Interval Pentapartitioned Neutrosophic Sets

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Abstract

Pentapartitioned neutrosophic set is a powerful mathematical tool, which is the extension of neutrosophic set and n-valued neutrosophic refined logic for better designing and modeling real-life problems. A generalization of the notion of pentapartitioned neutrosophic set is introduced. The new notion is called Interval Pentapartitioned Neutrosophic set (IPNS). Pentapartitioned neutrosophic set is developed by combining the pentapartitioned neutrosophic set and interval neutrosophic set. We define several set theoretic operations of IPNSs, namely, inclusion, complement, intersection. We also establish various properties of set-theoretic operators.

Keywords: Neutrosophic set, Single valued neutrosophic set, Interval neutrosophic set, Pentapartitioned neutrosophic set, Interval pentapartitioned neutrosophic set

1. Introduction

Smarandache [1] developed the Neutrosophic Set (NS) by extending fuzzy set [2] and intuitionistic fuzzy set [3] by introducing the degrees of indeterminacy and rejection (falsity or non-membership) as independent components. Wang et al. [4] defined Interval NS (INS) as a subclass of NS by considering that the truth membership degree, indeterminacy membership degree and falsity membership degree are independent and assume values from the subunitary interval of [0, 1]. In 2010, Wang et al. [5] defined the Single Valued NS (SVNS) by restricting the degrees of membership, indeterminacy and falsity in [01]. In 2013, Smarandache [6] presented n- valued neutrosophic refined logic.

Chatterjee et al. [7] defined Quadripartitioned SVNS (QSVNS) that involves degrees of truth, falsity, unknown and contradiction membership based on four valued logics [6].

Mallick and Pramanik [8] developed the theory of Pentapartitioned NS (PNS) by diving indeterminacy into three independent components, namely contradiction, ignorance, unknown. In this paper, we start the investigation of generalization of the notion-the Interval Pentapartitioned Neutrosophic Set (IPNS).We also establish some basic properties of the proposed set. The proposed structure is generalization of existing theories of INS and PNS.

The organization of the paper is as follows: Section 2 presents some preliminary results. Section 3 introduces the concept of IPNS and set-theoretic operations over IPNS. Section 4 concludes the paper by stating future scope of research.

2. Preliminary

Definition 2.1. Let a set W be fixed. An NS [1] D over W is defined as:

$D=\{w(T_D(w), I_D(w), F_D(w)): w\in W\}$ where $T_D, I_D, F_D: W\to]^-0,1^+[$ and

$^-0\leq T_D(w)+I_D(w)+F_D(w)\leq 3^+$.

Definition 2.2 Let a set W be fixed. An SVNS D over W is defined as:

$D=\{w(T_D(w), I_D(w), F_D(w)): w\in W\}$ where $T_D, I_D, F_D: W\to]^-0,1^+[$ and

$0\leq T_D(w)+I_D(w)+F_D(w)\leq 3$.

Definition 2.3. Let a set W be fixed. An INS D over W is defined as:

$D=\{(w,(T_D(w), I_D(w), F_D(w)): w\in W\}$

where for each $w\in W$, $T_D(w), I_D(w), F_D(w)\subseteq[0,1]$ are the membership functions of truth, indeterminacy, and falsity and
$T_D(w)=[\inf T_D(w), \sup T_D(w)], I_D(w)=[\inf I_D(w), \sup I_D(w)]), F_D(w)=[\inf F_D(w), \sup F_D(w)]$ and
$0\leq \sup T_D(w)+\sup I_D(w)+\sup F_D(w)\leq 3$.

D can be expressed as:

$D=\{w,([\inf T_D(w), \sup T_D(w)],[\inf I_D(w), \sup I_D(w)]),[\inf F_D(w), \sup F_D(w)]): w\in W\}$

3. The Basic Theory of IPNS

Definition 3.1. IPNS

Suppose that *W* be a fixed set. Then D, an IPNS over *W* is denoted as follows:

D= $\{(w, T_D(w), C_D(w), G_D(w), U_D(w), F_D(w)): w\in W\}$, where for each point $w\in W$, $T_D(w)$, $C_D(w)$, $G_D(w)$ U_D,(w) $F_D(w)$ $\subseteq[0, 1]$ are the membership functions of truth, contradiction, ignorance, unknown, and falsity and $T_D(w)$= [inf T_D(w, suf $T_D(w)$] , $C_D(w)$ =[inf $C_D(w)$, suf $C_D(w)$] , $G_D(w)$ =[inf $G_D(w)$, suf $G_D(w)$], U_D,(w) = [inf $U_D(w)$, suf $U_D(w)$], $F_D(w)$ = [inf $F_D(w)$, suf $F_D(w)$] $\subseteq$ [0, 1] and $0\leq$ sup $T_D(w)$ + suf $C_D(w)$ + suf $G_D(w)$ + suf $U_D(w)$ + suf $F_D(w)$ ≤ 5.

Example 3.1. Assume that W = [w_1, w_2, w_3], where w_1, w_2, and w_3 denote respectively capability, trustworthiness, and price. The values of w_1, w_2, and w_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be degree of truth (good), degree of contradiction, degree of ignorance, degree of unknown, G and degree of false (poor). D_1 is an IPNS of *W* defined by
D_1 = {[0.4, 0.7], [0.1, 0.2], [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.5, 0.8], [0.2, 0.3], [0.1, 0.2], [0.15, 0.25], [0.2, 0.3]}/w2+ [0.6, 0.8], [01, 0.2], [0.2, 0.3], [0.15, 0.25], [0.1, 0.2]}/ w_3
D_2 is an IPNS of *W* defined by

D_2 = {[0.5, 0.9], [0.15, 0.25], [0.15, 0.25], [0.2, 0.3], [0:2, 0:3]}/w_1 + {[0.5, 0.8], [0.25, 0.3], [0.1, 0.2], [0.15, 0.25], [0.1; 0.3]}/w2+ [0.4, 0.7]; [01; 0.2], [0.2; 0.3], [0.15, 0.25], [0.15; 0.2]}/ w_3

Definition 3.2 An IPNS is said to be empty (null) denoted by $\hat{0}$ if and only if its truth-membership, contradiction membership, ignorance membership, unknown membership and falsity membership function values are respectively defined as follows:

inf $T_D(w)$= sup $T_D(w)$ = 0, inf $C_D(w)$= sup $C_D(w)$ = 0, inf $G_D(w)$= sup $G_D(w)$= 1, inf $U_D(w)$ = sup $U_D(w)$= 1, inf $F_D(w)$= sup $F_D(w)$ = 1 ,

$$\hat{0} = \{[0,0],[0,0],[1,1],[1,1],[1,1]\}$$

Definition 3.3 An IPNS is said to be unity *denoted by* $\hat{1}$ if and only if its truth-membership, contradiction membership, ignorance membership, unknown membership and falsity membership function values are respectively defined as follows:

inf $T_D(w)$= sup $T_D(w)$ = 1, inf $C_D(w)$= sup $C_D(w)$ = 1, inf $G_D(w)$= sup $G_D(w)$= 0, inf $U_D(w)$ = sup $U_D(w)$= 0, inf $F_D(w)$= sup $F_D(w)$ = 0 ,

$$\hat{1} = \{[1,1],[1,1],[0,0],[0,0],[0,0]\}$$

Also, we have $\underline{0} = \langle 0,0,1,1,1\rangle \, and \, \underline{1} = \langle 1,1,0,0,0\rangle$

Definition 3.4. (Containment) Assume that D_1 and D_2 be any two IPNS over W, D_1 is said to be contained in D_2, denoted by $D_1 \subseteq D_2$ if and only if

$$\begin{aligned}
&\inf T_{D_1}(w) \le \inf T_{D_2}(w), \text{suf } T_{D_1}(w) \le \text{suf } T_{D_2}(w),\\
&\inf C_{D_1}(w) \le \inf C_{D_2}(w), \text{suf } C_{D_1}(w) \le \text{suf } C_{D_2}(w),\\
&\inf G_{D_1}(w) \ge \inf G_{D_2}(w), \text{suf } G_{D_1}(w) \ge \text{suf } G_{D_2}(w),\\
&\inf U_{D_1}(w) \ge \inf U_{D_2}(w), \text{suf } U_{D_1}(w) \ge \text{suf } U_{D_2}(w),\\
&\inf F_{D_1}(w) \ge \inf F_{D_2}(w), \text{suf } F_{D_1}(w) \ge \text{suf } F_{D_2}(w),
\end{aligned}$$

for any $w \in W$.

Definition 3.5. Two IPNS D_1 and D_2 are equal if and only if $D_1 \subseteq D_2$ and $D_1 \supseteq D_2$

Definition 3.6. (Complement) Let $D = \{(w, T_D(w), C_D(w), G_D(w), U_D(w), F_D(w)) : w \in W\}$ be an IPNS. The complement of D is denoted by D' and defined as:

$$\begin{aligned}
&T_{D'}(w) = F_D(w),\ C_{D'}(w) = U_D(w),\\
&\inf G_{D'}(w) = 1 - \text{suf } G_D(w),\\
&\text{suf } G_{D'}(w) = 1 - \inf G_D(w),\\
&U_{D'}(w) = C_D(w), F_{D'}(w) = T_D(w)\\
&D' = \{(w, [\inf F_D(w), \sup F_D(w)], [\inf U_D(w), \sup U_D(w)], [1 - \text{suf } G_D(w), 1 - \inf G_D(w)],\\
&\qquad [\inf C_D(w), \sup C_D(w)], [\inf T_D(w), \sup T_D(w)]) : w \in W\}
\end{aligned}$$

Example 3.2 . Consider an IPNS D of the form:

D = {[0.4, 0.75], [0.1, 0.25], [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.5, 0.8], [0.2, 0.3], [0.1, 0.2], [0.15, 0.25], [0.2, 0.35]}/w2+ [0.75, 0.85], [0.15, 0.25], [0.2, 0.35], [0.15, 0.25], [0.1, 0.25]}/ w_3

Then, complement of

D' == {[0.2, 0.4] , [0.2, 0.3], [0.8, 0.9], [0.1, 0.25], [0.4, 0.75]}/w_1 + {[0.2, 0.35], [0.15, 0.8], [0.8, 0.9], [0.2, 0.3], [0.5, 0.8]}/w2+ [0.1, 0.25], [0.15, 0.25], [0.65, 0.8], [0.15, 0.25], [0.75, 0.85]}/ w_3

Definition 3.7.(Intersection)

The intersection of any two IPNSs D_1 and D_2 is an IPNS D_3, written as $D_3 = D_1 \cap D_2$, whose truth-membership, contradiction-membership, ignorance- membership, unknown -membership and falsity -membership functions are related to those of D_1 and D_2 by

$$\begin{aligned}
&\inf T_{D_3}(w) = \min(\inf T_{D_1}(w), \inf T_{D_2}(w)),\\
&\sup T_{D_3} = \min(\mathrm{suf} T_{D_1}(w), \mathrm{suf} T_{D_2}(w)),\\
&\inf C_{D_3}(w) = \min(\inf C_{D_1}(w), \inf C_{D_2}(w)),\\
&\sup C_{D_3} = \min(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf} C_{D_2}(w)),\\
&\inf G_{D_3}(w) = \max(\inf G_{D_1}(w), \inf G_{D_2}(w)),\\
&\sup G_{D_3} = \max(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w)),\\
&\inf U_{D_3}(w) = \max(\inf U_{D_1}(w), \inf U_{D_2}(w)),\\
&\sup U_{D_3} = \max(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w)),\\
&\inf F_{D_3}(w) = \max(\inf F_{D_1}(w), \inf F_{D_2}(w)),\\
&\sup F_{D_3} = \max(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w)),\\
&\forall w \in W.
\end{aligned}$$

$D_3 = D_1 \cap D_2$

$$\begin{aligned}
&\{(w, [\inf T_{D_3}(w), \sup T_{D_3}(w)], [\inf C_{D_3}(w), \sup C_{D_3}(w)],\\
&[\inf G_{D_3}(w), \sup G_{D_3}(w)], [\inf U_{D_3}(w), \sup U_{D_3} w)],\\
&[\inf F_{D_3}(w), \sup F_{D_3}(w)]) : \forall w \in W\}.
\end{aligned}$$

$$\begin{aligned}
&\{(w, [\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\mathrm{suf} T_{D_1}(w), \mathrm{suf} T_{D_2}(w))],\\
&[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf} C_{D_2}(w))],\\
= &[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],\\
&[\max(\inf U_{D_1}(w), \inf U_{D_2}(w))], \max(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],\\
&[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : w \in W\}
\end{aligned}$$

Example 3.3. Let D_1 and D_2 be the IPNSs defined in Example 1.

Then, $D_1 \cap D_2$ ={[0.4, 0.7], [0.1, 0.2], [0.15, 0.25], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.5, 0.8], [0.2, 0.3], [0.1, 0.2], [0.15, 0.25], [0.2, 0.3]}/w2+ [0.4, 0.7], [0.1, 0.2], [0.2, 0.3], [0.15, 0.25], [0.15, 0.2]}/ w_3

Definition 3.8. (**Union**) The union of any two IPNSs D_1 and D_2 is denoted by an IPNS D_3, written as $D_1 \cup D_2$ and is defined by

$$\begin{aligned}
&\inf T_{D_3}(w) = \max(\inf T_{D_1}(w), \inf T_{D_2}(w)),\\
&\sup T_{D_3} = \max(\mathrm{suf} T_{D_1}(w), \mathrm{suf} T_{D_2}(w)),\\
&\inf C_{D_3}(w) = \max(\inf C_{D_1}(w), \inf C_{D_2}(w)),\\
&\sup C_{D_3} = \max(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf} C_{D_2}(w)),\\
&\inf G_{D_3}(w) = \min(\inf G_{D_1}(w), \inf G_{D_2}(w)),\\
&\sup G_{D_3} = \min(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w)),\\
&\inf U_{D_3}(w) = \min(\inf U_{D_1}(w), \inf U_{D_2}(w)),\\
&\sup U_{D_3} = \min(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w)),\\
&\inf F_{D_3}(w) = \min(\inf F_{D_1}(w), \inf F_{D_2}(w)),\\
&\sup F_{D_3} = \min(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w)),\\
&\forall w \in W.
\end{aligned}$$

$$D_3 = D_1 \cup D_2$$
$$\{(w,[\inf T_{D_3}(w), \sup T_{D_3}(w)],[\inf C_{D_3}(w), \sup C_{D_3}(w)],$$
$$[\inf G_{D_3}(w), \sup G_{D_3}(w)),[\inf U_{D_3}(w), \sup U_{D_3}(w)],[\inf F_{D_3}(w), \sup F_{D_3}]) : w \in W\}.$$
$$= \{(w,[\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))],[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf}C_{D_2}(w))],$$
$$[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],$$
$$[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : w \in W\}.$$

Example 3. 4. Let D_1 and D_2 be the IPNSs in example 1. Then

$D_1 \cup D_2$ ={[0.5, 0.9], [0.15, 0.25], [0.1, 0.2], [0.2, 0.3], [0.2, 0.3]}/w1 + {[0.5, 0.8], [0.25, 0.3], [0.1, 0.2], [0.15, 0.25], [0.1, 0.3]}/w2+ [0.4, 0.7], [0.1, 0.2], [0.2, 0.3], [0.15, 0.25], [0.15, 0.2]}/ w3

Theorem 3.1 For any two IPNS D_1 and D_2:

$(a) D_1 \cup D_2 = D_2 \cup D_1$

$(b) D_1 \cap D_2 = D_2 \cap D_1$

Proof: (a):

Assume that D_1 and D_2 be any two IPNSs over W *defined by*

$$D_i = \{(w, T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{Di_2}(w))]) : w \in W\}, i = 1, 2, \text{and } T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{Di_2}(w))] \subseteq [0,1]$$

We have,

$$D_1 \cup D_2 = \{(w,[\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))],$$
$$[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf}C_{D_2}(w))],$$
$$[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],$$
$$[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],$$
$$[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : w \in W\}$$
$$= \{w,[\max(\inf T_{D_2}(w), \inf T_{D_1}(w)), \max(\mathrm{suf}T_{D_2}(w), \mathrm{suf}T_{D_1}(w))],[\max(\inf C_{D_2}(w), \inf C_{D_1}(w)), \max(\mathrm{suf}\, C_{D_2}(w), \mathrm{suf}C_{D_1}(w))],$$
$$[\min(\inf G_{D_2}(w), \inf G_{D_1}(w)), \min(\mathrm{suf}\, G_{D_2}(w), \mathrm{suf}\, G_{D_1}(w))],[\min(\inf U_{D_2}(w), \inf U_{D_1}(w)),$$
$$\min(\mathrm{suf}\, U_{D_2}(w), \mathrm{suf}\, U_{D_1}(w))],[\min(\inf F_{D_2}(w), \inf F_{D_1}(w)), \min(\mathrm{suf}\, F_{D_2}(w), \mathrm{suf}\, F_{D_1}(w))] : w \in W\}$$
$$= D_2 \cup D_1$$

b) $D_1 \cap D_2 = D_2 \cap D_1$

$$D_1 \cap D_2 = \{(w,[\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))],$$
$$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf}C_{D_2}(w))],$$
$$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],$$
$$[\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],$$
$$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : \forall w \in W\}.$$
$$= \{(w,[\min(\inf T_{D_2}(w), \inf T_{D_1}(w)), \min(\mathrm{suf}T_{D_2}(w), \mathrm{suf}T_{D_1}(w))],$$
$$[\min(\inf C_{D_2}(w), \inf C_{D_1}(w)), \min(\mathrm{suf}\, C_{D_2}(w), \mathrm{suf}C_{D_1}(w))],$$
$$[\max(\inf G_{D_2}(w), \inf G_{D_1}(w)), \max(\mathrm{suf}\, G_{D_2}(w), \mathrm{suf}\, G_{D_1}(w))],$$
$$[\max(\inf U_{D_2}(w), \inf U_{D_1}(w)), \max(\mathrm{suf}\, U_{D_2}(w), \mathrm{suf}\, U_{D_1}(w))],$$
$$[\max(\inf F_{D_2}(w), \inf F_{D_1}(w)), \max(\mathrm{suf}\, F_{D_2}(w), \mathrm{suf}\, F_{D_1}(w))]) : \forall w \in W\}.$$
$$= D_2 \cap D_1$$

Theorem 3. 2. For any three IPNS, $D_1, D_2, \text{and } D_3$:

$(a) D_1 \cup (D_2 \cup D_3) = (D_1 \cup D_2) \cup D_3$

$(b) D_1 \cap (D_2 \cap D_3) = (D_1 \cap D_2) \cap D_3$

Proof (a): Assume that D_1, D_2 and D_3 be any three IPNSs over W *defined by*
$D_i = \{(w, T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{D_i}(w))]): w \in W\}, i = 1, 2, 3,$ and
$T_{D_i}(w), C_{D_i}(w), G_{D_i}(w), U_{D_i}(w), F_{D_i}(w))] \subseteq [0,1], i = 1, 2, 3.$

$$(D_1 \cup D_2) \cup D_3) = \{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\text{suf}T_{D_1}(w), \text{suf}T_{D_2}(w))],$$
$$[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\text{suf } C_{D_1}(w), \text{suf}C_{D_2}(w))],$$
$$[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\text{suf } G_{D_1}(w), \text{suf } G_{D_2}(w))], [\min(\inf U_{D_1}(w), \inf U_{D_2}(w)),$$
$$\min(\text{suf } U_{D_1}(w), \text{suf } U_{D_2}(w))], [\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\text{suf } F_{D_1}(w), \text{suf } F_{D_2}(w))]): w \in W\} \cup$$
$$\{w([\inf G_{D_3}(w), \sup G_{D_3}(w)), [\inf U_{D_3}(w), \sup U_{D_3}(w)], [\inf F_{D_3}(w), \sup F_{D_3}]): w \in W\}$$
$$=$$
$$\{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w), \inf T_{D_3}(w)), \max(\sup T_{D_1}(w), \sup T_{D_2}(w), \sup T_{D_3}(w))],$$
$$[\max(\inf C_{D_1}(w), \inf C_{D_2}(w), \inf C_{D_3}(w)), \max(\sup C_{D_1}(w), \sup C_{D_2}(w), \sup C_{D_3}(w))],$$
$$[\min(\inf G_{D_1}(w), \inf G_{D_2}(w), \inf G_{D_3}(w)), \min(\sup G_{D_1}(w), \sup G_{D_2}(w), \sup G_{D_3}(w))],$$
$$[(\inf U_{D_1}(w), \inf U_{D_2}(w), \inf U_{D_3}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w), \sup U_{D_3}(w))],$$
$$[(\inf F_{D_1}(w), \inf F_{D_2}(w), \inf F_{D_3}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w), \sup F_{D_3}(w))]): w \in W\}$$
$$=$$
$$\{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$$
$$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]): w \in W\} \cup$$
$$\{(w, [\max(\inf T_{D_2}(w), \inf T_{D_3}(w)), \max(\text{suf}T_{D_2}(w), \text{suf}T_{D_3}(w))],$$
$$[\max(\inf C_{D_2}(w), \inf C_{D_3}(w)), \max(\text{suf } C_{D_2}(w), \text{suf}C_{D_3}(w))],$$
$$[\min(\inf G_{D_2}(w), \inf G_{D_3}(w)), \min(\text{suf } G_{D_2}(w), \text{suf } G_{D_3}(w))],$$
$$[\min(\inf U_{D_2}(w), \inf U_{D_3}(w)), \min(\text{suf } U_{D_2}(w), \text{suf } U_{D_3}(w))],$$
$$[\min(\inf F_{D_2}(w), \inf F_{D_3}(w)), \min(\text{suf } F_{D_2}(w), \text{suf } F_{D_3}(w))]): w \in W\}$$
$$= D_1 \cup (D_2 \cup D_3)$$

Proof. (b):

$$D_1 \cap (D_2 \cap D_3)$$
$$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$$
$$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]): w \in W\} \cap$$
$$\{(w, [\min(\inf T_{D_2}(w), \inf T_{D_3}(w)), \min(\text{suf}T_{D_2}(w), \text{suf}T_{D_3}(w))],$$
$$[\min(\inf C_{D_2}(w), \inf C_{D_3}(w)), \min(\text{suf } C_{D_2}(w), \text{suf } C_{D_3}(w))],$$
$$[\max(\inf G_{D_2}(w), \inf G_{D_3}(w)), \max(\text{suf } G_{D_2}(w), \text{suf } G_{D_3}(w))],$$
$$[\max(\inf U_{D_2}(w), \inf U_{D_3}(w)), \max(\text{suf } U_{D_2}(w), \text{suf } U_{D_3}(w))],$$
$$[\max(\inf F_{D_2}(w), \inf F_{D_3}(w)), \max(\text{suf } F_{D_2}(w), \text{suf } F_{D_3}(w))]): w \in W\}.$$

$= \{(w,[\min(\inf T_{D_1}, \inf T_{D_2}(w), \inf T_{D_3}(w)), \min(\sup T_{D_1}(w), \mathrm{suf}T_{D_2}(w), \mathrm{suf}T_{D_3}(w))],$
$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w), \inf C_{D_3}(w)), \min(\sup C_{D_1}(w), \mathrm{suf}\, C_{D_2}(w), \mathrm{suf}\, C_{D_3}(w))],$
$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w), \inf G_{D_3}(w)), \max(\sup G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w), \mathrm{suf}\, G_{D_3}(w))],$
$[\max(\inf U_{D_1}\ \inf U_{D_2}(w), \inf U_{D_3}(w)), \max(\mathrm{suf}\, U_{D1}, \mathrm{suf}\, U_{D_2}(w), \mathrm{suf}\, U_{D_3}(w))],$
$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w), \inf F_{D_3}(w)), \max(\sup F_{D_1}(w)], \mathrm{suf}\, F_{D_2}(w), \mathrm{suf}\, F_{D_3}(w))]) : w \in W\}$
$=$
$\{(w,[\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))],$
$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf}C_{D_2}(w))],$
$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],$
$[\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],$
$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : w \in W\} \cap$
$\{w\,([\inf G_{D_3}(w), \sup G_{D_3}(w)), [\inf U_{D_3}(w), \sup U_{D_3}(w)], [\inf F_{D_3}(w), \sup F_{D_3}]) : w \in W\}$
$= (D_1 \cap D_2) \cap D_3$

Theorem 3.3. For any two IPNS, $D_1, \text{and}\, D_2$:

(a) $D_1 \cup (D_1 \cap D_2) = D_1$

(b) $D_1 \cap (D_1 \cup D_2) = D_1$

Proof .(a):

$D_1 \cup (D_1 \cap D_2) =$
$\{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$
$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$
$\cup$
$\{(w,[\min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \min(\mathrm{suf}T_{D_1}(w), \mathrm{suf}\, T_{D_2}(w))],$
$[\min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \min(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf}\, C_{D_2}(w))],$
$[\max(\inf G_{D_1}(w), \inf G_{D_2}(w)), \max(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],$
$[\max(\inf U_{D_1}(w), \inf U_{D_2}(w)), \max(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],$
$[\max(\inf F_{D_1}(w), \inf F_{D_2}(w)), \max(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : \forall w \in W\}.$
$=$
$\{w, ([\max(\inf T_{D_1}(w), \min(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\sup T_{D_1}(w),\ \min(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))],$
$[\max(\inf C_{D_1}(w), \min(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max((\sup C_{D_1}(w),\ \min(\mathrm{suf}\, C_{D_1}(w), \mathrm{suf}\, C_{D_2}(w))]),$
$[[\min(\inf (G_{D_1}(w), \max(\inf G_{D_1}(w), \inf G_{D_2}(w))),\ \min(\sup G_{D_1}(w), \max(\mathrm{suf}\, G_{D_1}(w), \mathrm{suf}\, G_{D_2}(w))],$
$[\min(\inf (U_{D_1}(w), \max(\inf U_{D_1}(w), \inf U_{D_2}(w))),\ \min(\sup U_{D_1}(w), \max(\mathrm{suf}\, U_{D_1}(w), \mathrm{suf}\, U_{D_2}(w))],$
$[\min(\inf (F_{D_1}(w), \max(\inf F_{D_1}(w), \inf F_{D_2}(w))),\ \min(\sup F_{D_1}(w), \max(\mathrm{suf}\, F_{D_1}(w), \mathrm{suf}\, F_{D_2}(w))]) : w \in W\}$
$=$
$\{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$
$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$
$= D_1$

Proof (b):

$$
\begin{aligned}
&D_1 \cap (D_1 \cup D_2) \\
&= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), \\
&[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in\} \cap \\
&\{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))], \\
&[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\mathrm{suf}\ C_{D_1}(w), \mathrm{suf}C_{D_2}(w))], \\
&[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\mathrm{suf}\ G_{D_1}(w), \mathrm{suf}\ G_{D_2}(w))], \\
&[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\mathrm{suf}\ U_{D_1}(w), \mathrm{suf}\ U_{D_2}(w))], \\
&[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\mathrm{suf}\ F_{D_1}(w), \mathrm{suf}\ F_{D_2}(w))]) : w \in W\} \\
&= \{w, [\min((\inf T_{D_1}(w), \max(\inf T_{D_1}(w), \inf T_{D_2}(w))), \min((\sup T_{D_1}(w), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))], \\
&[\min((\inf C_{D_1}(w), \max(\inf C_{D_1}(w), \inf C_{D_2}(w))), \min((\sup C_{D_1}(w), \max(\mathrm{suf}\ C_{D_1}(w), \mathrm{suf}C_{D_2}(w))], \\
&[\max((\inf G_{D_1}(w), \min(\inf G_{D_1}(w), \inf G_{D_2}(w))), \max((\sup G_{D_1}(w)), \min(\mathrm{suf}\ G_{D_1}(w), \mathrm{suf}\ G_{D_2}(w))], \\
&[\max((\inf U_{D_1}(w), \min(\inf U_{D_1}(w), \inf U_{D_2}(w))), \max((\sup U_{D_1}(w), \min(\mathrm{suf}\ U_{D_1}(w), \mathrm{suf}\ U_{D_2}(w))], \\
&[\max((\inf F_{D_1}(w), \min(\inf F_{D_1}(w), \inf F_{D_2}(w))), \max((\sup F_{D_1}(w), \min(\mathrm{suf}\ F_{D_1}(w), \mathrm{suf}\ F_{D_2}(w))] \\
&= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), \\
&[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in\} \\
&= D_1
\end{aligned}
$$

Theorem 3.4. For any IPNS D_1**:**

(a) $D_1 \cup D_1 = D_1$

(b) $D_1 \cap D_1 = D_1$

Proof. (a):

$$
\begin{aligned}
&D_1 \cup D_1 = \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)), \\
&[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cup \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], \\
&[\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], \\
&[\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \\
&= \{w, ([\max(\inf T_{D_1}(w), \inf T_{D_1}(w)), \max(\sup T_{D_1}(w), \sup T_{D_1}(w))], \\
&[\max(\inf C_{D_1}(w), (\inf C_{D_1}(w))), \max(\sup T_{D_1}(w), \sup C_{D_1}(w)]), \\
&[\min(\inf G_{D_1}(w), \sup G_{D_1}(w)), [\min(\inf U_{D_1}(w), \inf U_{D_1}(w)), (\min(\sup U_{D_1}(w), (\mathrm{suf}\ U_{D_1}(w))], \\
&[\min(\inf F_{D_1}(w), \inf F_{D_1}(w)), \min(\sup F_{D_1}(w), \sup F_{D_1}(w))]) : w \in W\} \\
&= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)), \\
&[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \\
&= D_1
\end{aligned}
$$

Proof. (b):

$$
\begin{aligned}
&D_1 \cap D_1 = \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)), \\
&[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cup \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], \\
&[\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], \\
&[\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}
\end{aligned}
$$

$$= \{(w,[\min(\inf T_{D_1}(w),\inf T_{D_1}(w)),\min(\text{suf}T_{D_1}(w),\text{suf}T_{D_1}(w))],$$
$$[\min(\inf C_{D_1}(w),\inf C_{D_1}(w)),\min(\text{suf } C_{D_1}(w),\text{suf } C_{D_1}(w))],$$
$$[\max(\inf G_{D_1}(w),\inf G_{D_1}(w)),\max(\text{suf } G_{D_1}(w),\text{suf } G_{D_1}(w))],$$
$$[\max(\inf U_{D_1}(w),\inf U_{D_1}(w)),\max(\text{suf } U_{D_1}(w),\text{suf } U_{D_1}(w))],$$
$$[\max(\inf F_{D_1}(w),\inf F_{D_1}(w)),\max(\text{suf } F_{D_1}(w),\text{suf } F_{D_1}(w))]) : \forall w \in W\}'$$

$$= \{w,([\inf T_{D_1}(w),\sup T_{D_1}(w)],[\inf C_{D_1}(w),\sup C_{D_1}(w)]),[\inf G_{D_1}(w),\sup G_{D_1}(w)),$$
$$[\inf U_{D_1}(w),\sup U_{D_1}(w)],[\inf F_{D_1}(w),\sup F_{D_1}(w)]) : w \in W\}$$
$$= D_1$$

Theorem 3.5 For any IPNS D_1,

$$(a) D_1 \cap \hat{0} = \hat{0}$$
$$(b) D_1 \cup \hat{1} = \hat{1}$$

Proof. (a):

$$D_1 \cap \hat{0}$$
$$= \{w,([\inf T_{D_1}(w),\sup T_{D_1}(w)],[\inf C_{D_1}(w),\sup C_{D_1}(w)]),$$
$$[\inf G_{D_1}(w),\sup G_{D_1}(w)),[\inf U_{D_1}(w),\sup U_{D_1}(w)],[\inf F_{D_1}(w),\sup F_{D_1}(w)]) : w \in W\} \cap \{[0,0],[0,0],[1,1],[1,1],[1,1]\}$$
$$= \{(w,[\min(\inf T_{D_1}(w),0),\min(\text{suf}T_{D_1}(w),0)],$$
$$[\min(\inf C_{D_1}(w),0),\min(\text{suf } C_{D_1}(w),0)],$$
$$[\max(\inf G_{D_1}(w),1),\max(\text{suf } G_{D_1}(w),1)],$$
$$[\max(\inf U_{D_1}(w),1),\max(\text{suf } U_{D_1}(w),1)],$$
$$[\max(\inf F_{D_1}(w),1),\max(\text{suf } F_{D_1}(w),1]) : \forall w \in W\}$$
$$= \{(w,[0,0)],[0,0)],[1,1],[1,1],[1,1]),\forall w \in W\}$$
$$= \hat{0}$$

Proof. (b):

$$D_1 \cup \hat{1}$$
$$= \{w,([\inf T_{D_1}(w),\sup T_{D_1}(w)],[\inf C_{D_1}(w),\sup C_{D_1}(w)]),$$
$$[\inf G_{D_1}(w),\sup G_{D_1}(w)),[\inf U_{D_1}(w),\sup U_{D_1}(w)],[\inf F_{D_1}(w),\sup F_{D_1}(w)]) : w \in W\} \cup \{[1,1],[1,1],[0,0],[0,0],[0,0]\}$$
$$= \{(w,[\max(\inf T_{D_1}(w),1),\max(\text{suf}T_{D_1}(w),1))],$$
$$[\max(\inf C_{D_1}(w),1),\max(\text{suf } C_{D_1}(w),1)],$$
$$[\min(\inf G_{D_1}(w),0),\min(\text{suf } G_{D_1}(w),0)],$$
$$[\min(\inf U_{D_1}(w),0),\min(\text{suf } U_{D_1}(w),0)],$$
$$[\min(\inf F_{D_1}(w),0)),\min(\text{suf } F_{D_1}(w),0)]) : w \in W\}$$
$$= \{w([1,1],[1,1],[0,0],[0,0],[0,0]) : w \in W\}$$
$$= \hat{1}$$

Theorem 3.6 For any IPNS D_1,

$$(a) D_1 \cup \hat{0} = D_1$$
$$(b) D_1 \cap \hat{1} = D_1$$

Proof. (a):

$D_1 \cup \hat{0}$
$== \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$
$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cup \{[0,0],[0,0],[1,1],[1,1],[1,1]\}$
$= \{(w, [\max(\inf T_{D_1}(w), 0), \max(\text{suf} T_{D_1}(w), 0))],$
$[\max(\inf C_{D_1}(w), 0), \max(\text{suf } C_{D_1}(w), 0)],$
$[\min(\inf G_{D_1}(w), 1), \min(\text{suf } G_{D_1}(w), 1)],$
$[\min(\inf U_{D_1}(w), 1), \min(\text{suf } U_{D_1}(w), 1)],$
$[\min(\inf F_{D_1}(w), 1)), \min(\text{suf } F_{D_1}(w), 1)]) : w \in W\}$
$w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]),$
$[\inf G_{D_1}(w), \sup G_{D_1}(w)), [\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$
$= D_1$

$D_1 \cap \hat{1}$
$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)),$
$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\} \cap \{[1,1,[1,1,[0,0],[0,0],[0,0]\}$
$= \{(w, [\min(\inf T_{D_1}(w), 1), \min(\text{suf} T_{D_1}(w), 1)], [\min(\inf C_{D_1}(w), 1), \min(\text{suf } C_{D_1}(w), 1)],$
$[\max(\inf G_{D_1}(w), 0), \max(\text{suf } G_{D_1}(w), 0)], [\max(\inf U_{D_1}(w), 0), \max(\text{suf } U_{D_1}(w), 0)],$
$[\max(\inf F_{D_1}(w), 1), \max(\text{suf } F_{D_1}(w), 1]) : \forall w \in W\}$
$= \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)),$
$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$
$= D_1$

Theorem 3.7 For any IPNS D_1, $(D_1')' = D_1$

Assume that $D_1 = \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)),$
$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$
$D_1' = \{w, ([\inf F_{D_1}(w), \sup F_{D_1}(w)], [\inf U_{D_1}(w), \sup U_{D_1}(w)], [1 - \text{suf } G_{D_1}(w), 1 - \inf G_{D_1}(w)],$
$[\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf T_{D_1}(w), \sup T_{D_1}(w)], : w \in W\}$
$\therefore (D_1')' = \{w, ([\inf T_{D_1}(w), \sup T_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), [\inf G_{D_1}(w), \sup G_{D_1}(w)),$
$[\inf U_{D_1}(w), \sup U_{D_1}(w)], [\inf F_{D_1}(w), \sup F_{D_1}(w)]) : w \in W\}$
$= D_1$

Theorem 3.8. For any two IPNSs, D_1 and D_2:

(a)$(D_1 \cup D_2)' = D_1' \cap D_2'$
(b)$(D_1 \cap D_2)' = D_1' \cup D_2'$

Proof. (a):
To prove the theorem 3.8, we need some propositions:
i. If $P \subset \mathbb{R}$ and $a \in \mathbb{R}$, then

$aP = \{y \in \mathbb{R} : y = ax \text{ for some } x \in P\}.$

Proposition 1. *If* $a \geq 0$,

$$\begin{array}{l} \text{then } \sup aP = a\sup P,\ \inf aP = a\inf P, \\ \text{if } a < 0, \text{then} \\ \sup aP = a\inf P,\ \text{Inf } aP = a\sup P. \end{array}$$

In particular, $\sup(-P) = -\inf P,\ Inf\,(-P) = -sup\,P.$

Proposition 2. If P and Q are nonempty set, then

sup(P + Q) = sup P+ sup Q, inf (P + Q) = inf P + inf Q

sup(P - Q) = sup P - inf Q, inf (P - Q) = inf P + inf Q

Now,

$$\begin{array}{l}
D_1 \cup D_2 = \{(w, [\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))], \\
[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\mathrm{suf}\ C_{D_1}(w), \mathrm{suf}C_{D_2}(w))], \\
[\min(\inf G_{D_1}(w), \inf G_{D_2}(w)), \min(\mathrm{suf}\ G_{D_1}(w), \mathrm{suf}\ G_{D_2}(w))], \\
[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\mathrm{suf}\ U_{D_1}(w), \mathrm{suf}\ U_{D_2}(w))], \\
[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\mathrm{suf}\ F_{D_1}(w), \mathrm{suf}\ F_{D_2}(w))]) : w \in W\} \\
(D_1 \cup D_2)' = \{(w, [\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\mathrm{suf}\ F_{D_1}(w), \mathrm{suf}\ F_{D_2}(w))], \\
[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\mathrm{suf}\ U_{D_1}(w), \mathrm{suf}\ U_{D_2}(w))], \\
[1 - \sup(\min(\mathrm{suf}\ G_{D_1}(w), \mathrm{suf}\ G_{D_2}(w)), 1 - \inf((\min(\inf G_{D_1}(w), \inf G_{D_2}(w))], \\
[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\mathrm{suf}\ C_{D_1}(w), \mathrm{suf}C_{D_2}(w))], \\
[\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))]) : w \in W\} \qquad (1)
\end{array}$$

$$\begin{array}{l}
D_1' \cap D_2' = \{(w,\ [\inf F_{D_1}(w), \sup F_{D_1}(w)],\ [\inf U_{D_1}(w), \sup U_{D_1}(w)], \\
[1 - \mathrm{suf}\ G_{D_1}(w), 1 - \inf G_{D_1}(w)], [\inf C_{D_1}(w), \sup C_{D_1}(w)]), \\
[\inf T_{D_1}(w), \sup T_{D_1}(w)], : w \in W\} \cap \{w, ([\inf F_{D_2}(w), \sup F_{D_2}(w)], \\
[\inf U_{D_2}(w), \sup U_{D_2}(w)], [1 - \mathrm{suf}\ G_{D_2}(w), 1 - \inf G_{D_2}(w)], \\
[\inf C_{D_2}(w), \sup C_{D_2}(w)]), [\inf T_{D_2}(w), \sup T_{D_2}(w)]) : w \in W\} \\
= \{(w, [\min(\inf F_{D_1}(w), \mathrm{minf}\ F_{D_2}(w), \min(\sup F_{D_1}(w), \sup F_{D_2}(w)], \\
[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\mathrm{suf}\ U_{D_1}(w), \mathrm{suf}\ U_{D_2}(w))], \\
[\max(\inf(1 - \mathrm{suf}\ G_{D_1}(w)), \inf(1 - \mathrm{suf}\ G_{D_2}(w))), \max(\sup(1 - \inf G_{D_1}(w)), \sup(1 - \inf G_{D_2}(w)))], \\
[\max(\inf C_{D_1}(w), \inf C_{D_2}(w)), \max(\mathrm{suf}\ C_{D_1}(w), \mathrm{suf}C_{D_2}(w))], \\
[\max(\inf T_{D_1}(w), \inf T_{D_2}(w)), \max(\mathrm{suf}T_{D_1}(w), \mathrm{suf}T_{D_2}(w))]) : w \in W\} \qquad (2)
\end{array}$$

Now, the theorem 3.8. (a) will be proved, if we can prove that

$$[1 - \sup(\min(suf\ G_{D_1}(w), suf\ G_{D_2}(w)) = \max(\inf(1 - suf\ G_{D_1}(w)), \inf(1 - suf\ G_{D_2}(w)))$$

$$1 - \inf((\min(\inf G_{D_1}(w), \inf G_{D_2}(w)) = \max(\sup(1 - \inf G_{D_1}(w)), \sup(1 - \inf G_{D_2}(w)))$$

Now, assume that

$$\begin{array}{l}
\inf G_{D_1}(w) = c_1, suf\ G_{D_1}(w) = d_1 \\
\inf G_{D_2}(w) = c_2, suf\ G_{D_1}(w) = d_2
\end{array}$$

$$1-\sup(\min(suf\ G_{D_1}(w), suf\ G_{D_2}(w)))$$
$$=1-\sup(\min(d_1, d_2))$$
$$=\begin{cases}1-\sup d_1, & if\ d_1 \geq d_2\\ 1-\sup d_2, & if\ d_1 \leq d_2\end{cases}$$
$$=\begin{cases}1-d_1, & if\ d_1 \geq d_2\\ 1-d_2, & if\ d_1 \leq d_2\end{cases} \quad (3)$$

$$\max(\inf (1-suf\ G_{D_1}(w)), \inf(1-suf\ G_{D_2}(w)))$$
$$= \max(\inf(1-d_1), \inf(1-d_2))$$
$$= \max(1-\sup d_1, 1-\sup d_2))\ by\ proposition\ 2.$$
$$=\max(1-d_1, 1-d_2))$$
$$=\begin{cases}1-d_1, & if\ d_1 \geq d_2\\ 1-d_2, & if\ d_1 \leq d_2\end{cases} \quad (4)$$

Therefore, from (3) and (4), we have

$$[1-\sup(\min(suf\ G_{D_1}(w), suf\ G_{D_2}(w)) = \max(\inf (1-suf\ G_{D_1}(w)), \inf(1-suf\ G_{D_2}(w))) \quad (5)$$

Now,

$$1-\inf((\min(\inf \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))$$
$$=1-\inf(\min(\mathrm{c}_1, \mathrm{c}_2))$$
$$=\begin{cases}1-\inf \mathrm{c}_1, \mathrm{if}\ \mathrm{c}_1 \geq \mathrm{c}_2\\ 1-\inf \mathrm{c}_2, \mathrm{if}\ \mathrm{c}_1 \leq \mathrm{c}_2\end{cases}$$
$$=\begin{cases}1-\mathrm{c}_1, \mathrm{if}\ \mathrm{c}_1 \geq \mathrm{c}_2\\ 1-\mathrm{c}_2, \mathrm{if}\ \mathrm{c}_1 \leq \mathrm{c}_2\end{cases} \quad (6)$$

$$\max(\sup(1-\inf \mathrm{G}_{\mathrm{D}_1}(\mathrm{w})), \sup(1-\inf \mathrm{G}_{\mathrm{D}_2}(\mathrm{w})))$$
$$= \max(\sup(1-\mathrm{c}_1), \sup(1-\mathrm{c}_2))$$
$$= \max(1-\inf \mathrm{c}_1, 1-\inf \mathrm{c}_2)$$
$$=\begin{cases}1-\inf \mathrm{c}_1, \mathrm{if}\ \mathrm{c}_1 \geq \mathrm{c}_2\\ 1-\inf \mathrm{c}_2, \mathrm{if}\ \mathrm{c}_1 \leq \mathrm{c}_2\end{cases}$$
$$=\begin{cases}1-\mathrm{c}_1, \mathrm{if}\ \mathrm{c}_1 \geq \mathrm{c}_2\\ 1-\mathrm{c}_2, \mathrm{if}\ \mathrm{c}_1 \leq \mathrm{c}_2\end{cases} \quad (7)$$

Therefore, from (6) and (7) , we have

$$1-\inf((\min(\inf G_{D_1}(w), \inf G_{D_2}(w)) = \max(\sup(1-\inf G_{D_1}(w)), \sup(1-\inf G_{D_2}(w))) \quad (8)$$

Therefore from (1), (2), (5) and (8), we prove that

$(D_1 \cup D_2)' = D_1' \cap D_2'$.

Proof. (b):

$$
\begin{aligned}
&(\mathrm{D}_1 \cap \mathrm{D}_2)' = \{(\mathrm{w}, [\min(\inf \mathrm{T}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{T}_{\mathrm{D}_2}(\mathrm{w})), \min(\mathrm{suf} \mathrm{T}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf} \mathrm{T}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\min(\inf \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{C}_{\mathrm{D}_2}(\mathrm{w})), \min(\mathrm{suf}\ \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf} \mathrm{C}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\max(\inf \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{G}_{\mathrm{D}_2}(\mathrm{w})), \max(\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\max(\inf \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{U}_{\mathrm{D}_2}(\mathrm{w})), \max(\mathrm{suf}\ \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{U}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\max(\inf \mathrm{F}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{F}_{\mathrm{D}_2}(\mathrm{w})), \max(\mathrm{suf}\ \mathrm{F}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{F}_{\mathrm{D}_2}(\mathrm{w}))]) : \forall \mathrm{w} \in \mathrm{W}\}'\\
&= \{(\mathrm{w}, [\max(\inf \mathrm{F}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{F}_{\mathrm{D}_2}(\mathrm{w})), \max(\mathrm{suf}\ \mathrm{F}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{F}_{\mathrm{D}_2}(\mathrm{w}))]),\\
&[\max(\inf \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{U}_{\mathrm{D}_2}(\mathrm{w})), \max(\mathrm{suf}\ \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{U}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[1-\sup(\max(\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w})), 1-\inf \max(\inf \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{U}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\min(\inf \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{C}_{\mathrm{D}_2}(\mathrm{w})), \min(\mathrm{suf}\ \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf} \mathrm{C}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\min(\inf \mathrm{T}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{T}_{\mathrm{D}_2}(\mathrm{w})), \min(\mathrm{suf} \mathrm{T}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf} \mathrm{T}_{\mathrm{D}_2}(\mathrm{w}))] : \forall \mathrm{w} \in \mathrm{W}\} \qquad (9)
\end{aligned}
$$

Now

$$
\begin{aligned}
D_1' \cup D_2' = &\{\mathrm{w}, ([\inf \mathrm{F}_{\mathrm{D}_1}(\mathrm{w}), \sup \mathrm{F}_{\mathrm{D}_1}(\mathrm{w})], [\inf \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \sup \mathrm{U}_{\mathrm{D}_1}(\mathrm{w})], [1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), 1-\inf \mathrm{G}_{\mathrm{D}_1}(\mathrm{w})], [\inf \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \sup \mathrm{C}_{\mathrm{D}_1}(\mathrm{w})]),\\
&[\inf \mathrm{T}_{\mathrm{D}_1}(\mathrm{w}), \sup \mathrm{T}_{\mathrm{D}_1}(\mathrm{w})]) : \mathrm{w} \in \mathrm{W}\} \cup \{\mathrm{w}, ([\inf \mathrm{F}_{\mathrm{D}_2}(\mathrm{w}), \sup \mathrm{F}_{\mathrm{D}_2}(\mathrm{w})], [\inf \mathrm{U}_{\mathrm{D}_2}(\mathrm{w}), \sup \mathrm{U}_{\mathrm{D}_2}(\mathrm{w})],\\
&[1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}), 1-\inf \mathrm{G}_{\mathrm{D}_2}(\mathrm{w})], [\inf \mathrm{C}_{\mathrm{D}_2}(\mathrm{w}), \sup \mathrm{C}_{\mathrm{D}_2}(\mathrm{w})]), [\inf \mathrm{T}_{\mathrm{D}_2}(\mathrm{w}), \sup \mathrm{T}_{\mathrm{D}_2}(\mathrm{w})]) : \mathrm{w} \in \mathrm{W}\}\\
&= \{\mathrm{w}, [\max(\inf \mathrm{F}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{F}_{\mathrm{D}_2}(\mathrm{w})), \max(\sup \mathrm{F}_{\mathrm{D}_1}(\mathrm{w})], \sup \mathrm{F}_{\mathrm{D}_2}(\mathrm{w})],\\
&[\max(\inf \mathrm{U}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{U}_{\mathrm{D}_2}(\mathrm{w})), \max(\sup \mathrm{U}_{\mathrm{D}_1}(\mathrm{w})], \sup \mathrm{U}_{\mathrm{D}_2}(\mathrm{w})],\\
&[\min((1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), 1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w})), \min((1-\inf \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), 1-\inf \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))],\\
&[\min((\inf \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \inf \mathrm{C}_{\mathrm{D}_2}(\mathrm{w})), \min((\mathrm{up}\, \mathrm{C}_{\mathrm{D}_1}(\mathrm{w}), \sup \mathrm{C}_{\mathrm{D}_2}(\mathrm{w}))]) : \mathrm{w} \in \mathrm{W}\} \qquad (10)
\end{aligned}
$$

To prove the theorem 3.8. (b), we are to prove

$$1-\sup(\max(\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w})) = \min(1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), 1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))$$

$$1-\inf \max(\inf U_{D_1}(w), \inf U_{D_2}(w))], = \min((1-\inf \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), 1-\inf \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))\ .$$

Now

$$
\begin{aligned}
&1-\sup(\max(\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), \mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))\\
&= 1-\sup(\max(\mathrm{d}_1, \mathrm{d}_2)\\
&= \begin{cases} \{1-\sup \mathrm{d}_1, \text{if } \mathrm{d}_1 \ge \mathrm{d}_2 \\ \{1-\sup \mathrm{d}_2, \text{if } \mathrm{d}_1 \ge \mathrm{d}_2 \end{cases}\\
&= \begin{cases} \{1-\mathrm{d}_1, \text{if } \mathrm{d}_1 \ge \mathrm{d}_2 \\ \{1-\mathrm{d}_2, \text{if } \mathrm{d}_1 \ge \mathrm{d}_2 \end{cases} \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\min((1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_1}(\mathrm{w}), 1-\mathrm{suf}\ \mathrm{G}_{\mathrm{D}_2}(\mathrm{w}))\\
&= \min(1-\mathrm{d}_1, 1-\mathrm{d}_2)\\
&= \begin{cases} 1-\mathrm{d}_1, \text{if } \mathrm{d}_1 \ge \mathrm{d}_2 \\ 1-\mathrm{d}_2, \text{if} \mathrm{d}_1 \le \mathrm{d}_2 \end{cases} \qquad (12)
\end{aligned}
$$

Therefore from (11) and (12), we have

$$1-\sup(\max(suf\ G_{D_1}(w), suf\ G_{D_2}(w)) = \min(1-suf\ G_{D_1}(w), 1-suf\ G_{D_2}(w)) \qquad (13)$$

Now

$$1-\inf \max(\inf U_{D_1}(w), \inf U_{D_2}(w))$$
$$=1-\inf(\max(c_1, c_2))$$
$$=\begin{cases}1-\inf c_1, \text{if } c_1 \geq c_2\\ 1-\inf c_2, \text{if } c_1 \leq c_2\end{cases}$$
$$=\begin{cases}1-c_1, \text{if } c_1 \geq c_2\\ 1-c_2, \text{if } c_1 \leq c_2\end{cases} \quad (14)$$

$$\min((1-\inf G_{D_1}(w), 1-\inf G_{D_2}(w))$$
$$=\min((1-c_1, 1-c_2)$$
$$=\begin{cases}1-c_1, \text{if } c_1 \geq c_2\\ 1-c_2, \text{if } c_1 \leq c_2\end{cases} \quad (15)$$

Therefore, from (14) and (15), we have

$$1-\inf \max(\inf U_{D_1}(w), \inf U_{D_2}(w))] = \min((1-\inf G_{D_1}(w), 1-\inf G_{D_2}(w)) \quad (16)$$

Therefore, from (9), (10,), (13) and (16),

$(D_1 \cap D_2)' = D_1' \cup D_2'$.

Theorem 3.9. For any two IPNS D_1, D_2,

$$D_1 \subseteq D_2 \Leftrightarrow D_2' \subseteq D_1'.$$

Proof.

$$D_1 \subseteq D_2 \Leftrightarrow$$
$$\inf T_{D_1}(w) \leq \inf T_{D_2}(w), \text{suf } T_{D_1}(w) \leq \text{suf } T_{D_2}(w),$$
$$\inf C_{D_1}(w) \leq \inf C_{D_2}(w), \text{suf } C_{D_1}(w) \leq \text{suf } C_{D_2}(w),$$
$$\inf G_{D_1}(w) \geq \inf G_{D_2}(w), \text{suf } G_{D_1}(w) \geq \text{suf } G_{D_2}(w),$$
$$\inf U_{D_1}(w) \geq \inf U_{D_2}(w), \text{suf } U_{D_1}(w) \geq \text{suf } U_{D_2}(w),$$
$$\inf F_{D_1}(w) \geq \inf F_{D_2}(w), \text{suf } F_{D_1}(w) \geq \text{suf } F_{D_2}(w),$$
$$\Leftrightarrow$$
$$\inf F_{D_2}(w) \leq \inf F_{D_1}(w), \text{suf } F_{D_2}(w) \leq \text{suf } F_{D_1}(w),$$
$$\inf U_{D_2}(w) \leq \inf U_{D_2}(w), \text{suf } U_{D_2}(w) \leq \text{suf } U_{D_1}(w),$$
$$1-\sup G_{D_2}(w) \geq 1-\sup\text{in } G_{D_1}(w), 1-\inf G_{D_2}(w) \geq 1-\text{suf } G_{D_1}(w),$$
$$\inf T_{D_2} \geq \inf T_{D_1}, \ \text{suf } T_{D_2}(w) \geq \text{suf } T_{D_1}(w)$$
$$\Leftrightarrow$$
$$D_2' \subseteq D_1'$$

Note: We establish the following properties of IPNSs:

1. Commutativity
2. Associativity
3. Idempotency
4. Absorption
5. De Morgan's laws
6. Involution

4. Conclusion

In this paper, we develop the notion of IPNS by combing the concept of PNS and INS. We define the notion of inclusion, complement, intersection, union of IPNSs. We prove some of the properties of IPNSs, namely, commutativity, associativity, idempotency, absorption, De Morgan's laws and involution. In the future, we shall develop the logic system based on the truth-value based IPNSs and utilize the theory to deal with practical applications in the areas such as information fusion, bioinformatics, military intelligence, web intelligence, etc.

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