Using the Partial Sums of the Alternating Harmonic Series to prove the Harmonic Series is divergent

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Abstract: Many proofs of the divergence of the harmonic series have been given since the first proof by Nicole Oresme (1323-1382). In this article we shall give a simple proof using the partial sums of the alternating harmonic series. We shall then show that every harmonic number greater than 1 is the sum of partial sums of the alternating harmonic series. Finally as a corollary we show that the sequence of subseries of every harmonic number greater than 1 is converging to $ln2$.

Keywords: Harmonic Series, Alternating Harmonic Series, Proof

Now let us consider the harmonic series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$

The first known person to show that this infinite series diverges was Nicole Oresme (1323-1382). His idea was to compare the harmonic series with another divergent series.

Proof:

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

$$\geq 1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$$

$$=1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\cdots$$

Therefore, since $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ diverges so must the harmonic series.

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For our purposes we shall merely state that a convergent series related to the harmonic series is the alternating harmonic series:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=ln2=0.693147\ldots$$

Now, the partial sums of the harmonic series are the (finite) harmonic numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad for\ n = 1, 2, 3, \dots.$$

Obviously the (finite) partial sums of the alternating harmonic series are related to the harmonic numbers. So we establish a connection between both partial sums.

Lemma:

$$\sum_{k=1}^{2^n} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{2^{n-1}} \frac{1}{k}, \quad for\ n = 1, 2, 3, \dots.$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2^n - 1} - \frac{1}{2^n}\right)$$

$$= (1 - 1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2^n - 1} - \frac{1}{2^n - 1}\right) + \left(\frac{1}{2^n} + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}}$$

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Theorem 1: The sequence $\{H_n\}$ for $n = 1, 2, 3, \dots$ is divergent.

Proof: As the identity holds for $n = 1, 2, 3, \dots$ we construct the subsequence $\{H_{2^n}\}$ as follows:

$$H_2 = 1 + \frac{1}{2} = \left(1 - \frac{1}{2}\right) + 1$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 + \frac{1}{2}\right)$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + H_2$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) + 1$$

$$H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + H_4$$

$$= \left(1 - \frac{1}{2} + \cdots + \frac{1}{7} - \frac{1}{8}\right) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) + 1$$

And the pattern continues for $H_{16}, H_{32}, H_{64}, \dots, H_{2^n}, \dots$. Therefore, as each consecutive harmonic number has an additional partial sum on the r.h.s. the subsequence $\{H_{2^n}\}$ is unbounded. Hence, the sequence $\{H_n\}$ is divergent.

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We can now show that *every* harmonic number greater than 1 is the sum of partial sums of the alternating harmonic series.

Lemma A:

$$\sum_{k=1}^{2n} \frac{1}{k} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{1}{k}, \quad for\ n = 1, 2, 3, \dots . \qquad (A)$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$= (1-1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n-1}\right) + \left(\frac{1}{2n} + \frac{1}{2n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

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Lemma B:

$$\sum_{k=1}^{2n+1} \frac{1}{k} = \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{1}{k}, \quad for\ n = 1, 2, 3, \dots . \qquad (B)$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n} + \frac{1}{2n+1}\right)$$

$$= (1-1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2n} + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} - \frac{1}{2n+1}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

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Theorem 2: Every harmonic number greater than 1 is the sum of partial sums of the alternating harmonic series.

Proof: Similar to the previous proof using A and B allows us to systematically construct the harmonic numbers as follows:

$$H_1 = 1 = \mathbf{1}$$

$(A): n = 1$ $$H_2 = 1 + \frac{1}{2} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$(B): n = 1$ $$H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$(A): n = 2 \qquad H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(1 + \frac{1}{2}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$$(B): n = 2 \quad H_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}}\right) + \left(1 + \frac{1}{2}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$$H_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$H_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}} - \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \cdots + \frac{\mathbf{1}}{\mathbf{7}} - \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

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It may be argued that we have gone to some length to prove the harmonic series diverges compared to other proofs, however, there is an interesting result that follows on from all this as a corollary.

Corollary: The sequence of subseries of every harmonic number greater than 1 is converging to $ln2$.

Proof: Using Lemmas A and B we have:

$$\frac{1}{2} = \left(1 - \frac{1}{2}\right)$$

$$\frac{1}{2} + \frac{1}{3} = \left(1 - \frac{1}{2} + \frac{1}{3}\right)$$

$$\frac{1}{3} + \frac{1}{4} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right)$$

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right)$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\right)$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\right)$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right)$$

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So we have the surprising result that whilst the harmonic series is divergent it contains an infinite number of subseries that are converging to $ln2$.