The Theodorus Constant

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Abstract.In this note we give some formulas related with the Theodorus constant T=1.860025....

Keywords. Theodorus constant , Spiral of Theodorus , Series , Integrals.

Theodorus of Cyrene (ca. 460-399 B.C.) ,teacher of Plato and Theaetetus, is known for his proof of the irrationality of $\sqrt{n}$, $n = 2, 3, 5, \ldots, 17$.

1. Introduction

The discrete spiral of theodorus is defined by

$$z_n = \left(1 + \frac{i}{\sqrt{n}}\right) z_{n-1} \ , \ z_0 = 1 \ , \ \ n = 1, 2, 3, \ldots; \ i = \sqrt{-1} \tag{1}$$

The points z_{n-1} and z_n determine a right triangle relative to the origin 0 , with legs 1 and $\sqrt{n}$. The polar coordinates (r_n, θ_n) of z_n are given by

$$r_n = \sqrt{n+1} \ , \ \ \theta_n = \sum_{k=0}^{n-1} \arctan\left(\frac{1}{\sqrt{k+1}}\right), \ n = 1, 2, 3, \ \ldots; \ \theta_0 = 0 \tag{2}$$

A closed-form expression for z_n is

$$z_n = \prod_{k=1}^{n} \left(1 + \frac{i}{\sqrt{k}}\right) \quad , \ n = 1, 2, 3, \ \ldots \tag{3}$$

The continuous spiral of theodorus is defined by

$$f(t) = \prod_{k=1}^{\infty} \frac{1 + \frac{i}{\sqrt{k}}}{1 + \frac{i}{\sqrt{k+t}}} = \sqrt{1+t} \ \exp\left(i \sum_{k=1}^{\infty} \left(\arctan\left(\sqrt{k+t}\right) - \arctan\left(\sqrt{k}\right)\right)\right) \ , \ -1 < t < \infty \tag{4}$$

and a polar representation is

$$\theta(r) = \sum_{k=0}^{\infty} \left(\arctan\left(\frac{1}{\sqrt{k+1}}\right) - \arctan\left(\frac{1}{\sqrt{k+r^2}}\right)\right) \ , \ r > 0 \tag{5}$$

The functional equation is

$$f(t) = \left(1 + \frac{i}{\sqrt{t}}\right) f(t-1) \ , \ f(0) = 1 \ , \ 0 < t < \infty \tag{6}$$

The slope of the spiral at the point 1 is

$$T = \left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = \left.\frac{d\theta}{dr}\right|_{(r,\theta)=(1,0)} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ (n+1)} = 1.860025 \ldots \tag{7}$$

This is called the constant of Theodorus (P.J. Davis 1993).

In this note we give some formulas for T .

2. Formulas for Theodorus constant

Entry 1. Series

$$T=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\,(n+1)}=\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2+4+6+\ldots+2n} \tag{8}$$

$$T=\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+1)}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \tag{9}$$

$$T=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)\sum_{k=1}^{n}\frac{1}{k+1} \tag{10}$$

$$T=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n(n+1)}}-\frac{1}{\sqrt{(n+1)(n+2)}}\right)\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}} \tag{11}$$

$$T=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{\sqrt{n}}{n+1}\right) \tag{12}$$

$$T=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}\left(\sqrt{\frac{n+1}{n}}-\sqrt{\frac{n}{n+1}}\right) \tag{13}$$

$$T=2\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}\sum_{k=1}^{n}\sqrt{k} \tag{14}$$

$$T=\sum_{n=1}^{\infty}n\left(\frac{1}{\sqrt{n}\,(n+1)}-\frac{1}{\sqrt{n+1}\,(n+2)}\right) \tag{15}$$

$$T=\frac{1}{2}+\sum_{n=2}^{\infty}(n-1)\left(\frac{1}{\sqrt{n}\,(n+1)}-\frac{1}{\sqrt{n+1}\,(n+2)}\right) \tag{16}$$

$$T=1+\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n+1}=1+\sum_{n=1}^{\infty}\frac{1}{(n+1)\left(\sqrt{n}+\sqrt{n+1}\right)} \tag{17}$$

$$T=\sum_{n=1}^{\infty}\frac{1}{(n+1)\sqrt{n}\left(1+\sqrt{n}\right)}-\sum_{n=1}^{\infty}\frac{1}{n(n+1)\left(1+\sqrt{n}\right)}+\sum_{n=1}^{\infty}\frac{1}{n\left(1+\sqrt{n}\right)} \tag{18}$$

$$T=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}-\frac{n}{(n+1)\sqrt{n+2}}\right)-\sum_{n=1}^{\infty}\frac{1}{n(n+1)\left(\sqrt{n}+\sqrt{n+1}\right)} \tag{19}$$

$$T=2\sum_{n=2}^{\infty}\frac{1}{n\left((n+1)\sqrt{n-1}+(n-1)\sqrt{n+1}\right)}+\sum_{n=2}^{\infty}(n-1)\left(\frac{1}{n\sqrt{n+1}}-\frac{1}{(n+1)\sqrt{n+2}}\right) \tag{20}$$

$$T=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}-\frac{n}{(n+1)\sqrt{n+2}}\right)-2\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,(n^2-1)} \tag{21}$$

$$T = 1 + \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - 1}{(n+1)(n+2)} = 1 + \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)\left(1 + \sqrt{n+1}\right)} \tag{22}$$

$$\begin{gathered} T = 2 \sum_{n=1}^{\infty} \tan^{-1}\left(\sqrt{1 + n(n+1)^2} - (n+1)\sqrt{n}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \sum_{n=1}^{\infty} \frac{1}{\left(\sqrt{n}\,(n+1)\right)^{2k+1}} = \\ 2 \sum_{m=n(n+1)^2, n\in\mathbb{N}} \tan^{-1}\left(\sqrt{1+m} - \sqrt{m}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \sum_{m=n(n+1)^2, n\in\mathbb{N}} \frac{1}{m^k \sqrt{m}} \end{gathered} \tag{23}$$

Entry 2. Zeta series

$$T = \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\zeta\left(n + \frac{1}{2}\right) - 1\right) \tag{24}$$

$$T = \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \zeta\left(k + \frac{3}{2}\right) \tag{25}$$

$$T = \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} \left(\zeta\left(n + \frac{3}{2}\right) - 1\right) \tag{26}$$

$$T = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\zeta\left(2n - \frac{1}{2}\right) - \zeta\left(2n + \frac{1}{2}\right)\right) \tag{27}$$

$$T = \sum_{n=1}^{\infty} \frac{1}{n^3 + n} + \sum_{n=0}^{\infty} (-1)^n \left(\zeta\left(n + \frac{3}{2}\right) - \zeta(2n+3)\right) \tag{28}$$

$$T = \zeta\left(\frac{3}{2}\right) - 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)\left(n\sqrt{n+1} + (n+1)\sqrt{n}\right)} \tag{29}$$

$$T = \zeta\left(\frac{3}{2}\right) - \sum_{n=1}^{\infty} \frac{1}{n\left(n\sqrt{n+1} + (n+1)\sqrt{n}\right)} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)\left(\sqrt{n} + \sqrt{n+1}\right)} \tag{30}$$

$$T = \sum_{n=0}^{\infty} \left(\zeta\left(n + \frac{3}{2}\right) - 1\right) - \sum_{n=1}^{\infty} \frac{1}{n(n+1)\left(\sqrt{n} + \sqrt{n+1}\right)} \tag{31}$$

$$T = \zeta\left(\frac{3}{2}\right) - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \sum_{k=1}^{n} \frac{1}{k^{3/2}} \tag{32}$$

$$T = \zeta\left(\frac{3}{2}\right) - \sum_{n=1}^{\infty} \frac{1}{n(n+1)\sqrt{n}} \tag{33}$$

$$T = \zeta\left(\frac{3}{2}\right) - \zeta\left(\frac{5}{2}\right) + \sum_{n=1}^{\infty} \frac{1}{n^2 (n+1)\sqrt{n}} \tag{34}$$

$$T = \zeta\left(\frac{3}{2}\right) - \zeta\left(\frac{5}{2}\right) + \zeta\left(\frac{7}{2}\right) - \sum_{n=1}^{\infty} \frac{1}{n^3 (n+1)\sqrt{n}} \tag{35}$$

$$T = \sum_{n=1}^{k} (-1)^{n-1} \zeta\left(\frac{2n+1}{2}\right) + (-1)^k \sum_{n=1}^{\infty} \frac{1}{n^k (n+1)\sqrt{n}}, \quad k = 1, 2, 3, \ldots \tag{36}$$

$$T=\zeta\left(\frac{3}{2}\right)-\frac{1}{2}-\sum_{n=0}^{\infty}(-1)^n\left(\zeta\left(n+\frac{5}{2}\right)-1\right) \tag{37}$$

$$T=\frac{1}{2}+\sum_{n=0}^{\infty}(-1)^n\,(n+1)\left(\frac{3}{2n+3}\left(\zeta\left(n+\frac{3}{2}\right)-1\right)+\frac{1}{2n+5}\left(\zeta\left(n+\frac{5}{2}\right)-1\right)\right) \tag{38}$$

$$T=\sum_{n=0}^{\infty}(-1)^n\,(n+1)\left(\zeta\left(n+\frac{3}{2}\right)+\zeta\left(n+\frac{5}{2}\right)\right) \tag{39}$$

$$T=\frac{1}{2}+\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\left(\zeta\left(n+\frac{3}{2}\right)-1-2^{-n-\frac{3}{2}}\right) \tag{40}$$

$$T=\frac{1}{2}+\frac{1}{3\sqrt{2}}+\frac{1}{4\sqrt{3}}+\sum_{n=0}^{\infty}(-2)^n\left(\zeta\left(n+\frac{3}{2}\right)-1-2^{-n-\frac{3}{2}}\right)\sum_{k=0}^{n}\binom{2k}{k}2^{-3k} \tag{41}$$

$$T=\frac{1}{2}+\sum_{n=0}^{\infty}(-1)^n\left(\zeta\left(3n+\frac{3}{2}\right)-\zeta\left(3n+\frac{5}{2}\right)+\zeta\left(3n+\frac{7}{2}\right)-1\right) \tag{42}$$

$$T=\frac{1}{2}\zeta\left(\frac{3}{2}\right)+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\zeta\left(n+\frac{3}{2}\right)-\zeta\left(n+\frac{5}{2}\right)\right) \tag{43}$$

$$T=\sum_{n=0}^{\infty}(-1)^n\left(1-2^{-n-\frac{3}{2}}\right)\zeta\left(n+\frac{3}{2}\right)+\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\left(\left(1-2^{-n-\frac{3}{2}}\right)\zeta\left(n+\frac{3}{2}\right)-1\right) \tag{44}$$

$$T=\sum_{n=0}^{\infty}\left(\binom{2n}{n}2^{-2n}+(-1)^n\right)2^{-n-\frac{3}{2}}\zeta\left(n+\frac{3}{2}\right) \tag{45}$$

$$T=\frac{1}{2}+\frac{1}{3\sqrt{2}}+\sum_{n=0}^{\infty}\binom{2n}{n}(-2)^{-n}\sum_{k=0}^{n}\binom{n}{k}2^{-k}\left(\zeta\left(n+k+\frac{3}{2}\right)-1-2^{-n-k-\frac{3}{2}}\right) \tag{46}$$

$$T=\frac{1}{2}+\sum_{n=0}^{\infty}\left(\binom{2n}{n}2^{-2n}+(-1)^n\right)\left(2^{-n-\frac{3}{2}}\Phi\left(1,n+\frac{3}{2},\frac{1}{2}\right)-1\right) \tag{47}$$

$$T=\left(\frac{3}{2\sqrt{2}}-1\right)\zeta\left(\frac{3}{2}\right)+\frac{1}{2\sqrt{2}}\Phi\left(1,\frac{3}{2},\frac{1}{2}\right)+\sum_{n=1}^{\infty}(-1)^n\,2^{-n-\frac{3}{2}}\zeta\left(n+\frac{3}{2}\right)+\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{1}{\sqrt{2n-1}}-\frac{1}{\sqrt{2n}}\right) \tag{48}$$

$$T=\sum_{n=0}^{\infty}(-1)^n\left(2^{-n}\Phi\left(1,n+\frac{3}{2},\frac{1}{2}\right)-2\sqrt{2}\right)\sum_{k=0}^{n}\binom{2k}{k}2^{-2k}(-1)^k \tag{49}$$

$$T=\sum_{n=0}^{\infty}(-1)^n\left(\Phi\left(1,n+\frac{3}{2},a\right)-a^{-n-\frac{3}{2}}\right)\sum_{k=0}^{n}\binom{2k}{k}2^{-2k}(-1)^k a^k(1-a)^{n-k}\ ,\ 0<a<1 \tag{50}$$

Remark 1: $\zeta(x)=\sum_{n=1}^{\infty}n^{-x}$, $x>1$, is the Riemann zeta function.

Remark 2: $\Phi(z,s,u)=\sum_{n=0}^{\infty}z^n(u+n)^{-s}$, is the Lerch transcendent function.

Entry 3. Integrals

$$T=\frac{1}{4}+\frac{\pi}{2}-\frac{1}{2}\int_1^{\infty}\frac{1+3x}{x^{3/2}(1+x)^2}\left(x-[x]-\frac{1}{2}\right)dx \tag{51}$$

$$T = \frac{1}{4} + \frac{\pi}{2} + \sqrt{2} \int_0^\infty \frac{x \sqrt{\sqrt{1+x^2} + 1} + 2\sqrt{\sqrt{1+x^2} - 1}}{(4+x^2)\sqrt{1+x^2}\,(e^{2\pi x} - 1)} \, dx \tag{52}$$

$$T = 2 \int_0^\infty \frac{\left(2x\sqrt{\sqrt{1+4x^2} + 1} + 3\sqrt{\sqrt{1+4x^2} - 1}\right)\tanh(\pi x)}{(9+4x^2)\sqrt{1+4x^2}} \, dx \tag{53}$$

$$T = \frac{2}{\sqrt{\pi}} \int_0^\infty \left(-1 - e^{x^2} \ln\left(1 - e^{-x^2}\right)\right) dx \tag{54}$$

$$T = \int_0^1 \zeta\left(\frac{3}{2}, 2 - x^2\right) dx \tag{55}$$

$$T = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{1}{e^x - 1} F\left(\sqrt{x}\right) dx \tag{56}$$

$$T = \frac{4}{\sqrt{\pi}} \int_0^\infty \int_0^1 \frac{x^2 e^{-x^2(1-y^2)}}{e^{x^2} - 1} \, dy \, dx \tag{57}$$

$$T = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n (2n+5)}{n!\,(2n+1)(2n+3)} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{2k+1} + \frac{4}{\sqrt{\pi}} \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^n B_{k+1}}{(2k+2)!\,(n-k)!\,(2n+2k+5)} \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{(-1)^m}{2m+1} + \frac{4}{\sqrt{\pi}} \int_1^\infty \int_0^1 \frac{x^2 e^{-x^2(1-y^2)}}{e^{x^2} - 1} \, dy \, dx \tag{58}$$

Remark 3: $F(x) = e^{-x^2} \int_0^x e^{t^2} \, dt$, is Dawson's integral.

Remark 4: $\zeta(s, u) = \sum_{n=0}^\infty (u+n)^{-s} = \Phi(1, s, u)$, is the Hurwitz zeta function.

Remark 5: $B_k = \{1/6, 1/30, 1/42, 1/30, 5/66, \ldots\}$, are the Bernoulli numbers.

Remark 6: $[x]$ is the integer part of x .

Entry 4. Inequalities and estimation

$$\pi - 2\tan^{-1}\left(\sqrt{n}\right) + \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\,(k+1)} \le T \le \pi - 2\tan^{-1}\left(\sqrt{n}\right) + \sum_{k=1}^{n} \frac{1}{\sqrt{k}\,(k+1)} \quad , n \in \mathbb{N} \tag{59}$$

$$2\tan^{-1}\left(\frac{1}{\sqrt{n}}\right) + \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\,(k+1)} \le T \le 2\tan^{-1}\left(\frac{1}{\sqrt{n}}\right) + \sum_{k=1}^{n} \frac{1}{\sqrt{k}\,(k+1)} \quad , n \in \mathbb{N} \tag{60}$$

$$\frac{\pi}{2} - 2\tan^{-1}\left(\frac{\sqrt{n} - 1}{\sqrt{n} + 1}\right) + \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\,(k+1)} \le T \le \frac{\pi}{2} - 2\tan^{-1}\left(\frac{\sqrt{n} - 1}{\sqrt{n} + 1}\right) + \sum_{k=1}^{n} \frac{1}{\sqrt{k}\,(k+1)} \quad , n \in \mathbb{N} \tag{61}$$

$$T = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\,(k+1)} + \pi - 2\tan^{-1}\left(\sqrt{n}\right) + \frac{1}{2\sqrt{n}\,(n+1)} \pm \frac{1}{\sqrt{n}}\left(-\frac{1}{8n^2} + \frac{5}{24n^3} - \frac{559}{1920n^4} + \frac{145}{384n^5} - \ldots\right), \quad n \gg 1 \tag{62}$$

3. Spiral of Theodorus

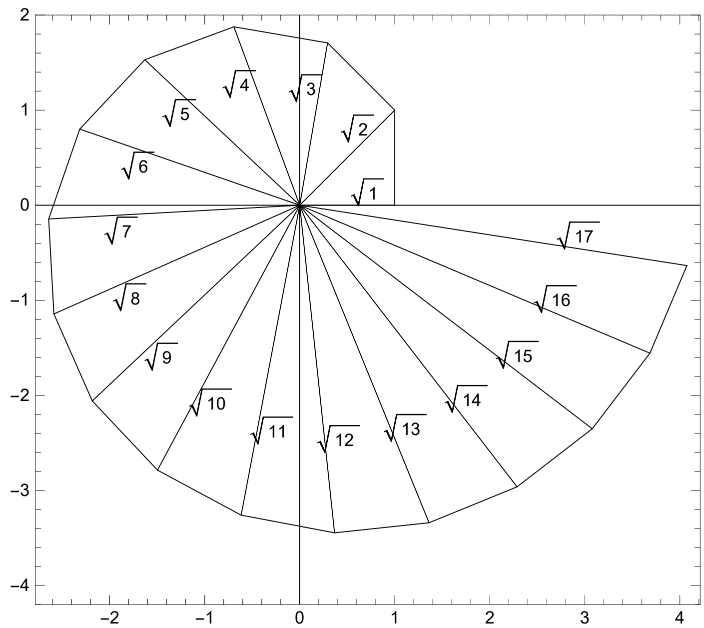

Fig.1 Discrete spiral of Theodorus

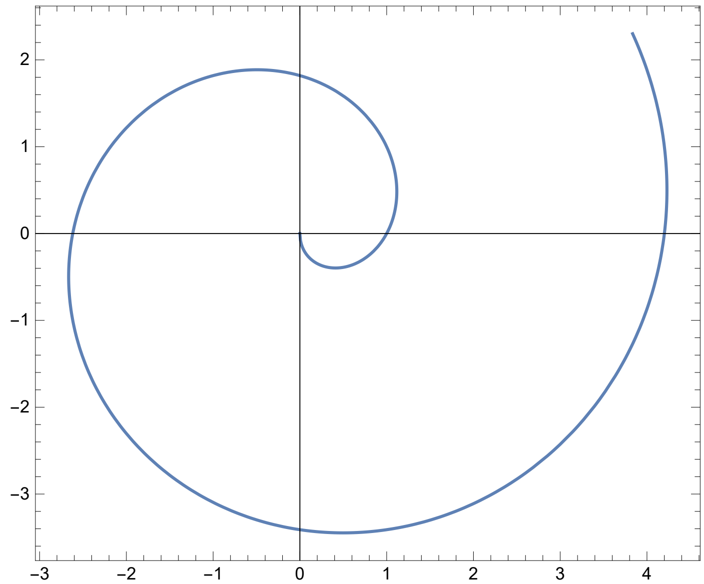

Fig.2 Continuous spiral of Theodorus

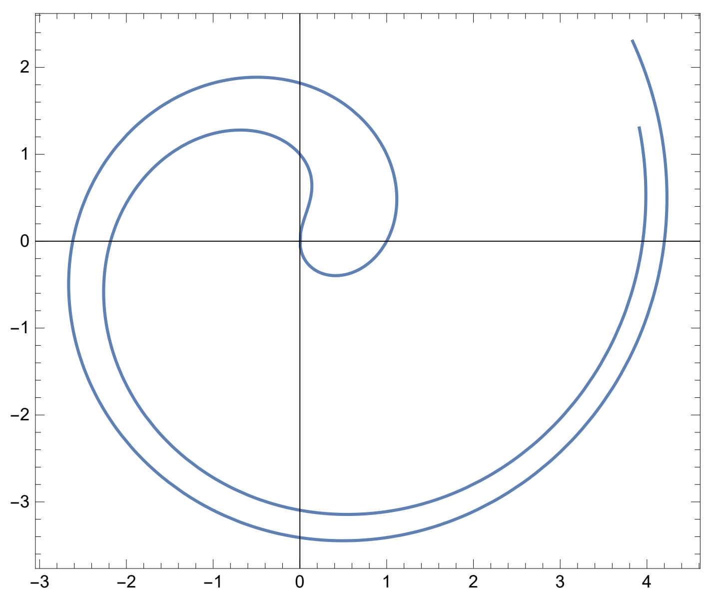

Fig.3 Twin - spiral of Theodorus

4. References

[1] Brink, D. , The spiral of Theodorus and sums of zeta-values at the half-integers, Amer. Math. Monthly 119, 2012

[2] Davis , P.J. , Spirals: From Theodorus to Chaos, A K Peters, 1993.

[3] Finch, S. , Constant of Theodorus , 2005 . https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.440.3922&rep=rep1&type=pdf .

[4] Gautschi, W. , The spiral of Theodorus, numerical analysis, and special functions, Journal of Computational and Applied Mathematics 235 , 2010 .

[5] Gronau, D. , The spiral of Theodorus, Amer. Math. Monthly 111, 2004.

[6] Valdebenito, E. , Theodorus constant analysis , unpublished work , 2016.

[7] Waldvogel, J. , Analytic Continuation of the Theodorus Spiral , http://www.sam.math.ethz.ch/~joergw/Papers/theopaper.pdf .