}|\vec{a}_N\rangle \tag{11.1.3}$$
$$=\langle\vec{a}_1|\vec{p}\rangle\exp\left(\vec{n}L\right)\langle\vec{p}|\vec{a}_N\rangle$$

$$L\,d\tau=\sum_{i=2}^{N}\langle\vec{p},\vec{a}_{i+1}-\vec{a}_i\rangle=\langle\vec{p},d\vec{q}\rangle \tag{11.1.4}$$

$$L=\langle\vec{p},\dot{\vec{q}}\rangle \tag{11.1.5}$$

L is known as the Lagrangian. $\vec{p}$ is the displacement generator. $\dot{\vec{q}}$ is the speed of the platform on which the particle resides. The image of the particle moves over the embedding field.

The equation (11.1.5) holds for the special condition in which $\vec{p}$ is constant. If $\vec{p}$ is not constant, then the Hamiltonian H varies with location.

$$\frac{\partial H}{\partial q_i}=-\dot{p}_i \tag{11.1.6}$$

$$\frac{\partial H}{\partial p_i}=\dot{q}_i \tag{11.1.7}$$

$$\frac{\partial L}{\partial q_i}=\dot{p}_i \tag{11.1.8}$$

$$\frac{\partial L}{\partial \dot{q}_i}=-\dot{p}_i \tag{11.1.9}$$

$$\frac{\partial H}{\partial \tau}=-\frac{\partial L}{\partial \tau} \tag{11.1.10}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{11.1.11}$$

$$H + L = \sum_{i=1}^{3} \dot{q}_i p_i \tag{11.1.12}$$

Here we used proper time τ rather than coordinate time t.

In the above reasoning the probability distribution of the steps is ignored. When that distribution is included, the influence of the stochastic generation will become clear.

References

More details and references to supporting theories are contained in "The Standard Model of Elementary Fermions and the Hilbert Repository"; https://vixra.org/abs/2106.0135