A New Approximation of Prime Counting Function Based on Modified Logarithmic Integral

Marko V. Jankovic

Institute of Electrical Engineering "Nikola Tesla", Belgrade, Serbia,

Abstract In this paper, a novel approximation of the prime counting function, based on modified Eulerian logarithmic integral, is going to be presented. Proposed approximation reduces the approximation error without increase of computational complexity when it is compared to approximation based on Eulerian logarithmic integral. Experimental results were used to support the claim.

1 Introduction

In this paper, a novel approximation method for twin counting function is going to be analyzed. It is known that Eulerian logarithmic integral Li(n) [1] represents a good approximation of the number of primes $\pi(n)$ smaller than some natural number n. *Li* function is defined by the following equation

$$Li(n) = \int_2^n \left(\frac{dx}{\ln(x)} \right).$$

However, it is well known that the error that is made by such approximation is significant for small numbers n. In order to reduce that error we are going to define a modified logarithmic integral (*MoLi*) which is given by the following equation

$$\pi(n) \sim MoLi(n) = \int_2^n \left(\frac{dx}{\ln(x + \sqrt{n})} \right).$$

In order to assess the quality of the proposed approximation, a number of experiments were conducted for numbers n smaller than one million.

2 Experimental results

In all experiments integration step was 0.01 and applied integration method was trapezoidal method. In all experiments result of approximation was rounded to the nearest integer. In Table 1, the results of experiments for $n = 10^k$, $k \in \{1, 2, 3, 4, 5, 6\}$, were presented.

Table 1. Comparison of the proposed method with some known methods

	n = 10	n = 100	n = 1000	n = 10000	n = 100000	n = 1000000
Li(n) - π(n)	1(2)	4(5)	8(10)	16(17)	36(38)	128(130)
Riemann(n) - π(n)	1	1	0	-2	-5	29
MoLi(n) - π(n)	0	0	0	-3	-6	24

From the Table 1 we can see that columns that represent *Li* function contain two values. Value in the bracket is value taken from the literature [2, 3], while the value in front of the bracket represents result obtained by the experiment (having in mind the value of integration step and method of integration, the obtained value is slightly lower, as it can be expected). From results it could be seen that proposed method produces very similar quality of approximation to the Riemann prime counting function [2] while, at the same time, it is less computationally demanding than Riemann prime counting function. From Table 1 is clear that proposed approximation outperforms the approximation based on *Li* function.

In order to assess the quality of the proposed approximation for some other values of n, results of another experiment are presented in Figures 1 and 2. Figure 3 and 4 present graphical interpretation of results presented in Table 1.

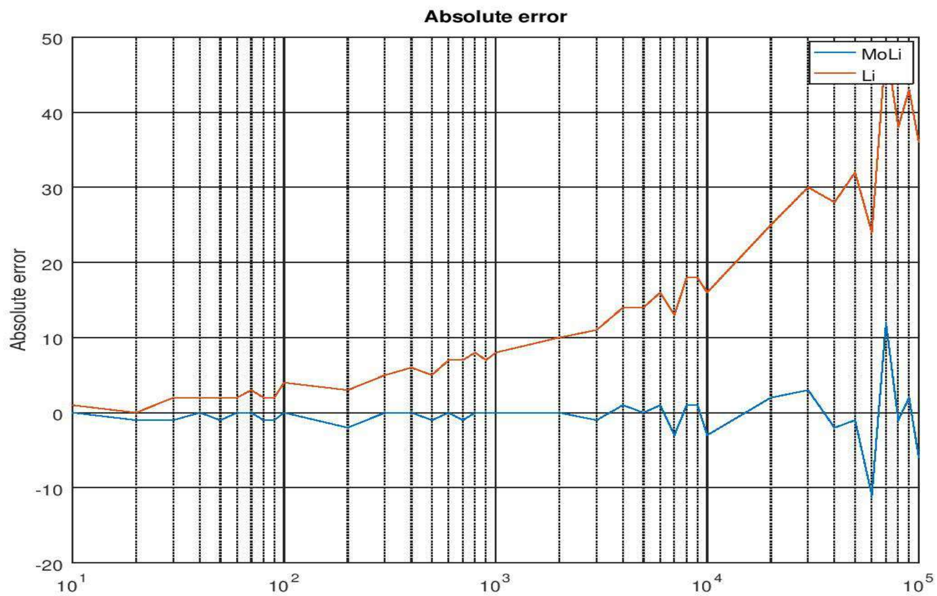

Figure 1. Absolute error of approximation (input points are defined as $s*10^k$, where s {1, 2, ...9} and k {1, 2, 3, 4, 5})

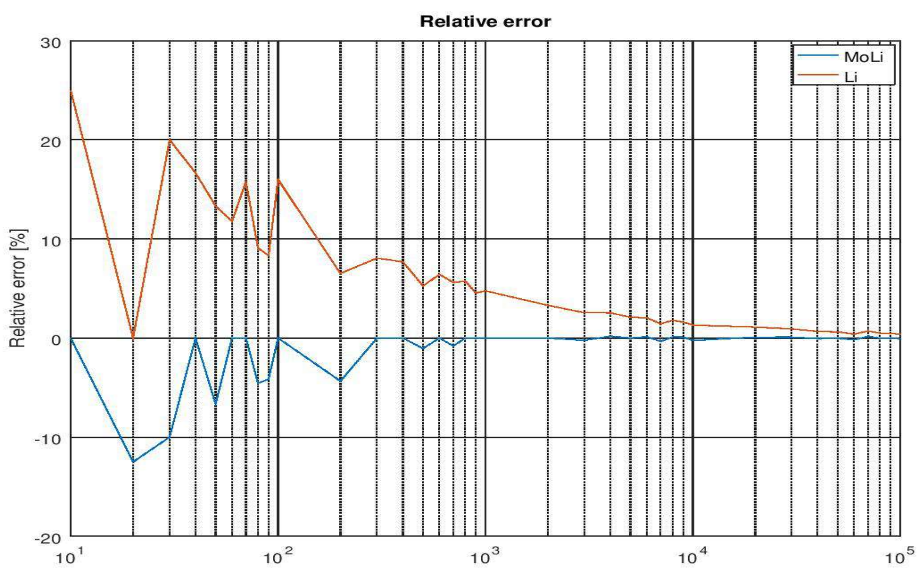

Figure 2. Relative error of approximation (input points are defined as $s*10^k$, where $s \in \{1, 2, ...9\}$ and $k \in \{1, 2, 3, 4, 5\}$)

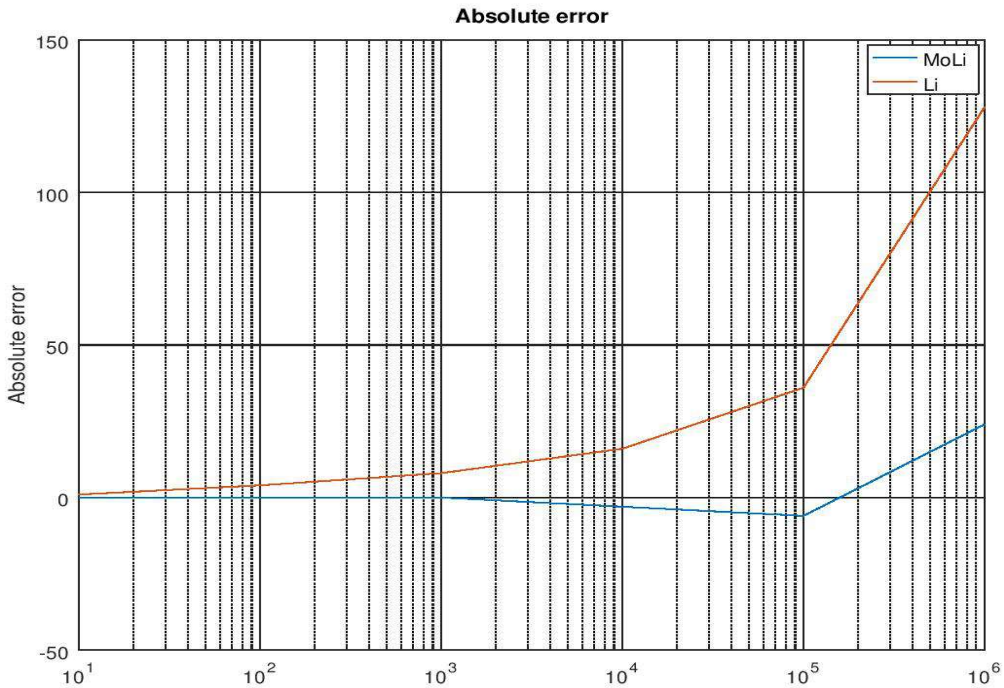

Figure 3. Absolute error of approximation (input points are defined as 10^k, where $k \in \{1, 2, 3, 4, 5, 6\}$)

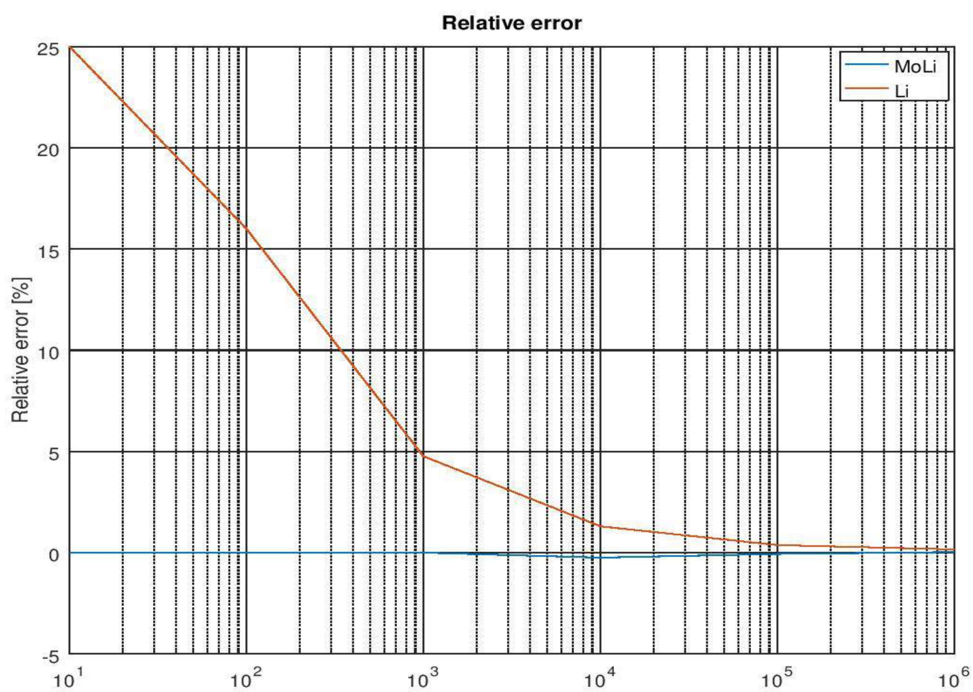

Figure 4. Relative error of approximation (input points are defined as 10^k, where $k \in \{1, 2, 3, 4, 5\}$)

From figures it could be concluded that proposed approximation outperforms the approximation of prime counting function based on *Li* function, in the range that is analyzed. Based on rough estimations (using big integration step), proposed approximation based on *MoLi* function outperforms approximation based on *Li* function in much broader range (at least till 10^{24}).

References

[1] Derbyshire, J. (2004) Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics. Ney York: Penguin.

[2] Weisstein, Eric W. “Prime Counting Function”, MathWorld.

[3] Prime Counting Function, Wikipedia.