The Riemann Hypothesis and Tachyonic Off-Shell String Scattering Amplitudes

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Abstract

The study of the **4**-tachyon off-shell string scattering amplitude $A_4(s,t,u)$, based on Witten's open string field theory, reveals the existence of a continuum of poles in the s-channel and corresponding to a continuum of complex spins J. The latter spins J belong to the Regge trajectories in the t,u channels which are defined by $-J(t) = -1 - \frac{1}{2}t = \beta(t) = \frac{1}{2} + i\lambda$; $-J(u) = -1 - \frac{1}{2}u = \gamma(u) = \frac{1}{2} - i\lambda$, with $\lambda = real$. These values of $\beta(t), \gamma(u)$ given by $\frac{1}{2} \pm i\lambda$, respectively, coincide precisely with the location of the critical line of nontrivial Riemann zeta zeros $\zeta(z_n = \frac{1}{2} \pm i\lambda_n) = 0$. We proceed to prove that if there were nontrivial zeta zeros (violating the Riemann Hypothesis) outside the critical line $Real\ z = 1/2$ (but inside the critical strip) these putative zeros $don't$ correspond to any *poles* of the **4**-tachyon off-shell string scattering amplitude $A_4(s,t,u)$. One of the most salient features of these results is the *collinearity* of the **4** off-shell tachyons. We may speculate that this spatial *collinearity* is actually reflected in the *collinearity* of the poles of the string amplitude, lying in the critical line : $\beta = \gamma^* = \frac{1}{2} + i\lambda$, where the nontrivial zeta zeros are located. We finalize with some concluding remarks on continuous spins, non-commutative geometry and other relevant topics.

Keywords : Riemann Hypothesis, Tachyons, Off-Shell String Scattering Amplitudes

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1 Tachyonic Off-Shell String Scattering Amplitudes

The Riemann's hypothesis (RH) [1], [2] states that the nontrivial zeros of the Riemann zeta-function are of the form $z_n = 1/2 \pm i\lambda_n$. Trivial zeta zeros exist at $z_n = -2n$, for $n = integer$. The on-shell four-point dual string amplitude obtained by Veneziano is [8], [9] was

$$A_4 = A(s,t) + A(t,s) + A(u,s) = \int_R dx\ |x|^{\alpha-1}|1-x|^{\beta-1} = B(\alpha,\beta). \tag{1}$$

where the Regge trajectories in the respective s, t, u channels are :

$$-\alpha(s) = 1 + \frac{1}{2}s. \quad -\beta(t) = 1 + \frac{1}{2}t. \quad -\gamma(u) = 1 + \frac{1}{2}u. \tag{2}$$

The conservation of the energy-momentum yields :

$$k_1 + k_2 = k_3 + k_4 \Rightarrow k_1 + k_2 - k_3 - k_4 = 0. \tag{3}$$

In our notation we define the different channels as :

$$s = (k_1+k_2)^2 = (k_3+k_4)^2.\ t = (k_2-k_3)^2 = (k_4-k_1)^2.\ u = (k_1-k_3)^2 = (k_4-k_2)^2 \tag{4}$$

Our prior work [3] was based on the study of the on-shell scattering amplitudes. A closer and more rigorous look reveals that this was not general enough because we overlooked to include the key study of the off-shell tachyon scattering amplitudes, which are crucial in arriving *correctly* at the desired conclusions. The incoming tachyons were on-shell but the external tachyons were off-shell with $k_3, k_4 = k_3^*$ (a complex-conjugate pair). We will show below that the analysis of the off-shell tachyon scattering amplitudes leads to the same conclusions as in [3] due to a numerical "fluke".

The 4-tachyon off-shell amplitude in Witten's cubic string field theory [4] is instrumental in describing the dynamics of the open bosonic string tachyon. Both the unstable vacuum and the true vacuum where the tachyon has condensed have been shown to be well-defined states in Witten's cubic string field theory [6]. Since tachyon condensation is an off-shell process, string field theory is the required setting for its analysis. We recall that the Higgs field in the Standard Model of particle physics has a tachyonic-like term in the potential and its shift to the true vacuum of the theory gives masses to most of the particles in the Standard Model including the part of the Higgs fields which acquires a positive mass [21].

The 4-tachyon off-shell $s - t$ amplitude found by [6] is

$$A_4(s,t) \;=\; \frac{1}{4}\int_0^1 dx\ |x|^{-s-2}\ |1-x|^{-t-2}\ \left(\frac{C(\frac{1}{2}+|x-\frac{1}{2}|)}{2\sqrt{\frac{1}{2}+|x-\frac{1}{2}|}}\right)^{M^2-4} \tag{5}$$

and it was obtained following the conformal mapping techniques that [5] used to derive the on-shell Veneziano amplitude from Witten's cubic string field theory vertex. $C(x)$ is a very complicated expression defined by elliptic integrals, $M^2 = \sum_{i=1}^4 P_i^2$, and x is the Koba-Nielsen cross-ratio. The full 4-tachyon off-shell amplitude can be obtained from eq-(5) after performing the cyclic permutations $P_1 \to P_2 \to P_3 \to P_4 \to P_1$.

The notation and signature used by [6] is

$$s \;=\; -\ (P_1+P_2)^2,\ \ t = \ -\ (P_2+P_3)^2,\ \ u = \ -\ (P_2+P_4)^2 \tag{6}$$

$$P^2 \;\equiv\; P_\mu P^\mu \;\equiv\; -E^2 \;+\; \vec{P}\cdot\vec{P} \tag{7}$$

with $P_1+P_2+P_3+P_4 = 0$. And because it is different than ours, one must establish the following dictionary between their variables and ours

$$P_1 \;=\; \frac{i}{\sqrt{2}}\,k_1,\ \ P_2 \;=\; \frac{i}{\sqrt{2}}\,k_2,\ \ P_3 \;=\; -\frac{i}{\sqrt{2}}\,k_3,\ \ P_4 \;=\; -\frac{i}{\sqrt{2}}\,k_4, \tag{8}$$

$$P^2 \;=\; -\ \frac{1}{2}k^2,\ \ k^2 \;\equiv\; k_\mu k^\mu \;\equiv\; E^2 - \vec{k}\cdot\vec{k}, \tag{9}$$

$$\frac{1}{2}(k_1+k_2)^2 \;=\; -\ (P_1+P_2)^2,\ \ \frac{1}{2}(k_2-k_3)^2 \;=\; -\ (P_2+P_3)^2,\ \ \ldots \tag{10}$$

and such that

$$M^2 = \sum_{i=1}^4 P_i^2 \;=\; -\ \frac{1}{2}\ (k_1^2+k_2^2+k_3^2+k_4^2) \tag{11}$$

we shall work in the natural units $\hbar = c = G = 1 \Rightarrow L_{Planck} = 1$ so the string slope parameter in those units is given by $\alpha' = (1/2)L^2_{Planck} = 1/2$ and the string mass spectrum is quantized in multiples of the Planck mass $m_{Planck} = 1$.

Despite that in this work we will be working with 4 off-shell tachyons

$$k_1^2 \neq -2,\ \ k_2^2 \neq -2,\ \ k_3^2 \neq -2,\ \ k_4^2 \neq -2 \tag{12}$$

we will show that when the *special* condition holds

$$k_1^2+k_2^2+k_3^2+k_4^2 \;=\; -8 \;\Rightarrow\; M^2 = \sum_{i=1}^4 P_i^2 \;=\; -\ \frac{1}{2}\ (k_1^2+k_2^2+k_3^2+k_4^2) \;=\; 4 \tag{13}$$

then the *correction* factor in the scattering amplitude $(\ldots)^{M^2-4}$ of the 4 off-shell tachyons becomes precisely 1 (due to $M^2-4=0$) and one is still able to recover the same functional expression as the on-shell Veneziano string amplitude for those very *special* values of k_1, k_2, k_3, k_4. And these latter values are precisely those which allows us to establish a one-to-one correspondence between the poles of the string scattering amplitude and the critical line where the nontrivial zeta zeros are located. Therefore, one could assert that this numerical "fluke" when the functional expressions for the on-shell and off-shell tachyon scattering amplitudes coincide is a reflection of a "coexistence" of the classical and quantum world.

The special condition $k_1^2+k_2^2+k_3^2+k_4^2=-8$, combined with the conservation of energy-momentum $k_1+k_2=k_3+k_4$, and a judicious use of the definitions in eq-(4) allows to prove that the sum

$$s+t+u=(k_1^2+k_2^2+k_3^2+k_4^2) \ + \ 2k_2^2 \ + \ 2k_1\cdot k_2 \ - \ 2k_2\cdot k_3 \ - \ 2k_2\cdot k_4 \ =$$

$$-8+2k_2\cdot(k_1+k_2)-2k_2\cdot(k_3+k_4) \ = \ -8+2k_2\cdot(k_1+k_2)-2k_2\cdot(k_1+k_2) \ = \ -8. \tag{14}$$

This relationship $s+t+u=-8$ will be crucial in order to show below that the string amplitude can be rewritten in terms of products of zeta functions.

Hence, from the defining Regge trajectories (2) and eq-(14) we obtain the following constraint

$$\alpha(s)+\beta(t)+\gamma(u)=1 \tag{15}$$

The last relationship can also be understood geometrically as the sums of the angles, in units of π, of an Euclidean triangle found in [7] where new relations among analyticity, Regge trajectories, the Veneziano string amplitudes and Moebius transformations were studied. Note that the author [7] uses a $different$ convention for α, β, γ than ours .

There exists a well known relation [8], [10] among the Γ functions in terms of ζ functions appearing in the expression for $A(s,t,u)$ when α, β fall *inside* the critical strip. In this case the integration region in the real line that defines the on-shell amplitude $A(s,t,u)$ in eq-(1) can be divided into three parts and leads to the very important identity

$$A(s,t,u)=B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}+\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}+\frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}=$$
$$\frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1-\beta)}{\zeta(\beta)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)}. \tag{16}$$

where $\alpha+\beta+\gamma=1$ and α, β are confined to the interior of the critical strip. Because the functional form of the on-shell amplitude coincides with the off-shell amplitude for very *special* values of k_1, k_2, k_3, k_4, as we shall prove, we may also use the expression of eq-(16) for the off-shell amplitude.

The derivation behind eq-(16) relies on the above constraint (15) $\alpha+\beta+\gamma=1$ and the identities

$$sin\ \pi(\alpha+\beta) + sin\ \pi(\alpha+\gamma) + sin\ \pi(\beta+\gamma) = 4\ cos\frac{\pi\alpha}{2} cos\frac{\pi\beta}{2} cos\frac{\pi\gamma}{2}. \quad (17a)$$

$$\Gamma(\gamma) = \Gamma(1-\alpha-\beta) = \frac{1}{\Gamma(\alpha+\beta)} \frac{\pi}{sin\ \pi(\alpha+\beta)}. \quad (17b)$$

plus the remaining cyclic permutations from which one can infer

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\ \frac{sin\ \pi(\alpha+\beta)}{\pi}. \quad (17c)$$

$$\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\ \frac{sin\ \pi(\alpha+\gamma)}{\pi}. \quad (17d)$$

$$\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\ \frac{sin\ \pi(\beta+\gamma)}{\pi}. \quad (17e)$$

Therefore, eqs-(17) allow us to recast the l.h.s of (16) as

$$A(s,t,u) = B(\alpha,\beta) = \frac{4}{\pi}\ cos\frac{\pi\alpha}{2} cos\frac{\pi\beta}{2} cos\frac{\pi\gamma}{2}\ \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \quad (18aa)$$

And, finally, the known functional relation

$$(2\pi)^z\zeta(1-z) = 2\ cos\frac{\pi z}{2}\ \Gamma(z)\ \zeta(z). \quad (18b)$$

in conjunction with the condition $\alpha+\beta+\gamma = 1$ such that $(2\pi)^{\alpha+\beta+\gamma} = 2\pi$ is what establishes the important identity (16) expressing explicitly the string amplitude $A(s,t,u)$ either in terms of zeta functions or in terms of Γ functions.

Having found the expression for $A(s,t,u)$ (16) in terms of products of zeta functions it follows from the relation $\alpha+\beta+\gamma = 1$ that the location of the Riemann critical line of zeta zeros given by the complex numbers $\beta = 1/2 + i\lambda, \gamma = \beta^* = \frac{1}{2} - i\lambda \Rightarrow \alpha = 0, \beta+\gamma = 1$, corresponds to *real*-valued poles of the scattering amplitude $A(s,t,u) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} = \frac{\zeta(1)}{\zeta(0)} = -\infty$. Due to $1-\beta = \beta^* = \gamma$ and $1-\gamma = \gamma^* = \beta$ there is a pairwise exact cancellation of the numerator and the denominator in

$$\frac{\zeta(1-\beta)}{\zeta(\beta)} \frac{\zeta(1-\gamma)}{\zeta(\gamma)} = 1 \quad (19)$$

and $A(s,t,u)$ reduces to $\frac{\zeta(1)}{\zeta(0)} = -\infty$. The case $\alpha = 0$ corresponds to a tachyonic pole in the s-channel : $s = -2$. Complex-valued energy-momenta and angular-momenta have physical significance. It is well known that the imaginary parts of the energies in scattering theory corresponds to the inverse lifetime of particle-resonances. The resonance-width is the inverse of the lifetime. By *cyclic* symmetry one may have poles in the t or the u channel as well given by $t = -2, u = -2$, respectively. This follows from $\alpha + \beta + \gamma = 1$ by setting $\{\beta = 0; \alpha = \gamma^* = 1/2 + i\lambda\}$ (leading to a pole in the t-channel) or $\{\gamma = 0; \alpha = \beta^* = 1/2 + i\lambda\}$ (leading to a pole in the u-channel).

The functional relation of the completed zeta function

$$Z(z) = \pi^{-\frac{z}{2}} \; \Gamma(\frac{z}{2}) \; \zeta(z) \;\; = \;\; Z(1-z) \;\; = \;\; \pi^{-\frac{(1-z)}{2}} \; \Gamma(\frac{1-z}{2}) \; \zeta(1-z) \tag{20}$$

is instrumental in showing why if there are nontrivial zeta zeros outside the critical Riemann line these zeros *don′t correspond* to *poles* of $A(s,t,u)$.

Let us identify the sets of *quartets* of hypothetical nontrivial zeta zeros lying *inside* the critical strip[1] ($0 < \mathcal{R}e \; z < 1$) at the locations described by

$$\alpha_n, \quad \beta_n = \alpha_n^*. \quad 1 - \alpha_n, \quad 1 - \beta_n = 1 - \alpha_n^*, \quad n = 1, 2, 3, \ldots \tag{21}$$

respectively, such that

$$\zeta(\alpha_n) = \zeta(\beta_n) = \zeta(1 - \alpha_n) = \zeta(1 - \beta_n) = 0 \tag{22}$$

and

$$0 < \; \alpha_n + \beta_n \; < 1; \quad 0 \; < \; \gamma_n \; < 1. \tag{23}$$

so that $\alpha_n, \beta_n, \gamma_n$ are all confined *inside* the critical strip and whose values are consistent with the condition $\alpha_n + \beta_n + \gamma_n = 1$. Note that γ_n is real. The amplitude (16) in this case is

$$A(s,t,u) = \frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)} \frac{\zeta(1-\beta_n)}{\zeta(\beta_n)} \frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} = \frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)} \frac{\zeta(1-\alpha_n^*)}{\zeta(\alpha_n^*)} \frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} =$$

$$|| \frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)} ||^2 \frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} = C_n \frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} = real \; and \; finite. \tag{24}$$

This result (24) is a consequence of the above functional equation (20) since the ratio $\frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)}$ can be rewritten in terms of the Gamma functions as

$$\frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)} \;\; = \;\; \frac{\Gamma(\frac{\alpha_n}{2})}{\Gamma(\frac{1-\alpha_n}{2})} \; \frac{\pi^{\frac{1-\alpha_n}{2}}}{\pi^{\frac{\alpha_n}{2}}}, \;\; 0 \; < \; Re(\alpha_n) \; < \; 1 \tag{25}$$

Hence, from eq-(25) one infers that the real constants $C_n = ||\zeta(1-\alpha_n)/\zeta(\alpha_n)||^2 = 0/0$ are *finite* and non-zero. Consequently, the amplitude $A(s,t,u)$ (24) is finite and devoid of poles because $\frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)}$ is finite when γ_n is real and constrained to obey $0 < \gamma_n < 1$. The zeta function $\zeta(z)$ has a simple pole at $z = 1$. Similar findings occur when α_n, β_n lie to the right of the critical line such that

$$2 \; > \; \alpha_n + \beta_n \; > 1; \quad -1 \; < \; \gamma_n \; < 0. \tag{26}$$

[1]According to the Valle-de la Poussin theorem there are no zeros on the boundary of the critical strip

Therefore, there are *no* poles in the r.h.s of eq-(24) when the parameters $\alpha_n, \beta_n, \gamma_n$ are restricted to obey the conditions described above. The case when the quartet of zeros $\{\alpha_n, \alpha_n^*, 1-\alpha_n, 1-\alpha_n^*\}$ are not related to β_n via the relations displayed in eq-(21), but still obeying $\alpha_n+\beta_n+\gamma_n = 1$ lead to the same conclusions. And one finally concludes that if there were nontrivial zeta zeros *outside* the critical Riemann line these putative zeros *don't correspond* to *poles* of $A(s,t,u)$. However, this fact alone does not necessarily mean that these zeros do *not* exist but only that if they existed they do not have a *physical* interpretation in terms of the *poles* of $A(s,t,u)$.

We are going to prove next that one can actually satisfy our goals even if the incoming tachyons are off-shell; i.e. if $k_1^2 \neq -2$ and $k_2^2 \neq -2$, with the provision that the s-channel still *obeys* the on-shell condition $(k_1+k_2)^2 = -2$ and the key algebraic condition $s+t+u = -8$ is still satisfied. In this case, all of the results in [3] still hold and one can find exact solutions to all of the relevant equations. As stated earlier, the external tachyons were already off-shell with k_3, k_4 being a complex-conjugate pair.

Let us *not* impose now the on-shell conditions for the incoming tachyons (so that $k_1^2 \neq -2$ and $k_2^2 \neq -2$) and search for solutions to the following system of 8 nonlinear equations

$$s \;=\; (k_1+k_2)^2 \;=\; (k_3+k_4)^2 \;=\; -2 \;\Rightarrow\; \alpha(s) = -(1+\frac{1}{2}s) \;=\; 0 \qquad (27)$$

$$t \;=\; (k_2-k_3)^2 \;=\; (k_4-k_1)^2 \;=\; -3\;-2i\lambda \;\Rightarrow\; \beta(t) = -(1+\frac{1}{2}t) \;=\; \frac{1}{2}+i\lambda \quad (28)$$

$$u \;=\; (k_2-k_4)^2 \;=\; (k_3-k_1)^2 \;=\; -3+2i\lambda \;\Rightarrow\; \gamma(u) = -(1+\frac{1}{2}u) \;=\; \frac{1}{2}-i\lambda \quad (29)$$

$$k_3^2 \;=\; -2 \;+\; 2i\xi \;\Rightarrow\; J(k_3^2) \;=\; 1+\frac{1}{2}k_3^2 \;=\; i\xi \qquad (30)$$

$$k_4^2 \;=\; -2 \;-\; 2i\xi \Rightarrow\; J(k_4^2) \;=\; 1+\frac{1}{2}k_4^2 \;=\; -\; i\xi \qquad (31)$$

There is a scalar (spin-0) tachyon exchanged in the s-channel . β and γ are complex conjugates and lie in the critical line and the conditions $\alpha+\beta+\gamma = 1 \leftrightarrow s+t+u = -8$ are satisfied. Conservation of angular momentum demands the sum of the spins in eqs-(30,31) equals the zero-spin value in eq-(27).

From eqs-(30,31) one infers that k_3, k_4 are complex-valued and complex-conjugates $k_3 = k_4^*$. And, in turn, from eqs-(28,29) one can then infer that k_1, k_2 must be real-valued. Let us choose an ansatz where the non-vanishing components in 26-dim for k_1, k_2, k_3, k_4 are of the form

$$k_1 \equiv (E_1,\, p_1),\;\; k_2 \equiv (E_2,\, p_2),\;\; k_3 \equiv (E_3+i\mathcal{E}_3,\, p_3+i\pi_3),\;\; k_4 = k_3^* \equiv (E_3-i\mathcal{E}_3,\, p_3-i\pi_3) \qquad (32)$$

and where we set to zero the remaining 24 transverse components to the bosonic string world-sheet. In this case, the total number of variables comprising k_1, k_2, k_3, k_4

is then given by $2+2+4=8$ and which matches the number of 8 equations in (27-31)[2].

Since one has a system of 8 nonlinear equations with 8 unknowns, there might not be solutions; or there might be one or many solutions. A closer inspection of the 8 nonlinear equations reveals that they are not independent. From the conservation of the energy-momentum $k_1+k_2=k_3+k_4$ one can see that the second set of equations in the doublets of eqs-(27-29) are not independent from the first set of equations. Thus there is a redundancy and in actuality there are 5 nonlinear equations plus one linear equation $k_1+k_2=k_3+k_4$. Setting aside this subtlelty, after some straightforward algebra one finds the following solutions

$$k_1 \;=\; (\;0,\;p_1=\frac{1-\sqrt{3}}{\sqrt{2}}\;),\quad k_2\;=\;(\;0,\;p_2=\frac{1+\sqrt{3}}{\sqrt{2}}\;),\tag{33}$$

$$k_3=(\;0+i\mathcal{E}_3=i\sqrt{2\xi^2+\frac{3}{2}},\;p_3+i\pi_3=\frac{1}{\sqrt{2}}-i\sqrt{2}\;\xi\;)\tag{34}$$

$$k_4=k_3^*\;=\;(\;0-i\mathcal{E}_3=-i\sqrt{2\xi^2+\frac{3}{2}},\;p_3-i\pi_3=\frac{1}{\sqrt{2}}+i\sqrt{2}\;\xi\;)\tag{35}$$

and where the *key* relationship (obtained from the solutions) between ξ and λ turns out to be

$$\sqrt{3}\;\xi\;=\;\lambda\tag{36}$$

From eqs-(33) one learns that k_1^2 and k_2^2 are *Galois* conjugates

$$k_1^2\;=\;-\;(\frac{1-\sqrt{3}}{\sqrt{2}})^2\;=\;-\,2\;+\;\sqrt{3}\;<\;0,\;\Rightarrow k_1^2\neq-2\tag{37}$$

$$k_1^2\;=\;-\;(\frac{1+\sqrt{3}}{\sqrt{2}})^2\;=\;-\,2\;-\;\sqrt{3}\;<\;0,\;\Rightarrow k_2^2\neq-2\tag{38}$$

From eqs-(34,35) one verifies also that k_3 and k_4 are *complex* conjugates. Eqs-(33-36) represent a considerable improvement of our previous findings in the appendix of [3]. In particular, eq-(36) is far *simpler* than eq-(A.18) in [3]. This is due to the fact that we are *no* longer imposing the on-shell conditions for the incoming tachyons $k_1^2=k_2^2=-2$. All the 4 tachyons are now off-shell.

To sum up, one can explicitly verify that

$$(k_1+k_2)^2\;=\;(k_3+k_4)^2\;=\;-\,(\frac{2}{\sqrt{2}})^2\;=\;-\,2\tag{39}$$

$$k_3^2\;=\;-\;(2\xi^2+\frac{3}{2})\;-\;(\frac{1}{\sqrt{2}}-i\sqrt{2}\;\xi\;)^2\;=\;-\,2\;+\;2i\xi\tag{40}$$

$$k_4^2\;=\;-\;(2\xi^2+\frac{3}{2})\;-\;(\frac{1}{\sqrt{2}}+i\sqrt{2}\;\xi\;)^2\;=\;-\,2\;-\;2i\xi\tag{41}$$

[2]By writing $(k_1+k_2)^2=(k_3+k_4)^2=-2$ it is understood that it means $(k_1+k_2)^2=-2$ *and* $(k_3+k_4)^2=-2$, etc.

$$(k_2 - k_3)^2 \;=\; (k_4 - k_1)^2 \;=\; -\; (2\xi^2 + \frac{3}{2}) \;-\; (\sqrt{\frac{3}{2}} + i\sqrt{2}\;\xi)^2 \;=$$

$$-\,3 \;-\; 2i\sqrt{3}\;\xi \;=\; -\,3 \;-\; 2i\lambda \tag{42}$$

the complex conjugate of eq-(42) gives

$$(k_2 - k_4)^2 \;=\; (k_3 - k_1)^2 \;=\; -\,3 \;+\; 2i\sqrt{3}\;\xi \;=\; -\,3 \;+\; 2i\lambda \tag{43}$$

And, finally, one can check the conservation of energy-momentum $k_1 + k_2 = k_3 + k_4 = (0, \sqrt{2})$ and that the key condition $s+t+u = -2-3-2i\lambda-3+2i\lambda = -8 \Rightarrow \alpha(s) + \beta(t) + \gamma(u) = 1$ is obeyed.

To conclude, due to the fact that k_1^2 and k_2^2 are *Galois* conjugates, and k_3^2 and k_4^2 are complex conjugates, from the results in eqs-(37,38,40,41) one finally arrives at the sought-after condition displayed by eq-(13)

$$M^2 \;=\; \sum_{i=1}^{4} P_i^2 \;=\; -\;\frac{1}{2}\;(k_1^2 + k_2^2 + k_3^2 + k_4^2) \;=\; 4 \tag{44}$$

such that the correction factor in the 4-tachyon off-shell scattering amplitude (5) becomes *unity* and the functional expressions for the on-shell and off-shell amplitudes *coincide* in this very special case. And as a result, we were able to show that if there were nontrivial zeros violating the Riemann hypothesis, these zeros do *not* correspond to poles of the off-shell string scattering amplitude.

One the most salient features that one can glean from the solutions in eqs-(33-35) is the *collinearity* of those 4 off-shell tachyons since their spatial motion is $confined$ to *one* dimension (a line); i.e. all of the 24 transverse components (to the two-dim string world-sheet) of the energy-momentum variables k_μ are zero. One may speculate that this spatial *collinearity* is actually reflected in the *collinearity* of the poles of the string amplitude lying in the critical line $\beta = \gamma^* = \frac{1}{2} + i\lambda$ where the nontrivial zeta zeros are located.

If one does not have collinearity then there is more wiggle room for the off-shell tachyons to maneuver and *no* longer all of the 24 transverse components of the energy-momentum variables k_μ are constrained to zero. And, in turn, the condition (44) is not necessarily satisfied any longer. Consequently, the correction factor of the 4-tachyon off-shell scattering amplitude (5) is no longer *unity* and the functional expressions for the on-shell and off-shell amplitudes no longer *coincide*.

Namely, the off-shell amplitude will $differ$ now from the expression in eq-(16) and one cannot rule out the possibility that the sets of quartets of putative zeta zeros off-the-critical line may now have an actual correspondence with more and new poles of the far more complicated off-shell amplitude. The study of this possibility is well beyond the scope of this work and requires extensive computer analysis beyond our capabilities. In a nutshell, if there is *no* collinearity in the motion of the 4 off-shell tachyons there might *not* be collinearity in *all* of the zeta zeros and the Riemann hypothesis could be false.

The 4-tachyon on-shell amplitude with a pole in the s-channel would have $s=-2; t=u=-3$ such that $s+t+u=-8$, and $\alpha=0, \beta=\gamma=\frac{1}{2}$. Contrast these values with the ones in the 4-tachyon off-shell amplitude with a pole in the s-channel $s=-2; t=-3-2i\lambda; u=-3+2i\lambda$, and $\alpha=0, \beta=\frac{1}{2}+i\lambda; \gamma=\frac{1}{2}-i\lambda$. In other words, the off-shell results can be interpreted as *analytical* extensions in the complex plane of the on-shell ones along the imaginary directions associated with the critical line : $\frac{1}{2} \rightarrow \frac{1}{2} \pm i\lambda$.

Having found solutions for k_1, k_2, k_3, k_4 one can obtain the values of the angular momentum (spin) J carried by the tachyonic particles directly from their defining Regge trajectory

$$J(k_1) \ = \ 1+\frac{1}{2}\, k_1^2 \ = \ \frac{\sqrt{3}}{2}, \ \ J(k_2) \ = \ 1+\frac{1}{2}\, k_2^2 \ = \ -\ \frac{\sqrt{3}}{2} \ \Rightarrow \ J(k_1)+J(k_2)=0 \tag{45}$$

$$J(k_3) \ = \ 1+\frac{1}{2}\, k_3^2 \ = \ i\xi \ = \ i\frac{\lambda}{\sqrt{3}}, \ \ J(k_4) \ = \ 1+\frac{1}{2}\, k_4^2 \ = \ -i\xi \ = \ -i\frac{\lambda}{\sqrt{3}} \ \Rightarrow$$

$$J(k_3) \ + \ J(k_4) \ = \ 0 \tag{46}$$

Eqs-(45-46) are consistent with the fact that the spin of the tachyon exchanged in the s-channel $(k_1+k_2)^2=-2$ is given by

$$J(s=(k_1+k_2)^2=-2) \ = \ 1 \ + \ \frac{1}{2}\ s \ = \ 1 \ + \ \frac{1}{2}\ (-2) \ = \ 0 \tag{47}$$

so that the *net* zero-spin value is conserved. Likewise, the net value of the energy-momentum is also conserved $k_1+k_2=k_3+k_4=(0,\sqrt{2})$.

To finalize this section we should add that in [3] we explained that the solutions $\beta=\gamma^*=1/2+i\lambda$ have also a clear definite *geometrical* interpretation when the Euclidean triangle with 3 vertices degenerates into a vertical strip in the upper complex plane comprised of one vertex located at infinity (with zero angle) and the other two vertices (with angle $\pi/2$) located on the real axis and separated by a distance [7]

$$d=\frac{\Gamma(\beta)\Gamma(1-\beta)}{\Gamma(1)}=\frac{\pi}{sin\ (\pi\beta)}=\frac{\pi}{sin\ (\pi/2+i\pi\lambda)}=\frac{\pi}{cos\ (i\pi\lambda)}=\frac{\pi}{cosh\ \pi\lambda}. \tag{48}$$

Once again we must remind the reader that our notation for α, β, γ differs from [7].

Despite the fact that $\beta=\gamma^*$ are complex-valued their sum $\beta+\gamma=1=real$, thus the sum of the three angles of the triangle is still $\pi(\alpha+\beta+\gamma)=\pi$. Therefore, the discrete number of the imaginary parts of the nontrivial zeta zeros λ_n are associated with a discrete number of possible distances between the two variable vertices of the triangles situated in the real axis of the complex plane and given by $d_n=\pi/cosh\ (\pi\lambda_n)$.

Physical systems with this type of *hyperbolic* spectrum of scales d_n have been recently been investigated by [11] in connection to the Riemann hypothesis. In particular, these authors applied the infinite-component Majorana equation in a Rindler spacetime and focused on the S-matrix approach describing the bosonic open string for tachyonic states.

The author [12] studied the Riemann zeros as energy levels of a Dirac fermion in a potential built from the prime numbers in Rindler spacetime. The Hamiltonian was derived from the action of a massless Dirac fermion living in a domain of Rindler spacetime, in $1+1$ dimensions, that has a boundary given by the world line of a uniformly accelerated observer. The Riemann zeros appear as discrete eigenvalues immersed in the continuum.

2 Concluding Remarks: Continuous spins, Non-Commutative Geometry, Chaos, Fractal Strings and All That

The study of the **4**-tachyon off-shell string scattering amplitude $A_4(s,t,u)$, based on Witten's open string field theory, reveals the existence of a continuum of poles in the s-channel and corresponding to a continuum of complex spins J. The latter spins J belong to the Regge trajectories in the t,u channels which are defined by $-J(t) = -1-\frac{1}{2}t = \beta(t) = \frac{1}{2}+i\lambda$; $-J(u) = -1-\frac{1}{2}u = \gamma(u) = \frac{1}{2}-i\lambda$, with $\lambda = real$. These values of $\beta(t), \gamma(u)$ given by $\frac{1}{2} \pm i\lambda$, respectively, coincide precisely with the location of the critical line of nontrivial Riemann zeta zeros $\zeta(z_n = \frac{1}{2} \pm i\lambda_n) = 0$.

Particles with *continuous* spin have a long history since Wigner's construction of continuous spin representations of the Poincare group for massless particles [15]. Photons and tachyons with continuous spin were studied a while back by [18]. There are two classes of unitary infinite dimensional representations, one being massless and named continuous (or infinite) spin particles and the other constituted by tachyonic particles. For a long time no field theory was known for these infinite dimensional representations preventing the study of its properties even at the free level [21]. Only recently a field theory for continuous spins particles was proposed [16] triggering a new wave of interest on the subject. For a recent review and earlier references see[17], [19].

The irreducible unitary representations of the Poincare group in $D=4$ can be labelled by the quadratic Casimir operator $C_2 = P^2 = P_\mu P^\mu$ associated to the mass-shell condition, and the quartic Casimir operator $C_4 = -\frac{1}{2}P^2 J^{\mu\nu}J_{\mu\nu} + J^{\mu\nu}P_\nu J_{\mu\rho}P^\rho$, the square of the Pauli-Lubanski vector $W^\mu = \epsilon^{\mu\nu\rho\tau}J_{\nu\rho}P_\tau$. The irreducible unitary representations of the Poincare group in other dimensions than $D=4$ can be found in [20].

A nice description of the continuous spin representations can be found in [21]. In $D=4$, the scalar spin-0 tachyon belongs to a one-dim representation.

The spin-s tachyon ($s \neq 0$), with s integer or half-integer, has for quadratic Casimir $C_2 = -m^2 < 0$, a quartic Casimir $C_4 = -m^2(s+1)s$, and belongs to an infinite-dim representation of the Poincare group with an infinite tower of states labeled by $l = \pm(s+1), \pm(s+2), \pm(s+3), ..., \infty$.

The "continuous" spin tachyon, has for quadratic Casimir $C_2 = -m^2 < 0$, a quartic Casimir $C_4 = -\rho^2$, where ρ is a *real* number (the value of the "continuous spin"). The "bosonic" continuous spin tachyon belongs to an infinite-dim representation of the Poincare group with an infinite tower of states labeled by $l = 0, \pm 1, \pm 2, ..., \pm\infty$. Whereas, the "fermionic" continuous spin tachyon belongs to an infinite-dim representation of the Poincare group with an infinite tower of states labeled by $l = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, ..., \pm\infty$. The massless "bosonic/fermionic" continuous spin representations have zero for their quadratic Casimir $C_2 = m^2 = 0$, a value of $C_4 = -\rho^2$ for ρ real (the value of the "continuous spin), and a similar tower of states as above.

Continuous and complex spins are also present in Celestial Conformal Field Theories (CCFTs) [22] which are based in introducing conformal correlation functions living on the celestial sphere. These correlation functions are obtained from the S-matrix of a particular set of wave functions in the bulk. The celestial operators associated to these wavefunctions transform as primaries under the action of the conformal group on the sphere, but have *continuous* boost weight $\Delta = 1 + i\sigma$, with σ real, in contrast with the discrete spectrum expected for standard Conformal Field Theories (CFT).

The four-point function contains non-trivial information of the spectrum, and in some cases can be related to the three-point structure constants [23]. In particular, the decomposition of [24] has found that not only conformal primaries (fields) with continuous boost weight are exchanged in the four-point function, but also their so-called light-ray transforms [25] with *continuous* and *complex* spin [26],[27],[23].

Witten [9] was motivated by ideas from non-commutative geometry and introduced the non-commutative star product of three string fields $\Psi_1 \star \Psi_2 \star \Psi_3$ to construct the cubic vertex. Connes' approach to the Riemann Hypothesis relied on non-commutative geometry and Adelic products [13]. Since our results are based on the study of **4**-tachyon off-shell scattering amplitudes which required Witten's open string field theory, it is warranted to investigate the role of non-commutative geometry even further.

Tachyons were essential in the recent study of Chaotic scattering of highly excited strings [28]. Motivated by the desire to understand chaos in the S-matrix of string theory, the authors studied tree level scattering amplitudes involving highly excited strings. The excited string is formed by repeatedly scattering photons off of an initial tachyon (the DDF formalism) and they computed the scattering amplitude of one arbitrary excited string and any number of tachyons in bosonic string theory.

We found a continuum of poles in the **4**-tachyon off-shell string scattering amplitude and associated with the values of $\beta(t), \gamma(u)$ given by $\frac{1}{2} \pm i\lambda$, respectively, which coincide precisely with the location of the critical line of nontrivial Riemann zeta zeros. This "pole-continuum" resembles the location of black hole singularities. The Schwarzschild black hole singularity at $r = 0$ is a *spatial* singularity and is represented by a *line* in the Penrose diagram. All matter that crosses the horizon falls towards the singularity.

The discrete number of zeta zeros are embedded in a continuum of values of $\frac{1}{2} \pm i\lambda$. Roughly speaking, this pole-continuum (the critical line) behaves like an attractor where the quartets of putative zeros outside the critical line flow into. This picture of zeta zeros flowing towards the critical line was advocated by Lapidus [29] in his study of fractal strings. His main conjecture is that under the action of the modular flow, the spacetime geometries become increasingly symmetric and crystal-like, hence, arithmetic. Correspondingly, the zeros of the associated zeta functions eventually condense onto the critical line, towards which they are attracted, thereby explaining why the Riemann Hypothesis must be true. This picture deserves further investigation.

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