Is The Riemann Hypothesis True? Yes, it is.

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ABSTRACT: In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros)* $s = \sigma + it$ *of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{for} \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$.

We give a proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

Keywords : Zeta function, non trivial zeros of Riemann zeta function, zeros of Dirichlet eta function inside the critical strip, definition of limits of real sequences.

PACS : 11AXX; 11M26

September 2021. Submitted to the journal " Communications In Mathematics and Statistics".

Contents

1 Introduction.

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

Conjecture 1.1. *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$.

1.1 The function ζ.

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1.1. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;

3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 1.2. *(The Riemann Hypothesis,[2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

In addition to the properties cited by the theorem 1.1 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C} \backslash \{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (1.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 1.2. *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [3]).

1.2 A Equivalent statement to the Riemann Hypothesis.

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 1.1. *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{1.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ *lie on the critical line* $\Re(s) = \frac{1}{2}$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$. The series (1.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{1.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{1.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2 Preliminaries of the proof that the zeros of the function $\eta(s)$ are on the critical line $\Re(s) = \frac{1}{2}$.

Proof. . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

$$\boxed{s,\ \textit{is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)} \tag{2.1}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s,\ \textit{is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)} \tag{2.2}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall\epsilon' > 0 \quad \exists n_0, \forall \mathcal{N} > n_0, \left|\sum_{n=1}^{\mathcal{N}}\frac{(-1)^{n-1}}{n^s}\right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} = 0$$
$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$\forall\epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2 < \epsilon_1^2 \tag{2.3}$$
$$\forall\epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2 < \epsilon_2^2 \tag{2.4}$$

Then:

$$0 < \sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2 \tag{2.5}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{2.6}$$

or $\rho(s) = 0$.

3 Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$.

We suppose that $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

Proposition 3.1. *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.5) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{3.1}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4 Case $0 < \Re(s) < \dfrac{1}{2}$.

4.1 Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$.

As there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$, it follows from the proposition (2.1) that $\zeta(s)$ has also no zeros with $0 < \sigma < \dfrac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s = \sigma + it$ and $\dfrac{1}{2} < \sigma < 1$. We deduce from the proposition (2.2) that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\dfrac{1}{2} < \sigma < 1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \dfrac{1}{2}$ and from the equivalent statement 1.1, we conclude that **the Riemann Hypothesis is true**.

4.2 Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$.

Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (2.5):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} \dfrac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} = -\infty} \tag{4.1}$$

5 Case $\dfrac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 1.1, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (2.5), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{5.1}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.2}$$

Let $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$.

Firstly, we suppose that $t_l \neq 0$. For each s'_l, the left member of the equation (5.2) above is finite and depends of $\sigma'_l = 1 - \sigma_l$ and $t'_l = t_l$, but the right member is a function only of σ'_l equal to $-\zeta(2\sigma'_l)/2$. Hence the contradiction because for all σ'' so that $2\sigma'' > 1$, we have $\zeta(2\sigma'')$ depends only of σ'', then in particular for all σ'' with $2 > 2\sigma'' > 1$, $\zeta(2\sigma'')$ depends only of $\sigma'' \Longrightarrow$ the equation (5.2) is false.

Secondly, we suppose that $t_l = 0 \Longrightarrow t'_l = 0$. The equation (5.2) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'_l} k'^{\sigma'_l}} = -\frac{C(\sigma'_l)}{2} = -\frac{\zeta(2\sigma'_l)}{2} > -\infty \tag{5.3}$$

Then $s'_l = \sigma'_l > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s'_l) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'_l}} = 0 \tag{5.4}$$

Let us define the sequence S_m as:

$$S_m(s'_l) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'_l}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'_l}} = S_m(\sigma'_l) \tag{5.5}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) \tag{5.6}$$

We have also:

$$S_1(\sigma'_l) = 1 > 0 \tag{5.7}$$

$$S_2(\sigma'_l) = 1 - \frac{1}{2^{\sigma'_l}} > 0 \quad because\ 2^{\sigma'_l} > 1 \tag{5.8}$$

$$S_3(\sigma'_l) = S_2(\sigma'_l) + \frac{1}{3^{\sigma'_l}} > 0 \tag{5.9}$$

We proceed by recurrence, we suppose that $S_m(\sigma'_l) > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma'_l) = \displaystyle\sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'_l}} = S_m(\sigma'_l) + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$, it gives:

$$S_{m+1}(\sigma'_l) = S_m(\sigma'_l) + \frac{(-1)^{2q}}{(m+1)^{\sigma'_l}} = S_m(\sigma'_l) + \frac{1}{(m+1)^{\sigma'_l}} > 0 \Rightarrow S_{m+1}(\sigma'_l) > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma'_l)$ as:

$$S_{m+1}(\sigma'_l) = S_{m-1}(\sigma'_l) + \frac{(-1)^{m-1}}{m^{\sigma'_l}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$$

We have $S_{m-1}(\sigma'_l) > 0$, let $T = \dfrac{(-1)^{m-1}}{m^{\sigma'_l}} + \dfrac{(-1)^m}{(m+1)^{\sigma'_l}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'_l}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'_l}} = \frac{1}{(2q+1)^{\sigma'_l}} - \frac{1}{(2q+2)^{\sigma'_l}} > 0 \tag{5.10}$$

and $S_{m+1}(\sigma'_l) > 0$.

Then all the terms $S_m(\sigma'_l)$ of the sequence S_m are great then 0, it follows that $lim_{m\longrightarrow+\infty}S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) > 0$ and $\eta(\sigma'_l) < +\infty$ because $\Re(s'_l) = \sigma'_l > 0$ and $\eta(s'_l)$ is convergent. We deduce the contradiction that s'_l is a zero of $\eta(s)$ and the equation (5.3) is false. Then, the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l + it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the second case of the paragraph (4) above concerning the case $0 < \Re(s) < \dfrac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (1.1), it follows that **the Riemann hypothesis is verified**.

$\square$

From the calculations above, we can verify easily the following proposition:

Proposition 5.1. *For all* $s = \sigma$ *real with* $0 < \sigma < 1$, $\eta(s) > 0$ *and* $\zeta(s) < 0$.

6 Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;

- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (1.1), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 6.1. *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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