On the prime distribution

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In this paper, the estimation formula of the number of primes in a given interval is obtained by using the prime distribution property. For any prime pairs $p > 5$ and $q > 5$, construct a disjoint infinite set sequence $A_1, A_2, \ldots, A_i \ldots$, such that the number of prime pairs (p_i and q_i, $p_i - q_i = p - q$) in A_i increases gradually, where $i > 0$. So twin prime conjecture is true. We also prove that for any even integer $m > 2700$, there exist more than 10 prime pairs (p, q), such that $p + q = m$. Thus Goldbach conjecture is true.

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1 Introduction

Like Goldbach conjecture, twin prime conjecture is also one of the famous unsolved problems in number theory. In 1973, Chen [1] proved that for any even number h, there are infinite prime numbers p, so that the number of prime factors of $p + h$ does not exceed 2. In 2008, Green and Tao [2] proved the existence of arbitrarily long arithmetic progressions in the primes. In 2014, Zhang [3] proved that bounded gaps between primes are all less than 70 million.

In this paper, the estimation formula of the number of primes in a given interval is given by using the prime distribution property. For any prime pairs $p > 5$ and $q > 5$, construct a disjoint infinite set sequence $A_1, A_2, \ldots, A_i \ldots$, such that the number of prime pairs (p_i and q_i, $p_i - q_i = p - q$) in A_i increases gradually, where $i > 0$. So the original conjecture is true. We also prove that for any even integer $m > 2700$, there exist more than 10 prime pairs (p, q), such that $p + q = m$. Thus Goldbach conjecture is true.

2 Three lemmas

In this section, three lemmas are proved.

Lemma 1. Given two coprime natural numbers p and q. If the remainder of natural numbers in the set A with respect to p is evenly distributed, then the remainder of

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natural numbers in the set $\{aq+c|a \in A\}$ is still evenly distributed, where $c \geq 0$ is an integer.

Proof. Prove by a contradiction.

Without loss of generality, suppose that $a_i \equiv i \mod p, a_i \in A,\ 0 \leq i < p$, but the remainder of natural numbers in the set $\{a_i q + c|a_i \in A, 0 \leq i < p\}$ is not evenly distributed. Let's say that $a_i q + c$ and $a_j q + c$ have the same remainder about p,

$\Longrightarrow a_i q + c - (a_j q + c) = kp, k$ is an integer. $\Longrightarrow (a_i - a_j)q = kp$.

As p and q are coprime natural numbers, $\Longrightarrow a_i \equiv a_j \mod p$. This is a contradiction. □

Based on Lemma 1, we prove the following Lemma 2.

Lemma 2. Given natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ pairwise prime, and the remainder of natural numbers in the set A with respect to $\alpha_i, 1 \leq i \leq n$, are evenly distributed, where $|A| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_n$. We conclude that

$$|\{a|a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq n\}| = (\alpha_1 - 1) \times (\alpha_2 - 1) \times \ldots \times (\alpha_n - 1)$$

Proof. If $n = 1$, then $|A| = \alpha_1$. Obviously, $|\{a|a \in A \text{ and } a \not\equiv 0 \mod \alpha_1\}| = \alpha_1 - 1$.

If $n = 2$, then $|A| = \alpha_1 \times \alpha_2$.

Without loss of generality, suppose that

$$A = \begin{pmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 \\ \alpha_1 + 1 & \alpha_1 + 2 & \cdots & 2\alpha_1 - 1 & 2\alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (\alpha_2 - 2)\alpha_1 + 1 & (\alpha_2 - 2)\alpha_1 + 2 & \cdots & (\alpha_2 - 1)\alpha_1 - 1 & (\alpha_2 - 1)\alpha_1 \\ (\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & \alpha_2\alpha_1 - 1 & \alpha_2\alpha_1 \end{pmatrix}$$

$\Longrightarrow$

$$\{a|a \in A \text{ and } a \not\equiv 0 \mod \alpha_1\} = \begin{pmatrix} 1 & 2 & \cdots & \alpha_1 - 1 \\ \alpha_1 + 1 & \alpha_1 + 2 & \cdots & 2\alpha_1 - 1 \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_2 - 2)\alpha_1 + 1 & (\alpha_2 - 2)\alpha_1 + 2 & \cdots & (\alpha_2 - 1)\alpha_1 - 1 \\ (\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & \alpha_2\alpha_1 - 1 \end{pmatrix}$$

As α_1 and α_2 are coprime natural numbers, and by Lemma 1, $\{i\alpha_1 + j|0 \leq i < \alpha_2\} \equiv \{0, 1, 2, \cdots, \alpha_2 - 1\} \mod \alpha_2$, where $0 < j < \alpha_1$.

$\Longrightarrow |\{a|a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq 2\}| = (\alpha_1 - 1) \times (\alpha_2 - 1)$.

If $n = k + 1$, then $|A| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k \times \alpha_{k+1}$.

As $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ are pairwise primes, and by Lemma 1, the remainder of natural numbers in the set $\{i \times \alpha_{k+1}|0 < i \leq \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k\}$ with respect to $\alpha_i, 1 \leq i \leq k$, is evenly distributed.

$\Longrightarrow$ the remainder of natural numbers in the set $\{a|a \in A \text{ and } a \not\equiv 0 \mod \alpha_{k+1}\}$ with respect to $\alpha_i, 1 \leq i \leq k$, is evenly distributed and $|\{a|a \in A \text{ and } a \not\equiv 0 \mod \alpha_{k+1}\}| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k \times (\alpha_{k+1} - 1)$.

Similarly, as $\alpha_1, \alpha_2, \ldots, \alpha_k$ are pairwise primes, and by Lemma 1, the remainder of natural numbers in the set $\{i \times \alpha_k | 0 < i \leq \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{k-1} \times (\alpha_{k+1} - 1)\}$ with respect to $\alpha_i, 1 \leq i \leq k-1$, is evenly distributed.

$\Longrightarrow$ the remainder of natural numbers in the set $\{a | a \in A$ and $a \not\equiv 0 \mod \alpha_x, x = k, k+1\}$ with respect to $\alpha_i, 1 \leq i \leq k-1$, is evenly distributed and $|\{a | a \in A$ and $a \not\equiv 0 \mod \alpha_x, x = k, k+1\}| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{k-1} \times (\alpha_k - 1) \times (\alpha_{k+1} - 1)$.

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Finally, we have

$$|\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq k+1\}| = (\alpha_1 - 1) \times (\alpha_2 - 1) \times \ldots \times (\alpha_{k+1} - 1)$$

□

The following examples is helpful to understand Lemma 2.

It is known that the natural numbers $2, 3, 5$ are mutually prime. Let $A = \{1, 2, 3, \ldots, 29, 30\}$. Obviously the remainder of the natural numbers in A about $2, 3, 5$ is evenly distributed. According to Lemma 2, after removing all natural numbers with 0 remainder about $2, 3, 5$, the number of natural numbers in A becomes $(2-1) \times (3-1) \times (5-1) = 8$. Namely, $\{1, 7, 11, 13, 17, 19, 23, 29\}$.

Let $A = \{7 * i + 1 | 1 \leq i \leq 30\}$. According to Lemma 1, the remainder of natural numbers about $2, 3, 5$ in A is still evenly distributed. Then, according to Lemma 2, after removing all natural numbers with 0 remainder about $2, 3, 5$, the number of natural numbers in A becomes $(2-1) \times (3-1) \times (5-1) = 8$.

The following proves Lemma 3

Lemma 3.

$$(1 - \frac{d}{30 \times x + c}) < (1 - \frac{d}{30 \times (2x + e + 1) + c})(1 - \frac{d}{30 \times (2x + e + 2) + c})$$

$$x \geq 0, e \geq 0, 0 < c < 32, 0 < d < 30x.$$

Proof. First prove: $\frac{d}{30 \times (2x+1) + c} + \frac{d}{30 \times (2x+2) + c} < \frac{d}{30 \times x + c}$

$$\Longleftrightarrow (b + c)(2b + c + 30 + 2b + c + 60) < (2b + c + 30)(2b + c + 60), b = 30x$$

$$\Longleftrightarrow (b + c)(4b + 2c + 90) < (2b + c + 30)(2b + c + 60)$$

$$\Longleftrightarrow 4b^2 + 6bc + 90b + 90c + 2c^2 < 4b^2 + 4bc + 180b + c^2 + 90c + 1800$$

From both sides of the inequality remove $4b^2 + 4bc + 90b + 90c + c^2$

The original inequality $\Longleftrightarrow c^2 + 2bc < 90b + 1800 \Longleftrightarrow 0 < (90 - 2c)b + 1800 - c^2$.

As $c < 32$, thus

$$\frac{d}{30 \times (2x + 1) + c} + \frac{d}{30 \times (2x + 2) + c} < \frac{d}{30 \times x + c}$$

$$\Longrightarrow \frac{d}{30 \times (2x + e + 1) + c} + \frac{d}{30 \times (2x + e + 2) + c} < \frac{d}{30 \times x + c}$$

$$\Longrightarrow (1 - \frac{d}{30 \times (2x+e+1)+c})(1 - \frac{d}{30 \times (2x+e+2)+c})$$
$$> 1 - (\frac{d}{30 \times (2x+e+1)+c} + \frac{d}{30 \times (2x+e+2)+c}) > 1 - \frac{d}{30 \times x + c}$$

□

3 Possible form of prime numbers $\{11 + 30 * x | x \geq 0\}$ and $\{13 + 30 * x | x \geq 0\}$

Introduce some basic properties of prime numbers. All prime numbers are in odd numbers with single digits of 1, 3, 7 and 9 (except 2 and 5). Now let's take a look at the prime number whose single digit is 1. It is easy to find that there are only two possible forms:

$\{11 + 30 * x | x \geq 0\}$ and $\{31 + 30 * x | x \geq 0\}$

For prime number whose single digit is 3, there are only two possible forms:

$\{13 + 30 * x | x \geq 0\}$ and $\{23 + 30 * x | x \geq 0\}$

For prime number whose single digit is 7, there are only two possible forms:

$\{7 + 30 * x | x \geq 0\}$ and $\{17 + 30 * x | x \geq 0\}$

For prime number whose single digit is 9, there are only two possible forms:

$\{19 + 30 * x | x \geq 0\}$ and $\{29 + 30 * x | x \geq 0\}$

Among possible form of prime numbers $\{11 + 30 * x | x \geq 0\}$, if any $11 + 30 * x$ is not a prime number, then there are only four possible decomposition forms:

$H_1 = [7 + 30a][23 + 30b], a \geq 0, b \geq 0$; $H_2 = [13 + 30a][17 + 30b], a \geq 0, b \geq 0$;

$H_3 = [11 + 30a][31 + 30b], a \geq 0, b \geq 0$; $H_4 = [19 + 30a][29 + 30b], a \geq 0, b \geq 0$.

As $(19 + 30a)^2 = 30c + 1$, thus $(19 + 30a)^2$ is not a possible decomposition form of $11 + 30 * x$.

In Table 1, the number in the cell indicates that the corresponding $11 + 30 * x$ is not a prime number, and the number in the cell is a factor.

For example, $11 + 30 * 33 = 1001 = 11 \times 7 \times 13$;

$11 + 30 * 47 = 1421 = 29 \times 49 = 7 \times 203$.

Based on Lemma 2, the number of primes in 30 consecutive natural numbers $\{6, 7, \dots, 34, 35\}$ is $(2-1)(3-1)(5-1) = 30(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5}) = 8$. As $30 = 2 \times 3 \times 5$, the formula is accurate. These specific prime numbers are: $7, 11, 13, 17, 19, 23, 29, 31$.

We are concerned about the following two issues here.

1. Number counting formula of primes $(2-1)(3-1)(5-1)$ is valid for 30 consecutive natural numbers less than 49. Because 49 is not a multiple of $2, 3, 5$, but $49 = 7 \times 7$. For example, the number of primes among 30 consecutive natural numbers $\{20, 21, \dots, 48, 49\}$ is 7. These specific prime numbers are: $23, 29, 31, 37, 41, 43, 47$. That is, the actual number of primes is less than $(2-1)(3-1)(5-1)$.

2. Note that number counting formula of primes

$$30(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7}) \approx 6.86$$

Table 1: The possible form of prime numbers $\{11+30*x|0 \le x < 210\}$

11	41	71	101	131	161	191	221	251	281	311	341	371	401	431
					7*23		13*17				11*31	7*53		
461	491	521	551	581	611	641	671	701	731	761	791	821	851	881
			19*29	7*83	13*47		11*61		17*43		7*113		23*37	
911	941	971	1001	1031	1061	1091	1121	1151	1181	1211	1241	1271	1301	1331
			7				19			7	17	31		11
1361	1391	1421	1451	1481	1511	1541	1571	1601	1631	1661	1691	1721	1751	1781
	13	7				23			7	11	19		17	13
1811	1841	1871	1901	1931	1961	1991	2021	2051	2081	2111	2141	2171	2201	2231
	7				37	11	43	7				13	31	23
2261	2291	2321	2351	2381	2411	2441	2471	2501	2531	2561	2591	2621	2651	2681
7	29	11					7	41		13			11	7
2711	2741	2771	2801	2831	2861	2891	2921	2951	2981	3011	3041	3071	3101	3131
		17		19		7	23	13	11			37	7	31
3161	3191	3221	3251	3281	3311	3341	3371	3401	3431	3461	3491	3521	3551	3581
29				17	7	13		19	47			7	53	
3611	3641	3671	3701	3731	3761	3791	3821	3851	3881	3911	3941	3971	4001	4031
23	11			7		17					7	11		29
4061	4091	4121	4151	4181	4211	4241	4271	4301	4331	4361	4391	4421	4451	4481
31		13	7	37				11	61	7				
4511	4541	4571	4601	4631	4661	4691	4721	4751	4781	4811	4841	4871	4901	4931
13	19	7	43	11	59				7	17	47		13	
4961	4991	5021	5051	5081	5111	5141	5171	5201	5231	5261	5291	5321	5351	5381
11	7				19	53		7			11	17		
5411	5441	5471	5501	5531	5561	5591	5621	5651	5681	5711	5741	5771	5801	5831
7					67		7		13			29		7
5861	5891	5921	5951	5981	6011	6041	6071	6101	6131	6161	6191	6221	6251	6281
	43	31	11			7	13			61	41		7	11

is less than the actual number 8 of primes in 30 consecutive natural numbers $\{6, 7, \ldots, 34, 35\}$, and also less than the actual number 7 of primes in 30 consecutive natural numbers $\{20, 21, \ldots, 48, 49\}$.

The number counting formula of primes $30(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7})$ is valid for 30 consecutive natural numbers less than 11^2. This paper mainly considers the estimation formula of primes lower than the actual number of primes.

Based on Lemma 2, we consider a formula for estimating the number of primes. In Table 1, consider first 30 natural numbers $\{11+30*x|0 \le x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of primes:

$$30(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})(1-\frac{1}{17})(1-\frac{1}{19})(1-\frac{1}{23})(1-\frac{1}{29})(1-\frac{1}{31}) \approx 17.20$$

The actual number of primes in $\{11+30*x|0 \le x < 30\}$ is 19.

If $a \in \{11+30*x|0 \le x < 30\}$ and a is a composite number, then a must has one factor in $\{7, 11, 13, 17, 19, 23\}$, so the above estimation formula is lower than the actual number of primes.

As $(a+30)(b+30) = 30(30+a+b)+ab$, the above estimation formula is valid for $11+30*x < 53 \times 37 = 1961 = 11+30 \times 65$.

Among possible form of prime numbers $\{13+30*x|x \ge 0\}$, if any $13+30*x$ is not a prime number, then there are only four possible decomposition forms:

$H_5 = [7+30a][19+30b], a \ge 0, b \ge 0$; $H_6 = [13+30a][31+30b], a \ge 0, b \ge 0$;

$H_7 = [11+30a][23+30b], a \ge 0, b \ge 0$; $H_8 = [17+30a][29+30b], a \ge 0, b \ge 0$.

In Table 2, consider first 30 natural numbers $\{11+30*x|0 \le x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of primes:

Table 2: The possible form of prime numbers $\{13 + 30 * x | 0 \leq x < 210\}$

13	43	73	103	133	163	193	223	253	283	313	343	373	403	433
				7*19				11*23			7*49		13*31	
463	493	523	553	583	613	643	673	703	733	763	793	823	853	883
	17		7	11				19		7	13			
913	943	973	1003	1033	1063	1093	1123	1153	1183	1213	1243	1273	1303	1333
11	23	7	17						7		11	19		31
1363	1393	1423	1453	1483	1513	1543	1573	1603	1633	1663	1693	1723	1753	1783
29	7				17		11	7	23					
1813	1843	1873	1903	1933	1963	1993	2023	2053	2083	2113	2143	2173	2203	2233
7	19		11		13		7					41		7
2263	2293	2323	2353	2383	2413	2443	2473	2503	2533	2563	2593	2623	2653	2683
31		23	13		19	7			17	11		43	7	
2713	2743	2773	2803	2833	2863	2893	2923	2953	2983	3013	3043	3073	3103	3133
	13	47			7	11	37		19	23	17	7	29	13
3163	3193	3223	3253	3283	3313	3343	3373	3403	3433	3463	3493	3523	3553	3583
	31	11		7				41			7	13	11	
3613	3643	3673	3703	3733	3763	3793	3823	3853	3883	3913	3943	3973	4003	4033
			7		53				11	7		29		37
4063	4093	4123	4153	4183	4213	4243	4273	4303	4333	4363	4393	4423	4453	4483
17		7		47	11			13	7		23		61	
4513	4543	4573	4603	4633	4663	4693	4723	4753	4783	4813	4843	4873	4903	4933
	7	17		41		13		7			29	11		
4963	4993	5023	5053	5083	5113	5143	5173	5203	5233	5263	5293	5323	5353	5383
7			31	13		37	7	11		19	67		53	7
5413	5443	5473	5503	5533	5563	5593	5623	5653	5683	5713	5743	5773	5803	5833
		13		11		7				29		23	7	19
5863	5893	5923	5953	5983	6013	6043	6073	6103	6133	6163	6193	6223	6253	6283
11	71			31	7			17			11	7	13	61

$$30(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})(1-\frac{1}{17})(1-\frac{1}{19})(1-\frac{1}{23})(1-\frac{1}{29})(1-\frac{1}{31}) \approx 17.20$$

The actual number of primes in $\{13 + 30 * x | 0 \leq x < 30\}$ is 20.

If $a \in \{13 + 30 * x | 0 \leq x < 30\}$ and a is a composite number, then a must has one factor in $\{7, 11, 13, 17, 19\}$, so the above estimation formula is lower than the actual number of primes. The above estimation formula is valid for $13 + 30 * x < 49 \times 37 = 1813 = 13 + 30 \times 60$.

4 Twin prime conjecture

We further consider possible form of twin prime numbers $\{(11+30*x, 13+30*x) | x \geq 0\}$. From Table 1 and Table 2, (6131, 6133) is twin prime numbers.

Consider first 30 pairs of natural numbers $\{(13 + 30 * x, 13 + 30 * x) | 0 \leq x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of twin prime numbers $C(30)$:

$$30(1-\frac{2}{7})(1-\frac{2}{11})(1-\frac{2}{13})(1-\frac{2}{17})(1-\frac{2}{19})(1-\frac{2}{23})(1-\frac{2}{29})(1-\frac{2}{31}) \approx 9.31$$

The actual number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x) | 0 \leq x < 30\}$ is 13.

The above estimation formula can be understood in this way (take prime number 7 as an example): for every 7 consecutive cells, there must be one cell of $11 + 30 * x_1$ can be divided by 7, and another cell of $13 + 30 * x_2$ can be divided by 7, where $x_1 \neq x_2$. Therefore, one term in the above formula is $\frac{7-2}{7}$.

When $x = 19$, $11 + 30 * 19 = 581$ can be divided by 7, and $13 + 30 * 19 = 583$ can be divided by 11. Therefore, this cell is counted twice, so the estimation formula $C(30)$ is lower than the actual number of twin primes.

The estimation formula for the number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|30 \leq x < 90\}$ is $C(60)$:

$$60(1-\frac{2}{7})(1-\frac{2}{11})(1-\frac{2}{13})(1-\frac{2}{17})(1-\frac{2}{19})(1-\frac{2}{23})(1-\frac{2}{29})(1-\frac{2}{31})$$
$$(1-\frac{2}{37})(1-\frac{2}{67})(1-\frac{2}{41})(1-\frac{2}{71})(1-\frac{2}{43})(1-\frac{2}{73})(1-\frac{2}{47})(1-\frac{2}{77})(1-\frac{2}{49})(1-\frac{2}{79})(1-\frac{2}{53})(1-\frac{2}{83})(1-\frac{2}{59})(1-\frac{2}{89})(1-\frac{2}{61})(1-\frac{2}{91})$$
$$\approx 10.72$$

The actual number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|30 \leq x < 90\}$ is 15.

In the above formula, $77, 49$ may not appear in the formula because they are multiples of 7. For the completeness of the formula, $77, 49$ are still retained, which only make the valuation smaller.

Because $(a + 30)(b + 30) = 30(30 + a + b) + ab$, the effective range of the above estimation formula is: $13 + 30 * x < 109 \times 97 = 10573 = 13 + 30 \times 352$, where $352 > 90$.

The actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|90 \leq x < 210\}$ is 24. The estimation formula for the number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|90 \leq x < 210\}$ is $C(120)$ (only the part about $\{7 + 30 * x|x \geq 0\}$ is given here):

$$(1-\frac{2}{7})$$
$$(1-\frac{2}{37})(1-\frac{2}{67})$$
$$(1-\frac{2}{97})(1-\frac{2}{127})(1-\frac{2}{157})(1-\frac{2}{187})$$

The actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|210 \leq x < 450\}$ is 29. The estimation formula for the number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|210 \leq x < 450\}$ is $C(240)$(only the part about $\{7 + 30 * x|x \geq 0\}$ is given here):

$$(1-\frac{2}{7})$$
$$(1-\frac{2}{37})(1-\frac{2}{67})$$
$$(1-\frac{2}{97})(1-\frac{2}{127})(1-\frac{2}{157})(1-\frac{2}{187})$$
$$(1-\frac{2}{217})(1-\frac{2}{247})(1-\frac{2}{277})(1-\frac{2}{307})(1-\frac{2}{337})(1-\frac{2}{367})(1-\frac{2}{397})(1-\frac{2}{427})$$

It is easy to obtain the following (only the part about $\{7 + 30 * x|x \geq 0\}$ is given here):

$$\frac{C(60)}{C(30)} = 2(1-\frac{2}{37})(1-\frac{2}{67})$$

$$\frac{C(120)}{C(60)} = 2(1-\frac{2}{97})(1-\frac{2}{127})(1-\frac{2}{157})(1-\frac{2}{187})$$

$$\frac{C(240)}{C(120)} = 2(1-\frac{2}{217})(1-\frac{2}{247})(1-\frac{2}{277})(1-\frac{2}{307})(1-\frac{2}{337})(1-\frac{2}{367})(1-\frac{2}{397})(1-\frac{2}{427})$$

$$\cdots\cdots$$

By Lemma 3,

$$(1-\frac{2}{217})(1-\frac{2}{247}) > (1-\frac{2}{97}), (1-\frac{2}{277})(1-\frac{2}{307}) > (1-\frac{2}{127}), \cdots\cdots,$$

Thus

$$\cdots\cdots > \frac{C(240)}{C(120)} > \frac{C(120)}{C(60)} > \frac{C(60)}{C(30)}$$

The complete $\frac{C(60)}{C(30)}$ is as follows,

$$2(1-\frac{2}{37})(1-\frac{2}{67})(1-\frac{2}{41})(1-\frac{2}{71})(1-\frac{2}{43})(1-\frac{2}{73})(1-\frac{2}{47})(1-\frac{2}{77})(1-\frac{2}{49})(1-\frac{2}{79})(1-\frac{2}{53})(1-\frac{2}{83})(1-\frac{2}{59})(1-\frac{2}{89})(1-\frac{2}{61})(1-\frac{2}{91})$$

$$= 2 \times 0.575 > 1$$

Therefore

$$\cdots\cdots > C(240) > C(120) > C(60) > C(30) \approx 9.31$$

In fact, the actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|450 \leq x < 930\}$ is 71, and the actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|930 \leq x < 1890\}$ is $113, \ldots\ldots,$

Similarly, we can always find another larger interval, which has more than 9.31 twin prime numbers. Therefore, there are infinite twin primes.

Among possible form of prime numbers $\{17+30*x|x \geq 0\}$, if any $17+30*x$ is not a prime number, then there are only four possible decomposition forms:

$H_9 = [7+30a][11+30b], a \geq 0, b \geq 0$; $H_{10} = [13+30a][29+30b], a \geq 0, b \geq 0$;

$H_{11} = [17+30a][31+30b], a \geq 0, b \geq 0$; $H_{12} = [19+30a][23+30b], a \geq 0, b \geq 0$.

Consider an estimation formula lower than the actual number of primes in $\{17+30*x|0 \leq x < 30\}$:

$$30(1-\frac{1}{7})(1-\frac{1}{11})(1-\frac{1}{13})(1-\frac{1}{17})(1-\frac{1}{19})(1-\frac{1}{23})(1-\frac{1}{29})(1-\frac{1}{31}) \approx 17.20$$

By considering an estimation formula lower than the actual number of prime pairs in $\{(11+30*x, 17+30*x)|x \geq 0\}$, similarly we can prove that there are infinite prime pairs in $\{(11+30*x, 17+30*x)|x \geq 0\}$.

To sum up, we get the following theorem.

Theorem 1. For any two prime numbers $p_0 > 5$ and $q_0 > 5$, there are infinite prime pairs p_i and $q_i, i \geq 1$, such that $p_i - q_i = p_0 - q_0$.

5 Goldbach conjecture

Very similar to the case of twin prime conjecture, we further consider possible form of prime pairs $\{(11+30*x, 13+30*(n-x))|x \geq 0\}$. From Table 1 and Table 2, (131, 6163) is a pair of primes, where $n=209, x=4$.

Consider 30 pairs of natural numbers $\{(13+30*x, 13+30*(29-x))|0 \leq x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of twin prime numbers $C(30)$:

$$30(1-\frac{2}{7})(1-\frac{2}{11})(1-\frac{2}{13})(1-\frac{2}{17})(1-\frac{2}{19})(1-\frac{2}{23})(1-\frac{2}{29})(1-\frac{2}{31}) \approx 9.31$$

The actual number of prime pairs in $\{(11+30*x, 13+30*(29-x))|0 \leq x < 30\}$ is 11.

The above estimation formula can be understood in this way (take prime number 7 as an example): for every 7 consecutive cells, there must be one cell of $11+30*x_1$ can be divided by 7, and another cell of $13+30*(29-x_2)$ can be divided by 7, where $x_1 \neq x_2$. Therefore, one term in the above formula is $\frac{7-2}{7}$.

When $x=11$, $11+30*11=341$ can be divided by 11, and $13+30*(29-11)=553$ can be divided by 7. Therefore, this cell is counted twice, so the estimation formula $C(30)$ is lower than the actual number of prime pairs.

The estimation formula for the number of prime pairs in $\{(11+30*x, 13+30*(89-x))|30 \leq x < 90\}$ is $C(60)$:

$$60(1-\frac{2}{7})(1-\frac{2}{11})(1-\frac{2}{13})(1-\frac{2}{17})(1-\frac{2}{19})(1-\frac{2}{23})(1-\frac{2}{29})(1-\frac{2}{31})$$
$$(1-\frac{2}{37})(1-\frac{2}{67})(1-\frac{2}{41})(1-\frac{2}{71})(1-\frac{2}{43})(1-\frac{2}{73})(1-\frac{2}{47})(1-\frac{2}{77})(1-\frac{2}{49})(1-\frac{2}{79})(1-\frac{2}{53})(1-\frac{2}{83})(1-\frac{2}{59})(1-\frac{2}{89})(1-\frac{2}{61})(1-\frac{2}{91})$$
$$\approx 10.72$$

The actual number of prime pairs in $\{(11+30*x, 13+30*(89-x))|30 \leq x < 90\}$ is 16.

In the above formula, $77, 49$ may not appear in the formula because they are multiples of 7. For the completeness of the formula, $77, 49$ are still retained, which only make the valuation smaller.

The effective range of the above estimation formula is: $91^2 > 30*270 > 30*210$.

The actual number of prime pairs in $\{(11+30*x, 13+30*(209-x))|90 \leq x < 210\}$ is 29. The estimation formula for the number of prime pairs in $\{(11+30*x, 13+30*(209-x))|90 \leq x < 210\}$ is $C(120)$ (only the part about $\{7+30*x|x \geq 0\}$ is given here):

$$(1-\frac{2}{7})$$
$$(1-\frac{2}{37})(1-\frac{2}{67})$$
$$(1-\frac{2}{97})(1-\frac{2}{127})(1-\frac{2}{157})(1-\frac{2}{187})$$

The effective range of the above estimation formula is: $211^2 > 30*70*210 > 30*450$.

More generally,

$$(30*2^k-30)^2-30*(30*2^{k+1}-30)=900*2^{2k}-900*2^{k+2}+2*900$$
$$=900*2^k(2^k-2^2)+2*900>0$$
$$\Longrightarrow (30*2^k-30)^2>30*(30*2^{k+1}-30)$$

where $k\geq 2$.

When $k=2$, the inequality means $90^2>30*210$.

When $k=3$, the inequality means $210^2>30*450$.

The actual number of prime pairs in $\{(11+30*x,13+30*(449-x))|210\leq x<450\}$ is 44. The estimation formula for the number of prime pairs in $\{(11+30*x,13+30*(449-x))|210\leq x<450\}$ is $C(240)$(only the part about $\{7+30*x|x\geq 0\}$ is given here):

$$(1-\frac{2}{7})$$
$$(1-\frac{2}{37})(1-\frac{2}{67})$$
$$(1-\frac{2}{97})(1-\frac{2}{127})(1-\frac{2}{157})(1-\frac{2}{187})$$
$$(1-\frac{2}{217})(1-\frac{2}{247})(1-\frac{2}{277})(1-\frac{2}{307})(1-\frac{2}{337})(1-\frac{2}{367})(1-\frac{2}{397})(1-\frac{2}{427})$$

It is easy to obtain the following (only the part about $\{7+30*x|x\geq 0\}$ is given here):

$$\frac{C(60)}{C(30)}=2(1-\frac{2}{37})(1-\frac{2}{67})$$
$$\frac{C(120)}{C(60)}=2(1-\frac{2}{97})(1-\frac{2}{127})(1-\frac{2}{157})(1-\frac{2}{187})$$
$$\frac{C(240)}{C(120)}=2(1-\frac{2}{217})(1-\frac{2}{247})(1-\frac{2}{277})(1-\frac{2}{307})(1-\frac{2}{337})(1-\frac{2}{367})(1-\frac{2}{397})(1-\frac{2}{427})$$
$$\cdots\cdots$$

By Lemma 3,

$$(1-\frac{2}{217})(1-\frac{2}{247})>(1-\frac{2}{97}),(1-\frac{2}{277})(1-\frac{2}{307})>(1-\frac{2}{127}),\cdots\cdots$$

Thus

$$\cdots\cdots>\frac{C(240)}{C(120)}>\frac{C(120)}{C(60)}>\frac{C(60)}{C(30)}$$

The complete $\frac{C(60)}{C(30)}$ is as follows,

$$2(1-\frac{2}{37})(1-\frac{2}{67})(1-\frac{2}{41})(1-\frac{2}{71})(1-\frac{2}{43})(1-\frac{2}{73})(1-\frac{2}{47})(1-\frac{2}{77})(1-\frac{2}{49})(1-\frac{2}{79})(1-\frac{2}{53})(1-\frac{2}{83})(1-\frac{2}{59})(1-\frac{2}{89})(1-\frac{2}{61})(1-\frac{2}{91})$$

$$= 2 \times 0.575 > 1$$

Therefore

$$\cdots\cdots > C(240) > C(120) > C(60) \approx 10.72$$

In fact, the actual number of prime pairs in $\{(11+30*x, 13+30*(929-x))|450 \le x < 930\}$ is 73, and the actual number of prime pairs in $\{(11+30*x, 13+30*(1889-x))|930 \le x < 1890\}$ is $136, \ldots\ldots$

Similarly, we can always find another larger interval, which has more than 10.72 prime pairs (p, q), such that $p+q = 24+30*(30*2^k-30-1)$, where $k > 1$.

For any $24+30*a, a \ge 90$, if $30*2^k-30-1 < a < 30*2^{k+1}-30-1$, by Lemma 2, the estimation formula for the number of prime pairs in $\{(11+30*x, 13+30*(a-x))|0 \le x \le a\}$ is greater than $C(30*2^{k-1}) \ge C(60)$.

Since we can also consider the prime pairs in $\{(7+30*x, 17+30*(a-x))|0 \le x \le a\}$, thus there exist much more than 10 prime pairs (p, q), such that $p+q = 24+30*a$,

For example, for $24+30*99$, since $30*2^2-30-1 < 99 < 30*2^{2+1}-30-1$, so the estimation formula for the number of prime pairs in $\{(11+30*x, 13+30*(99-x))|0 \le x \le 99\}$ is greater than $C(30*2^{2-1}) = C(60)$. In fact, the actual number of prime pairs in $\{(11+30*x, 13+30*(99-x))|0 \le x \le 99\}$ is 27, and the actual number of prime pairs in $\{(7+30*x, 17+30*(99-x))|0 \le x \le 99\}$ is 32.

It is easy to show that for $a > 0$,
$\{0+30*a = 7+30*(a-1)+23\}$;
$\{2+30*a = 13+30*(a-1)+19\}$;
$\{4+30*a = 11+30*(a-1)+23\}$;
$\{6+30*a = 13+30*(a-1)+23\}$;
$\{8+30*a = 19+30*(a-1)+19\}$;
$\{10+30*a = 17+30*(a-1)+23\}$;
$\{12+30*a = 19+30*(a-1)+23\}$;
$\{14+30*a = 13+30*(a-1)+31\}$;
$\{16+30*a = 17+30*(a-1)+29\}$;
$\{18+30*a = 7+30*a+11\}$;
$\{20+30*a = 7+30*a+13\}$;
$\{22+30*a = 11+30*a+11\}$;
$\{26+30*a = 13+30*a+13\}$;
$\{28+30*a = 11+30*a+17\}$.

Very similar to the case of $\{24+30*a = 11+30*a+13\}$, we can prove that for any $2l+30*a, 0 \le l < 15, a \ge 90$, there exist more than 10 prime pairs (p, q), such that $p+q = 2l+30*a$.

To sum up, we get the following theorem.

Theorem 2. For any even integer $m > 30*90$, there exist more than 10 prime pairs (p, q), such that $p+q = m$.

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