A New Approximation of Prime Counting Function Based on Modified Logarithmic Integral

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Abstract In this paper, a novel approximation of the prime counting function, based on modified Eulerian logarithmic integral, is going to be presented. Proposed approximation reduces the approximation error without increase of computational complexity when it is compared to approximation based on Eulerian logarithmic integral. Experimental results were used to support the claim. Combining proposed method with |Riemannian approximation of prime counting function it is possible to design the new approximation function that outperforms Riemannian approximation for all values that were analyzed.

1 Introduction

In this paper, a novel approximation method for twin counting function is going to be analyzed. It is known that Eulerian logarithmic integral *Li*(*n*) [1] represents a good approximation of the number of primes $\pi(n)$ smaller than some natural number *n*. *Li* function is defined by the following equation

$$Li(n)=\int_{2}^{n}\left(\frac{dx}{\ln(x)}\right).$$

However, it is well known that the error that is made by such approximation is significant for small numbers *n*. In order to reduce that error we are going to define a modified logarithmic integral (*MoLi*) which is given by the following equation

$$\pi(n) \sim MoLi(n)=\int_{2}^{n}\left(\frac{dx}{\ln(x+\sqrt{n})}\right).$$

In order to assess the quality of the proposed approximation, a number of experiments were conducted for numbers n smaller than one million.

2 Experimental results

In all experiments integration step was 0.01 and applied integration method was trapezoidal method. In all experiments result of approximation was rounded to the nearest integer. In Table 1, the results of experiments for $n = 10^k$, $k \in \{1, 2, 3, 4, 5, 6\}$, were presented.

Table 1. Comparison of the proposed method with some known methods

	$n = 10$	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$	$n = 1000000$
Li(n) - π(n)	1(2)	4(5)	8(10)	16(17)	36(38)	128(130)
Riemann(n) - π(n)	-1	-1	0	-2	5	-29
MoLi(n) - π(n)	0	0	0	-3	-6	24

From the Table 1 we can see that columns that represent *Li* function contain two values. Value in the bracket is value taken from the literature [2, 3], while the value in front of the bracket represents result obtained by the experiment (having in mind the value of integration step and method of integration, the obtained value is slightly lower, as it can be expected). From results it could be seen that proposed method produces very similar quality of approximation to the Riemann prime counting function [2] while, at the same time, it is less computationally demanding than Riemann prime counting function. From Table 1 is clear that proposed approximation outperforms the approximation based on *Li* function.

In order to assess the quality of the proposed approximation for some other values of n, results of another experiment are presented in Figures 1 and 2. Figure 3 and 4 present graphical interpretation of results presented in Table 1.

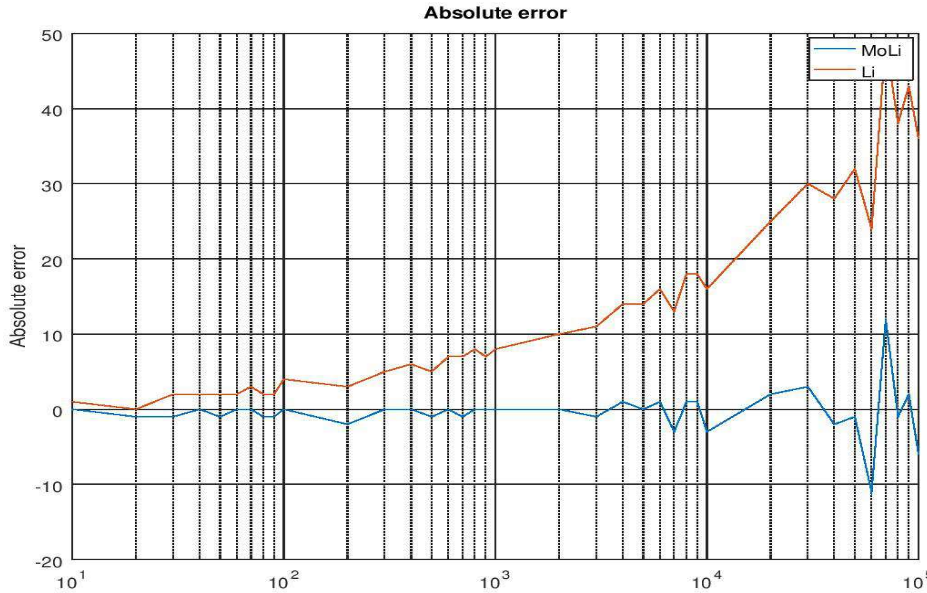

Figure 1. Absolute error of approximation (input points are defined as s*10^k, where s {1, 2, ...9} and k {1, 2, 3, 4, 5})

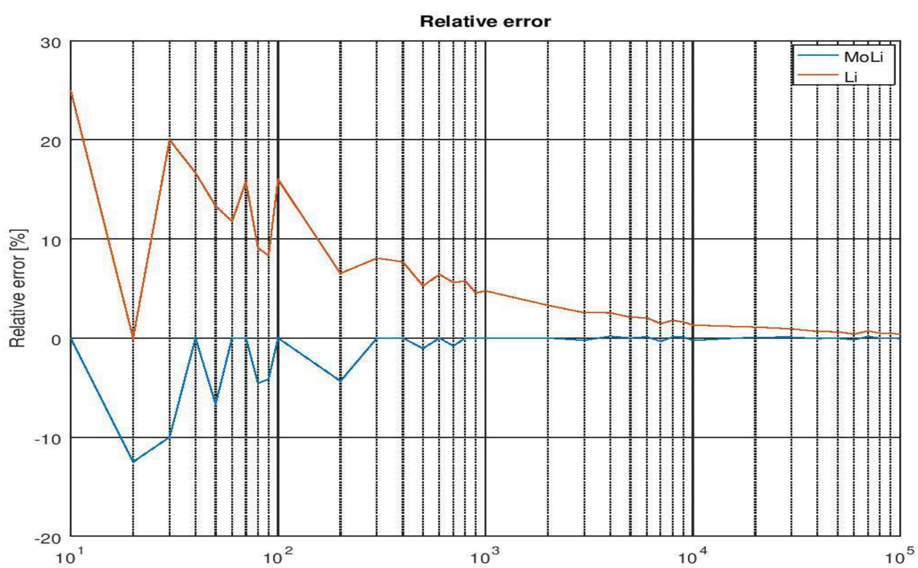

Figure 2. Relative error of approximation (input points are defined as s*10^k, where $s \in \{1, 2, ...9\}$ and $k \in \{1, 2, 3, 4, 5\}$)

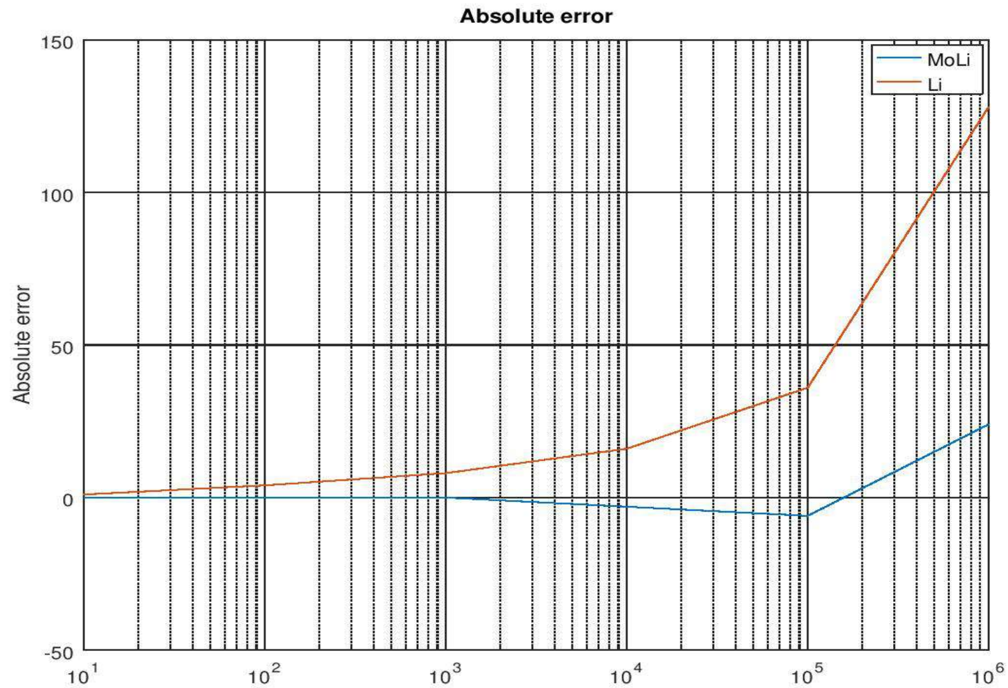

Figure 3. Absolute error of approximation (input points are defined as 10^k, where $k \in \{1, 2, 3, 4, 5, 6\}$)

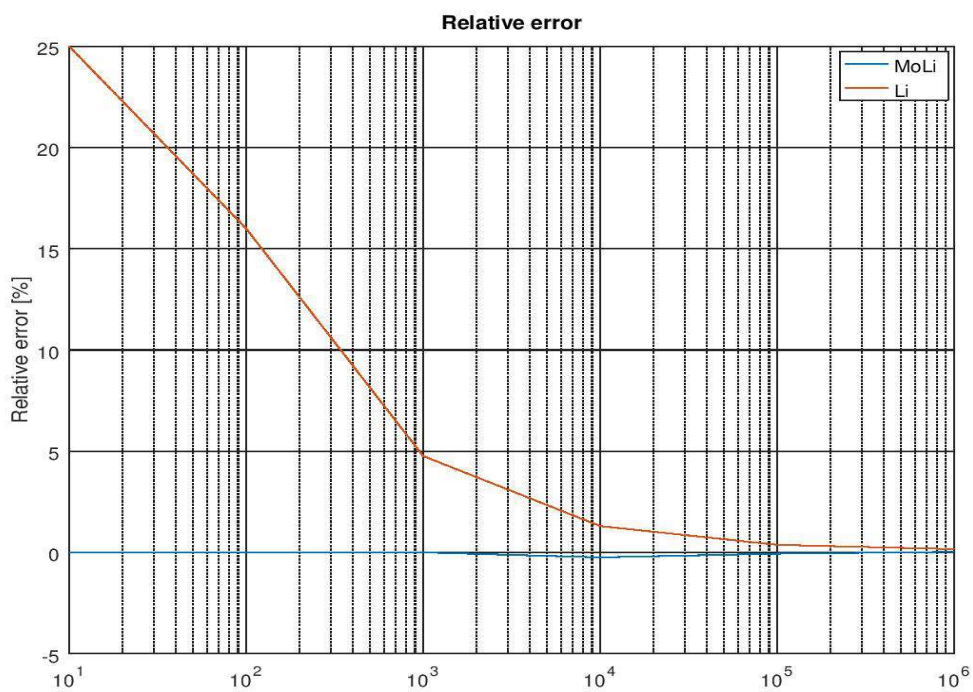

Figure 4. Relative error of approximation (input points are defined as 10^k, where $k \in \{1, 2, 3, 4, 5\}$)

From figures it could be concluded that proposed approximation outperforms the approximation of prime counting function based on *Li* function, in the range that is analyzed. Based on rough estimations (using big integration step), proposed approximation based on *MoLi* function outperforms approximation based on *Li* function in much broader range (at least till 10^{24}).

The next Table 2, gives comparison of error of *Li*(*n*) function, Riemann *R*(*n*), *MoLi*(*n*) and newly created function *RMoLi*(*n*) =(2**R*(*n*)+*MoLi*(*n*))/3 in much broader range. The results for function *MoLi*(*n*) were obtained by using CASIO computation service [4] for *li* function [1].

Table 2. Comparison of several methods for prime counting function approximation

	Li(n) - π(n)	***R(n) - π(n)***	***MoLi(n) - π(n)***	***RMoLi(n)-π(n)***
$n = 10^1$	2	-1	0	-1
$n = 10^2$	5	-1	0	-1
$n = 10^3$	10	0	0	0
$n = 10^4$	17	-2	-3	-2
$n = 10^5$	38	5	-6	1
$n = 10^6$	130	-29	24	-11
$n = 10^7$	339	-88	72	-35
$n = 10^8$	754	-97	51	-48
$n = 10^9$	1701	79	-207	-16
$n = 10^{10}$	3104	1828	-2184	491
$n = 10^{11}$	11588	2318	-3299	446
$n = 10^{12}$	38263	1476	-4174	407
$n = 10^{13}$	108971	5773	-13218	-557
$n = 10^{14}$	314890	19200	-39818	-473
$n = 10^{15}$	1052619	-73218	15868	-43523
$n = 10^{16}$	3214632	-327052	166763	-162447
$n = 10^{17}$	7956589	598255	-1048475	49345
$n = 10^{18}$	21949555	3501366	-4772209	743508
$n = 10^{19}$	99877775	-23884333	20279712	-9162985
$n = 10^{20}$	222744644	4891825	-15163730	-1793360
$n = 10^{21}$	597394254	86432204	-115833916	19010164
$n = 10^{22}$	1932355208	127132665	-211645365	14206655

From Table 2 it can be noticed that proposed function *MoLi*(n) outperforms function *Li*(n) in the whole range that was analyzed. Also, it can be noticed that newly proposed *RMoLi*(n) function is equal to (up to $n = 10^4$) or outperforms Riemannian function for all values that were analyzed. That could rise some interesting questions related to Riemann hypothesis, but this will not be further elaborated here.

References

[1] Derbyshire, J. (2004) Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics. Ney York: Penguin.

[2] Weisstein, E. W. "Prime Counting Function", From MathWorld – A Wolfram Web Resource. https://mathworld.wolfram.com/PrimeCountingFunction.html

[3] Prime Counting Function, Wikipedia.

[4] Web Resource keisan.casio.com – logarithmic integral li(x) Calculator