On the prime distribution

Yong Zhao   Jianqin Zhou*

Department of Computer Science, Anhui University of Technology
Ma'anshan 243002, P. R. China

In this paper, the estimation formula of the number of primes in a given interval is obtained by using the prime distribution property. For any prime pairs $p > 5$ and $q > 5$, construct a disjoint infinite set sequence $A_1, A_2, \ldots, A_i \ldots$, such that the number of prime pairs (p_i and q_i, $p_i - q_i = p - q$) in A_i increases gradually, where $i > 0$. So twin prime conjecture is true. We also prove that for any even integer $m > 2700$, there exist more than 10 prime pairs (p, q), such that $p + q = m$. Thus Goldbach conjecture is true.

1 Introduction

Like Goldbach conjecture, twin prime conjecture is also one of the famous unsolved problems in number theory. In 1973, Chen [1] proved that for any even number h, there are infinite prime numbers p, so that the number of prime factors of $p + h$ does not exceed 2. In 2008, Green and Tao [2] proved the existence of arbitrarily long arithmetic progressions in the primes. In 2014, Zhang [3] proved that bounded gaps between primes are all less than 70 million.

In this paper, the estimation formula of the number of primes in a given interval is given by using the prime distribution property. For any prime pairs $p > 5$ and $q > 5$, construct a disjoint infinite set sequence $A_1, A_2, \ldots, A_i \ldots$, such that the number of prime pairs (p_i and q_i, $p_i - q_i = p - q$) in A_i increases gradually, where $i > 0$. So the original conjecture is true. We also prove that for any even integer $m > 2700$, there exist more than 10 prime pairs (p, q), such that $p + q = m$. Thus Goldbach conjecture is true. For any triple primes $p > 7$, $q > 7$ and $r > 7$, construct a disjoint infinite set sequence $A_1, A_2, \ldots, A_i \ldots$, such that the number of triple primes (p_i q_i and r_i, $p_i - q_i = p - q$, $p_i - r_i = p - r$) in A_i increases gradually, where $i > 0$. So triple primes conjecture is true.

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*Corresponding author. Email: zhou63@ahut.edu.cn

2   Three lemmas

In this section, three lemmas are proved.

Lemma 1. Given two coprime natural numbers p and q. If the remainder of natural numbers in the set A with respect to p is evenly distributed, then the remainder of natural numbers in the set $\{aq + c | a \in A\}$ is still evenly distributed, where $c \geq 0$ is an integer.

Proof. Prove by a contradiction.

Without loss of generality, suppose that $a_i \equiv i \mod p, a_i \in A$, $0 \leq i < p$, but the remainder of natural numbers in the set $\{a_i q + c | a_i \in A, 0 \leq i < p\}$ is not evenly distributed. Let's say that $a_i q + c$ and $a_j q + c$ have the same remainder about p,

$\implies a_i q + c - (a_j q + c) = kp, k$ is an integer. $\implies (a_i - a_j)q = kp$.

As p and q are coprime natural numbers, $\implies a_i \equiv a_j \mod p$. This is a contradiction.
$\square$

Based on Lemma 1, we prove the following Lemma 2.

Lemma 2. Given natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ pairwise prime, and the remainder of natural numbers in the set A with respect to $\alpha_i, 1 \leq i \leq n$, are evenly distributed, where $|A| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_n$. We conclude that

$$|\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq n\}| = (\alpha_1 - 1) \times (\alpha_2 - 1) \times \ldots \times (\alpha_n - 1)$$

Proof. If $n = 1$, then $|A| = \alpha_1$. Obviously, $|\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_1\}| = \alpha_1 - 1$.

If $n = 2$, then $|A| = \alpha_1 \times \alpha_2$.

Without loss of generality, suppose that

$$A = \begin{pmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 \\ \alpha_1 + 1 & \alpha_1 + 2 & \cdots & 2\alpha_1 - 1 & 2\alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (\alpha_2 - 2)\alpha_1 + 1 & (\alpha_2 - 2)\alpha_1 + 2 & \cdots & (\alpha_2 - 1)\alpha_1 - 1 & (\alpha_2 - 1)\alpha_1 \\ (\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & \alpha_2\alpha_1 - 1 & \alpha_2\alpha_1 \end{pmatrix}$$

$\implies$

$$\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_1\} = \begin{pmatrix} 1 & 2 & \cdots & \alpha_1 - 1 \\ \alpha_1 + 1 & \alpha_1 + 2 & \cdots & 2\alpha_1 - 1 \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_2 - 2)\alpha_1 + 1 & (\alpha_2 - 2)\alpha_1 + 2 & \cdots & (\alpha_2 - 1)\alpha_1 - 1 \\ (\alpha_2 - 1)\alpha_1 + 1 & (\alpha_2 - 1)\alpha_1 + 2 & \cdots & \alpha_2\alpha_1 - 1 \end{pmatrix}$$

As α_1 and α_2 are coprime natural numbers, and by Lemma 1, $\{i\alpha_1 + j | 0 \leq i < \alpha_2\} \equiv \{0, 1, 2, \cdots, \alpha_2 - 1\} \mod \alpha_2$, where $0 < j < \alpha_1$.

$\implies |\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq 2\}| = (\alpha_1 - 1) \times (\alpha_2 - 1)$.

If $n = k + 1$, then $|A| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k \times \alpha_{k+1}$.

As $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ are pairwise primes, and by Lemma 1, the remainder of natural numbers in the set $\{i \times \alpha_{k+1} | 0 < i \leq \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k\}$ with respect to $\alpha_i, 1 \leq i \leq k$, is evenly distributed.

$\Longrightarrow$ the remainder of natural numbers in the set $\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_{k+1}\}$ with respect to $\alpha_i, 1 \leq i \leq k$, is evenly distributed and $|\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_{k+1}\}| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_k \times (\alpha_{k+1} - 1)$.

Similarly, as $\alpha_1, \alpha_2, \ldots, \alpha_k$ are pairwise primes, and by Lemma 1, the remainder of natural numbers in the set $\{i \times \alpha_k | 0 < i \leq \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{k-1} \times (\alpha_{k+1} - 1)\}$ with respect to $\alpha_i, 1 \leq i \leq k - 1$, is evenly distributed.

$\Longrightarrow$ the remainder of natural numbers in the set $\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_x, x = k, k+1\}$ with respect to $\alpha_i, 1 \leq i \leq k-1$, is evenly distributed and $|\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_x, x = k, k+1\}| = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{k-1} \times (\alpha_k - 1) \times (\alpha_{k+1} - 1)$.

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Finally, we have

$$|\{a | a \in A \text{ and } a \not\equiv 0 \mod \alpha_i, 1 \leq i \leq k+1\}| = (\alpha_1 - 1) \times (\alpha_2 - 1) \times \ldots \times (\alpha_{k+1} - 1)$$

$\square$

The following examples is helpful to understand Lemma 2.

It is known that the natural numbers $2, 3, 5$ are mutually prime. Let $A = \{1, 2, 3, \ldots, 29, 30\}$. Obviously the remainder of the natural numbers in A about $2, 3, 5$ is evenly distributed. According to Lemma 2, after removing all natural numbers with 0 remainder about $2, 3, 5$, the number of natural numbers in A becomes $(2 - 1) \times (3 - 1) \times (5 - 1) = 8$. Namely, $\{1, 7, 11, 13, 17, 19, 23, 29\}$.

Let $A = \{7 * i + 1 | 1 \leq i \leq 30\}$. According to Lemma 1, the remainder of natural numbers about $2, 3, 5$ in A is still evenly distributed. Then, according to Lemma 2, after removing all natural numbers with 0 remainder about $2, 3, 5$, the number of natural numbers in A becomes $(2 - 1) \times (3 - 1) \times (5 - 1) = 8$.

Based on Lemma 2, the number of primes in 30 consecutive natural numbers $\{6, 7, \ldots, 34, 35\}$ is $(2 - 1)(3 - 1)(5 - 1) = 30(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 8$. As $30 = 2 \times 3 \times 5$, the formula is accurate. These specific prime numbers are: $7, 11, 13, 17, 19, 23, 29, 31$.

We are concerned about the following two issues here.

1. Number counting formula of primes $(2 - 1)(3 - 1)(5 - 1)$ is valid for 30 consecutive natural numbers less than 49. Because 49 is not a multiple of $2, 3, 5$, but $49 = 7 \times 7$. For example, the number of primes among 30 consecutive natural numbers $\{20, 21, \ldots, 48, 49\}$ is 7. These specific prime numbers are: $23, 29, 31, 37, 41, 43, 47$. That is, the actual number of primes is less than $(2 - 1)(3 - 1)(5 - 1)$.

2. Note that number counting formula of primes

$$30(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7}) \approx 6.86$$

is less than the actual number 8 of primes in 30 consecutive natural numbers $\{6, 7, \ldots, 34, 35\}$, and also less than the actual number 7 of primes in 30 consecutive natural numbers $\{20, 21, \ldots, 48, 49\}$.

The number counting formula of primes $30(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})$ is valid for 30 consecutive natural numbers less than 11^2. However, it is well known that the prime distribution is not even, and the prime numbers in the front part of the effective range are dense and the prime numbers behind are sparse. For example, the actual number 6 of primes in 30 consecutive natural numbers $\{71, 72, \ldots, 99, 100\}$ is less than 6.86. These specific prime numbers are: $71, 73, 79, 83, 89, 97$.

This paper mainly considers the case that **the estimation formula for the number of primes lower than the actual number of primes.**

The following proves Lemma 3

Lemma 3.

$$(1 - \frac{d}{30 \times x + c}) < (1 - \frac{d}{30 \times (2x + e + 1) + c})(1 - \frac{d}{30 \times (2x + e + 2) + c})$$

$$x \geq 0, e \geq 0, 0 < c < 32, 0 < d < 30x.$$

Proof. First prove: $\frac{d}{30 \times (2x+1)+c} + \frac{d}{30 \times (2x+2)+c} < \frac{d}{30 \times x+c}$

$$\iff (b + c)(2b + c + 30 + 2b + c + 60) < (2b + c + 30)(2b + c + 60), b = 30x$$

$$\iff (b + c)(4b + 2c + 90) < (2b + c + 30)(2b + c + 60)$$

$$\iff 4b^2 + 6bc + 90b + 90c + 2c^2 < 4b^2 + 4bc + 180b + c^2 + 90c + 1800$$

From both sides of the inequality remove $4b^2 + 4bc + 90b + 90c + c^2$
The original inequality $\iff c^2 + 2bc < 90b + 1800 \iff 0 < (90 - 2c)b + 1800 - c^2$.
As $c < 32$, thus

$$\frac{d}{30 \times (2x + 1) + c} + \frac{d}{30 \times (2x + 2) + c} < \frac{d}{30 \times x + c}$$

$$\implies \frac{d}{30 \times (2x + e + 1) + c} + \frac{d}{30 \times (2x + e + 2) + c} < \frac{d}{30 \times x + c}$$

$$\implies (1 - \frac{d}{30 \times (2x + e + 1) + c})(1 - \frac{d}{30 \times (2x + e + 2) + c})$$

$$> 1 - (\frac{d}{30 \times (2x + e + 1) + c} + \frac{d}{30 \times (2x + e + 2) + c}) > 1 - \frac{d}{30 \times x + c}$$

$\square$

3 Possible form of prime numbers $\{11 + 30 * x | x \geq 0\}$ and $\{13 + 30 * x | x \geq 0\}$

Introduce some basic properties of prime numbers. All prime numbers are in odd numbers with single digits of 1, 3, 7 and 9 (except 2 and 5). Now let's take a look at the prime number whose single digit is 1. It is easy to find that there are only two possible forms:

$\{11 + 30 * x | x \geq 0\}$ and $\{31 + 30 * x | x \geq 0\}$
For prime number whose single digit is 3, there are only two possible forms:
$\{13 + 30 * x | x \geq 0\}$ and $\{23 + 30 * x | x \geq 0\}$
For prime number whose single digit is 7, there are only two possible forms:
$\{7 + 30 * x | x \geq 0\}$ and $\{17 + 30 * x | x \geq 0\}$
For prime number whose single digit is 9, there are only two possible forms:
$\{19 + 30 * x | x \geq 0\}$ and $\{29 + 30 * x | x \geq 0\}$

Among possible form of prime numbers $\{11 + 30 * x | x \geq 0\}$, if any $11 + 30 * x$ is not a prime number, then there are only four possible decomposition forms:
$H_1 = [7 + 30a][23 + 30b], a \geq 0, b \geq 0$; $H_2 = [13 + 30a][17 + 30b], a \geq 0, b \geq 0$;
$H_3 = [11 + 30a][31 + 30b], a \geq 0, b \geq 0$; $H_4 = [19 + 30a][29 + 30b], a \geq 0, b \geq 0$.
As $(19 + 30a)^2 = 30c + 1$, thus $(19 + 30a)^2$ is not a possible decomposition form of $11 + 30 * x$.

Table 1:   The possible form of prime numbers $\{11 + 30 * x | 0 \leq x < 210\}$

11	41	71	101	131	161	191	221	251	281	311	341	371	401	431
					7*23		13*17				11*31	7*53		
461	491	521	551	581	611	641	671	701	731	761	791	821	851	881
			19*29	7*83	13*47		11*61		17*43		7*113		23*37	
911	941	971	1001	1031	1061	1091	1121	1151	1181	1211	1241	1271	1301	1331
			7				19			7	17	31		11
1361	1391	1421	1451	1481	1511	1541	1571	1601	1631	1661	1691	1721	1751	1781
	13	7				23			7	11	19		17	13
1811	1841	1871	1901	1931	1961	1991	2021	2051	2081	2111	2141	2171	2201	2231
	7				37	11	43	7				13	31	23
2261	2291	2321	2351	2381	2411	2441	2471	2501	2531	2561	2591	2621	2651	2681
7	29	11					7	41		13			11	7
2711	2741	2771	2801	2831	2861	2891	2921	2951	2981	3011	3041	3071	3101	3131
		17		19		7	23	13	11			37	7	31
3161	3191	3221	3251	3281	3311	3341	3371	3401	3431	3461	3491	3521	3551	3581
29				17	7	13		19	47			7	53	
3611	3641	3671	3701	3731	3761	3791	3821	3851	3881	3911	3941	3971	4001	4031
23	11			7		17					7	11		29
4061	4091	4121	4151	4181	4211	4241	4271	4301	4331	4361	4391	4421	4451	4481
31		13	7	37				11	61	7				
4511	4541	4571	4601	4631	4661	4691	4721	4751	4781	4811	4841	4871	4901	4931
13	19	7	43	11	59				7	17	47		13	
4961	4991	5021	5051	5081	5111	5141	5171	5201	5231	5261	5291	5321	5351	5381
11	7				19	53		7			11	17		
5411	5441	5471	5501	5531	5561	5591	5621	5651	5681	5711	5741	5771	5801	5831
7					67		7		13			29		7
5861	5891	5921	5951	5981	6011	6041	6071	6101	6131	6161	6191	6221	6251	6281
	43	31	11			7	13			61	41		7	11

In Table 1, the number in the cell indicates that the corresponding $11 + 30 * x$ is not a prime number, and the number in the cell is a factor.
For example, $11 + 30 * 33 = 1001 = 11 \times 7 \times 13$;
$11 + 30 * 47 = 1421 = 29 \times 49 = 7 \times 203$.
Based on Lemma 2, we consider a formula for estimating the number of primes. In Table 1, consider first 30 natural numbers $\{11 + 30 * x | 0 \leq x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of primes:

$$30(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31}) \approx 17.20$$

The actual number of primes in $\{11 + 30 * x | 0 \leq x < 30\}$ is 19.

If $a \in \{11 + 30 * x | 0 \le x < 30\}$ and a is a composite number, then a must has one factor in $\{7, 11, 13, 17, 19, 23\}$, so the above estimation formula is lower than the actual number of primes.

As $(a + 30)(b + 30) = 30(30 + a + b) + ab$, the above estimation formula is valid for $11 + 30 * x < 53 \times 37 = 1961 = 11 + 30 \times 65$. So the above estimation formula is applied to the front part of the effective range, which is the reason that the above estimation formula is lower than the actual number of primes.

Table 2:   The possible form of prime numbers $\{13 + 30 * x | 0 \le x < 210\}$

13	43	73	103	133	163	193	223	253	283	313	343	373	403	433
				7*19				11*23			7*49		13*31	
463	493	523	553	583	613	643	673	703	733	763	793	823	853	883
	17		7	11				19		7	13			
913	943	973	1003	1033	1063	1093	1123	1153	1183	1213	1243	1273	1303	1333
11	23	7	17						7		11	19		31
1363	1393	1423	1453	1483	1513	1543	1573	1603	1633	1663	1693	1723	1753	1783
29	7				17		11	7	23					
1813	1843	1873	1903	1933	1963	1993	2023	2053	2083	2113	2143	2173	2203	2233
7	19		11		13	7						41		7
2263	2293	2323	2353	2383	2413	2443	2473	2503	2533	2563	2593	2623	2653	2683
31		23	13		19	7			17	11		43	7	
2713	2743	2773	2803	2833	2863	2893	2923	2953	2983	3013	3043	3073	3103	3133
	13	47			7	11	37		19	23	17	7	29	13
3163	3193	3223	3253	3283	3313	3343	3373	3403	3433	3463	3493	3523	3553	3583
	31	11		7				41			7	13	11	
3613	3643	3673	3703	3733	3763	3793	3823	3853	3883	3913	3943	3973	4003	4033
			7		53				11	7		29		37
4063	4093	4123	4153	4183	4213	4243	4273	4303	4333	4363	4393	4423	4453	4483
17		7		47	11			13	7			23		61
4513	4543	4573	4603	4633	4663	4693	4723	4753	4783	4813	4843	4873	4903	4933
	7	17		41		13		7			29	11		
4963	4993	5023	5053	5083	5113	5143	5173	5203	5233	5263	5293	5323	5353	5383
7			31	13		37	7	11		19	67		53	7
5413	5443	5473	5503	5533	5563	5593	5623	5653	5683	5713	5743	5773	5803	5833
		13		11		7				29		23	7	19
5863	5893	5923	5953	5983	6013	6043	6073	6103	6133	6163	6193	6223	6253	6283
11	71			31	7			17			11	7	13	61

Among possible form of prime numbers $\{13 + 30 * x | x \ge 0\}$, if any $13 + 30 * x$ is not a prime number, then there are only four possible decomposition forms:

$H_5 = [7 + 30a][19 + 30b], a \ge 0, b \ge 0$; $H_6 = [13 + 30a][31 + 30b], a \ge 0, b \ge 0$;

$H_7 = [11 + 30a][23 + 30b], a \ge 0, b \ge 0$; $H_8 = [17 + 30a][29 + 30b], a \ge 0, b \ge 0$.

In Table 2, consider first 30 natural numbers $\{11 + 30 * x | 0 \le x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of primes:

$$30(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31}) \approx 17.20$$

The actual number of primes in $\{13 + 30 * x | 0 \le x < 30\}$ is 20.

If $a \in \{13 + 30 * x | 0 \le x < 30\}$ and a is a composite number, then a must has one factor in $\{7, 11, 13, 17, 19\}$, so the above estimation formula is lower than the actual number of primes. As $19 + 30$ is a composite number, the above estimation formula is valid for $13 + 30 * x < 41 \times 53 = 2173 = 13 + 30 \times 72$. So the above estimation formula is applied to the front part of the effective range, which is the reason that the above estimation formula is lower than the actual number of primes.

4 Twin prime conjecture

We further consider possible form of twin prime numbers $\{(11+30*x, 13+30*x)|x \geq 0\}$. From Table 1 and Table 2, $(6131, 6133)$ is twin prime numbers.

Consider first 30 pairs of natural numbers $\{(11 + 30 * x, 13 + 30 * x)|0 \leq x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of twin prime numbers $C(30)$:

$$30(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31}) \approx 9.31$$

The actual number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|0 \leq x < 30\}$ is 13.

The above estimation formula can be understood in this way (take prime number 7 as an example): for every 7 consecutive cells, there must be one cell of $11 + 30 * x_1$ can be divided by 7, and another cell of $13 + 30 * x_2$ can be divided by 7, where $x_1 \neq x_2$. Therefore, one term in the above formula is $\frac{7-2}{7}$.

When $x = 19$, $11 + 30 * 19 = 581$ can be divided by 7, and $13 + 30 * 19 = 583$ can be divided by 11. Therefore, this cell is counted twice, so the estimation formula $C(30)$ is lower than the actual number of twin primes.

The estimation formula for the number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|30 \leq x < 90\}$ is $C(60)$:

$$60(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31})$$

$$(1 - \frac{2}{37})(1 - \frac{2}{67})(1 - \frac{2}{41})(1 - \frac{2}{71})(1 - \frac{2}{43})(1 - \frac{2}{73})(1 - \frac{2}{47})(1 - \frac{2}{77})(1 - \frac{2}{49})(1 - \frac{2}{79})(1 - \frac{2}{53})(1 - \frac{2}{83})(1 - \frac{2}{59})(1 - \frac{2}{89})(1 - \frac{2}{61})(1 - \frac{2}{91})$$

$$\approx 10.72$$

The actual number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|30 \leq x < 90\}$ is 15.

In the above formula, $77, 49$ may not appear in the formula because they are multiples of 7. For the completeness of the formula, $77, 49$ are still retained, which only make the valuation smaller.

Because $91^2 > 13 + 270 * 30$, the effective range of the above estimation formula is: $13 + 30 * x \leq 13 + 30 \times 270$, where $270 > 90$. So the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

The actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|90 \leq x < 210\}$ is 24. The estimation formula for the number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|90 \leq x < 210\}$ is $C(120)$:

$$120(1 - \frac{2}{7}) \cdots (1 - \frac{2}{31})$$

$$(1 - \frac{2}{37})(1 - \frac{2}{67}) \cdots (1 - \frac{2}{61})(1 - \frac{2}{91})$$

$$(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \cdots (1 - \frac{2}{121})(1 - \frac{2}{151})(1 - \frac{2}{181})(1 - \frac{2}{211})$$

The effective range of the above estimation formula is: $211^2 > 30*70*210 > 30*450$. More generally,

$$(30 * 2^k - 30)^2 - 30 * (30 * 2^{k+1} - 30) = 900 * (2^{2k} - 2^{k+2} + 2) > 0$$

$$\implies (30 * 2^k - 30)^2 > 30 * (30 * 2^{k+1} - 30)$$

where $k \geq 2$.

When $k = 2$, the inequality means $90^2 > 30 * 210$.

When $k = 3$, the inequality means $210^2 > 30 * 450$.

When $k = 4$, the inequality means $450^2 > 30 * 930$.

$\cdots\cdots$

Therefore, the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

The actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|210 \leq x < 450\}$ is 29. The estimation formula for the number of twin prime numbers in $\{(11 + 30 * x, 13 + 30 * x)|210 \leq x < 450\}$ is $C(240)$(only the part about $\{7 + 30 * x | x \geq 0\}$ is given here):

$$(1 - \frac{2}{7})$$

$$(1 - \frac{2}{37})(1 - \frac{2}{67})$$

$$(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187})$$

$$(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})$$

It is easy to obtain the following (only the part about $\{7 + 30 * x | x \geq 0\}$ is given here):

$$\frac{C(60)}{C(30)} = 2(1 - \frac{2}{37})(1 - \frac{2}{67})$$

$$\frac{C(120)}{C(60)} = 2(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187})$$

$$\frac{C(240)}{C(120)} = 2(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})$$

$$\cdots\cdots$$

By Lemma 3,

$$(1 - \frac{2}{217})(1 - \frac{2}{247}) > (1 - \frac{2}{97}), (1 - \frac{2}{277})(1 - \frac{2}{307}) > (1 - \frac{2}{127}), \cdots\cdots,$$

Thus

$$\cdots\cdots > \frac{C(240)}{C(120)} > \frac{C(120)}{C(60)} > \frac{C(60)}{C(30)}$$

The complete $\frac{C(60)}{C(30)}$ is as follows,

$$2(1-\frac{2}{37})(1-\frac{2}{67})(1-\frac{2}{41})(1-\frac{2}{71})(1-\frac{2}{43})(1-\frac{2}{73})(1-\frac{2}{47})(1-\frac{2}{77})(1-\frac{2}{49})(1-\frac{2}{79})(1-\frac{2}{53})(1-\frac{2}{83})(1-\frac{2}{59})(1-\frac{2}{89})(1-\frac{2}{61})(1-\frac{2}{91})$$

$$= 2 \times 0.575 > 1$$

Therefore

$$\cdots\cdots > C(240) > C(120) > C(60) > C(30) \approx 9.31$$

In fact, the actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|450 \leq x < 930\}$ is 71, and the actual number of twin prime numbers in $\{(11+30*x, 13+30*x)|930 \leq x < 1890\}$ is $113, \ldots\ldots$,

Similarly, we can always find another larger interval, which has more than 9.31 twin prime numbers. Therefore, there are infinite twin primes.

Among possible form of prime numbers $\{17 + 30*x | x \geq 0\}$, if any $17 + 30*x$ is not a prime number, then there are only four possible decomposition forms:

$H_9 = [7 + 30a][11 + 30b], a \geq 0, b \geq 0$; $H_{10} = [13 + 30a][29 + 30b], a \geq 0, b \geq 0$;

$H_{11} = [17 + 30a][31 + 30b], a \geq 0, b \geq 0$; $H_{12} = [19 + 30a][23 + 30b], a \geq 0, b \geq 0$.

Consider an estimation formula lower than the actual number of primes in $\{17 + 30 * x | 0 \leq x < 30\}$:

$$30(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31}) \approx 17.20$$

By considering an estimation formula lower than the actual number of prime pairs in $\{(11 + 30 * x, 17 + 30 * x)|x \geq 0\}$, similarly we can prove that there are infinite prime pairs in $\{(11 + 30 * x, 17 + 30 * x)|x \geq 0\}$.

To sum up, we get the following theorem.

Theorem 1. For any two prime numbers $p_0 > 5$ and $q_0 > 5$, there are infinite prime pairs p_i and $q_i, i \geq 1$, such that $p_i - q_i = p_0 - q_0$.

5 Goldbach conjecture

Very similar to the case of twin prime conjecture, we further consider possible form of prime pairs $\{(11 + 30 * x, 13 + 30 * (n - x))|x \geq 0\}$. From Table 1 and Table 2, $(131, 6163)$ is a pair of primes, where $n = 209, x = 4$.

Consider 30 pairs of natural numbers $\{(13+30*x, 13+30*(29-x))|0 \leq x < 30\}$. As 30 is not the multiple of $7, 11, 13, 17, 19, 23, 29, 31$, thus consider an estimation formula lower than the actual number of twin prime numbers $C(30)$:

$$30(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31}) \approx 9.31$$

The actual number of prime pairs in $\{(11 + 30 * x, 13 + 30 * (29 - x))|0 \leq x < 30\}$ is 11.

The above estimation formula can be understood in this way (take prime number 7 as an example): for every 7 consecutive cells, there must be one cell of $11 + 30 * x_1$ can be divided by 7, and another cell of $13 + 30 * (29 - x_2)$ can be divided by 7, where $x_1 \neq x_2$. Therefore, one term in the above formula is $\frac{7-2}{7}$.

When $x = 11$, $11 + 30 * 11 = 341$ can be divided by 11, and $13 + 30 * (29 - 11) = 553$ can be divided by 7. Therefore, this cell is counted twice, so the estimation formula $C(30)$ is lower than the actual number of prime pairs.

The estimation formula for the number of prime pairs in $\{(11 + 30 * x, 13 + 30 * (89 - x)) | 30 \leq x < 90\}$ is $C(60)$:

$$60(1 - \frac{2}{7})(1 - \frac{2}{11})(1 - \frac{2}{13})(1 - \frac{2}{17})(1 - \frac{2}{19})(1 - \frac{2}{23})(1 - \frac{2}{29})(1 - \frac{2}{31})$$

$$(1 - \frac{2}{37})(1 - \frac{2}{67})(1 - \frac{2}{41})(1 - \frac{2}{71})(1 - \frac{2}{43})(1 - \frac{2}{73})(1 - \frac{2}{47})(1 - \frac{2}{77})(1 - \frac{2}{49})(1 - \frac{2}{79})(1 - \frac{2}{53})(1 - \frac{2}{83})(1 - \frac{2}{59})(1 - \frac{2}{89})(1 - \frac{2}{61})(1 - \frac{2}{91})$$

$$\approx 10.72$$

The actual number of prime pairs in $\{(11 + 30 * x, 13 + 30 * (89 - x)) | 30 \leq x < 90\}$ is 16.

In the above formula, $77, 49$ may not appear in the formula because they are multiples of 7. For the completeness of the formula, $77, 49$ are still retained, which only make the valuation smaller.

The effective range of the above estimation formula is: $91^2 > 30 * 270 > 30 * 210$.

The actual number of prime pairs in $\{(11 + 30 * x, 13 + 30 * (209 - x)) | 90 \leq x < 210\}$ is 29. The estimation formula for the number of prime pairs in $\{(11 + 30 * x, 13 + 30 * (209 - x)) | 90 \leq x < 210\}$ is $C(120)$:

$$120(1 - \frac{2}{7}) \cdots (1 - \frac{2}{31})$$

$$(1 - \frac{2}{37})(1 - \frac{2}{67}) \cdots (1 - \frac{2}{61})(1 - \frac{2}{91})$$

$$(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187}) \cdots (1 - \frac{2}{121})(1 - \frac{2}{151})(1 - \frac{2}{181})(1 - \frac{2}{211})$$

The effective range of the above estimation formula is: $211^2 > 30 * 70 * 210 > 30 * 450$. More generally,

$$(30 * 2^k - 30)^2 - 30 * (30 * 2^{k+1} - 30) = 900 * (2^{2k} - 2^{k+2} + 2) > 0$$

$$\implies (30 * 2^k - 30)^2 > 30 * (30 * 2^{k+1} - 30)$$

where $k \geq 2$.

When $k = 2$, the inequality means $90^2 > 30 * 210$.

When $k = 3$, the inequality means $210^2 > 30 * 450$.

Therefore, the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

The actual number of prime pairs in $\{(11 + 30 * x, 13 + 30 * (449 - x)) | 210 \leq x < 450\}$ is 44. The estimation formula for the number of prime pairs in $\{(11 + 30 * x, 13 + 30 *

$(449 - x))|210 \le x < 450\}$ is $C(240)$(only the part about $\{7 + 30 * x|x \ge 0\}$ is given here):

$$(1 - \frac{2}{7})$$

$$(1 - \frac{2}{37})(1 - \frac{2}{67})$$

$$(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187})$$

$$(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})$$

It is easy to obtain the following (only the part about $\{7 + 30 * x|x \ge 0\}$ is given here):

$$\frac{C(60)}{C(30)} = 2(1 - \frac{2}{37})(1 - \frac{2}{67})$$

$$\frac{C(120)}{C(60)} = 2(1 - \frac{2}{97})(1 - \frac{2}{127})(1 - \frac{2}{157})(1 - \frac{2}{187})$$

$$\frac{C(240)}{C(120)} = 2(1 - \frac{2}{217})(1 - \frac{2}{247})(1 - \frac{2}{277})(1 - \frac{2}{307})(1 - \frac{2}{337})(1 - \frac{2}{367})(1 - \frac{2}{397})(1 - \frac{2}{427})$$

$$\cdots\cdots$$

By Lemma 3,

$$(1 - \frac{2}{217})(1 - \frac{2}{247}) > (1 - \frac{2}{97}), (1 - \frac{2}{277})(1 - \frac{2}{307}) > (1 - \frac{2}{127}), \cdots\cdots$$

Thus

$$\cdots\cdots > \frac{C(240)}{C(120)} > \frac{C(120)}{C(60)} > \frac{C(60)}{C(30)}$$

The complete $\frac{C(60)}{C(30)}$ is as follows,

$$2(1 - \frac{2}{37})(1 - \frac{2}{67})(1 - \frac{2}{41})(1 - \frac{2}{71})(1 - \frac{2}{43})(1 - \frac{2}{73})(1 - \frac{2}{47})(1 - \frac{2}{77})(1 - \frac{2}{49})(1 - \frac{2}{79})(1 - \frac{2}{53})(1 - \frac{2}{83})(1 - \frac{2}{59})(1 - \frac{2}{89})(1 - \frac{2}{61})(1 - \frac{2}{91})$$

$$= 2 \times 0.575 > 1$$

Therefore

$$\cdots\cdots > C(240) > C(120) > C(60) \approx 10.72$$

In fact, the actual number of prime pairs in $\{(11+30*x, 13+30*(929-x))|450 \le x < 930\}$ is 73, and the actual number of prime pairs in $\{(11+30*x, 13+30*(1889-x))|930 \le x < 1890\}$ is $136, \cdots\cdots$.

Similarly, we can always find another larger interval, which has more than 10.72 prime pairs (p, q), such that $p + q = 24 + 30 * (30 * 2^k - 30 - 1)$, where $k > 1$.

For any $24+30*a, a \ge 90$, if $30*2^k - 30 - 1 < a < 30*2^{k+1} - 30 - 1$, by Lemma 2, the estimation formula for the number of prime pairs in $\{(11+30*x, 13+30*(a-x))|0 \le x \le a\}$ is greater than $C(30*2^{k-1}) \ge C(60)$.

Since we can also consider the prime pairs in $\{(7+30*x, 17+30*(a-x))|0 \leq x \leq a\}$, thus there exist much more than 10 prime pairs (p,q), such that $p+q = 24+30*a$,

For example, for $24+30*99$, since $30*2^2 - 30 - 1 < 99 < 30*2^{2+1} - 30 - 1$, so the estimation formula for the number of prime pairs in $\{(11+30*x, 13+30*(99-x))|0 \leq x \leq 99\}$ is greater than $C(30*2^{2-1}) = C(60)$. In fact, the actual number of prime pairs in $\{(11+30*x, 13+30*(99-x))|0 \leq x \leq 99\}$ is 27, and the actual number of prime pairs in $\{(7+30*x, 17+30*(99-x))|0 \leq x \leq 99\}$ is 32.

It is easy to show that for $a > 0$,
$\{0 + 30*a = 7 + 30*(a-1) + 23\};$
$\{2 + 30*a = 13 + 30*(a-1) + 19\};$
$\{4 + 30*a = 11 + 30*(a-1) + 23\};$
$\{6 + 30*a = 13 + 30*(a-1) + 23\};$
$\{8 + 30*a = 19 + 30*(a-1) + 19\};$
$\{10 + 30*a = 17 + 30*(a-1) + 23\};$
$\{12 + 30*a = 19 + 30*(a-1) + 23\};$
$\{14 + 30*a = 13 + 30*(a-1) + 31\};$
$\{16 + 30*a = 17 + 30*(a-1) + 29\};$
$\{18 + 30*a = 7 + 30*a + 11\};$
$\{20 + 30*a = 7 + 30*a + 13\};$
$\{22 + 30*a = 11 + 30*a + 11\};$
$\{26 + 30*a = 13 + 30*a + 13\};$
$\{28 + 30*a = 11 + 30*a + 17\}.$

Very similar to the case of $\{24+30*a = 11+30*a+13\}$, we can prove that for any $2l + 30*a, 0 \leq l < 15, a \geq 90$, there exist more than 10 prime pairs (p,q), such that $p+q = 2l + 30*a$.

To sum up, we get the following theorem.

Theorem 2. For any even integer $m > 30*90$, there exist more than 10 prime pairs (p,q), such that $p+q = m$.

6   Triple prime conjecture

Based on Lemma 2, we consider a formula for estimating the number of primes. Consider an estimation formula for the number of primes in first 210 natural numbers $\{11 + 210*x|0 \leq x < 210\}$:

$$210(1-\frac{1}{11})(1-\frac{1}{13})(1-\frac{1}{17})(1-\frac{1}{19})(1-\frac{1}{23})(1-\frac{1}{29})(1-\frac{1}{31})(1-\frac{1}{37})(1-\frac{1}{41})(1-\frac{1}{43})$$

$$(1-\frac{1}{47})(1-\frac{1}{53})(1-\frac{1}{59})(1-\frac{1}{61})(1-\frac{1}{67})(1-\frac{1}{71})(1-\frac{1}{73})(1-\frac{1}{79})(1-\frac{1}{83})(1-\frac{1}{89})$$

$$(1-\frac{1}{97})(1-\frac{1}{101})(1-\frac{1}{103})(1-\frac{1}{107})(1-\frac{1}{109})(1-\frac{1}{113})(1-\frac{1}{127})(1-\frac{1}{131})(1-\frac{1}{137})(1-\frac{1}{139})(1-\frac{1}{149})$$

$$(1-\frac{1}{151})(1-\frac{1}{157})(1-\frac{1}{163})(1-\frac{1}{167})(1-\frac{1}{173})(1-\frac{1}{179})(1-\frac{1}{181})(1-\frac{1}{191})(1-\frac{1}{193})(1-\frac{1}{197})(1-\frac{1}{199})(1-\frac{1}{211})$$

$$\approx 95.00$$

The actual number of primes in $\{11 + 210 * x | 0 \leq x < 210\}$ is 98.

Similarly, consider an estimation formula for the number of primes in first 210 natural numbers $\{17 + 210 * x | 0 \leq x < 210\}$:

$$210(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31})(1 - \frac{1}{37})(1 - \frac{1}{41})(1 - \frac{1}{43})$$

$$(1 - \frac{1}{47})(1 - \frac{1}{53})(1 - \frac{1}{59})(1 - \frac{1}{61})(1 - \frac{1}{67})(1 - \frac{1}{71})(1 - \frac{1}{73})(1 - \frac{1}{79})(1 - \frac{1}{83})(1 - \frac{1}{89})$$

$$(1 - \frac{1}{97})(1 - \frac{1}{101})(1 - \frac{1}{103})(1 - \frac{1}{107})(1 - \frac{1}{109})(1 - \frac{1}{113})(1 - \frac{1}{127})(1 - \frac{1}{131})(1 - \frac{1}{137})(1 - \frac{1}{139})(1 - \frac{1}{149})$$

$$(1 - \frac{1}{151})(1 - \frac{1}{157})(1 - \frac{1}{163})(1 - \frac{1}{167})(1 - \frac{1}{173})(1 - \frac{1}{179})(1 - \frac{1}{181})(1 - \frac{1}{191})(1 - \frac{1}{193})(1 - \frac{1}{197})(1 - \frac{1}{199})(1 - \frac{1}{211})$$

$$\approx 95.00$$

The actual number of primes in $\{17 + 210 * x | 0 \leq x < 210\}$ is 96.

Further consider an estimation formula for the number of primes in first 210 natural numbers $\{23 + 210 * x | 0 \leq x < 210\}$:

$$210(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{19})(1 - \frac{1}{23})(1 - \frac{1}{29})(1 - \frac{1}{31})(1 - \frac{1}{37})(1 - \frac{1}{41})(1 - \frac{1}{43})$$

$$(1 - \frac{1}{47})(1 - \frac{1}{53})(1 - \frac{1}{59})(1 - \frac{1}{61})(1 - \frac{1}{67})(1 - \frac{1}{71})(1 - \frac{1}{73})(1 - \frac{1}{79})(1 - \frac{1}{83})(1 - \frac{1}{89})$$

$$(1 - \frac{1}{97})(1 - \frac{1}{101})(1 - \frac{1}{103})(1 - \frac{1}{107})(1 - \frac{1}{109})(1 - \frac{1}{113})(1 - \frac{1}{127})(1 - \frac{1}{131})(1 - \frac{1}{137})(1 - \frac{1}{139})(1 - \frac{1}{149})$$

$$(1 - \frac{1}{151})(1 - \frac{1}{157})(1 - \frac{1}{163})(1 - \frac{1}{167})(1 - \frac{1}{173})(1 - \frac{1}{179})(1 - \frac{1}{181})(1 - \frac{1}{191})(1 - \frac{1}{193})(1 - \frac{1}{197})(1 - \frac{1}{199})(1 - \frac{1}{211})$$

$$\approx 95.00$$

The actual number of primes in $\{23 + 210 * x | 0 \leq x < 210\}$ is 94.

We further consider possible form of triple primes $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | x \geq 0\}$. For example, when $x = 100$, 21011, 21017 and 21023 are all primes.

Consider an estimation formula $C(210)$ lower than the actual number of triple primes in first 210 triplets of natural numbers $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | 0 \leq x < 210\}$:

$$210(1 - \frac{3}{11})(1 - \frac{3}{13})(1 - \frac{3}{17})(1 - \frac{3}{19})(1 - \frac{3}{23})(1 - \frac{3}{29})(1 - \frac{3}{31})(1 - \frac{3}{37})(1 - \frac{3}{41})(1 - \frac{3}{43})$$

$$(1 - \frac{3}{47})(1 - \frac{3}{53})(1 - \frac{3}{59})(1 - \frac{3}{61})(1 - \frac{3}{67})(1 - \frac{3}{71})(1 - \frac{3}{73})(1 - \frac{3}{79})(1 - \frac{3}{83})(1 - \frac{3}{89})$$

$$(1 - \frac{3}{97})(1 - \frac{3}{101})(1 - \frac{3}{103})(1 - \frac{3}{107})(1 - \frac{3}{109})(1 - \frac{3}{113})(1 - \frac{3}{127})(1 - \frac{3}{131})(1 - \frac{3}{137})(1 - \frac{3}{139})(1 - \frac{3}{149})$$

$$(1 - \frac{3}{151})(1 - \frac{3}{157})(1 - \frac{3}{163})(1 - \frac{3}{167})(1 - \frac{3}{173})(1 - \frac{3}{179})(1 - \frac{3}{181})(1 - \frac{3}{191})(1 - \frac{3}{193})(1 - \frac{3}{197})(1 - \frac{3}{199})(1 - \frac{3}{211})$$

$$\approx 17.46$$

The actual number of triple primes in $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | 0 \le x < 210\}$ is 19.

Based on Lemma 1, the above estimation formula can be understood in this way (take prime number 11 as an example): for every 11 consecutive cells, there must be one cell of $11 + 210 * x_1$ can be divided by 11, another cell of $17 + 210 * x_2$ can be divided by 11 and the third cell of $23 + 210 * x_3$ can be divided by 11, where x_1, x_2, x_3 are distinct numbers. Therefore, one term in the above formula is $\frac{11-3}{11}$.

However, when $x = 23$, $11 + 210 * 23 = 4841$ can be divided by 47, $17 + 210 * 23 = 4847$ can be divided by 37, and $23 + 210 * 23 = 4853$ can be divided by 23. Therefore, this cell is counted three times, so the estimation formula $C(210)$ is lower than the actual number of triple primes.

The estimation formula for the number of triple prime numbers in $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | 210 \le x < 630\}$ is $C(420)$:

$$420(1 - \frac{3}{11})(1 - \frac{3}{13}) \cdots (1 - \frac{3}{211})$$

$$(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{223})(1 - \frac{3}{433}) \cdots (1 - \frac{3}{421})(1 - \frac{3}{631})$$

$$\approx 17.66$$

The actual number of triple prime numbers in $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | 210 \le x < 630\}$ is 22.

In the above formula, $221, 247, 253, \cdots\cdots$ may not appear in the formula because $221 = 13 \times 17, 247 = 13 \times 19, 253 = 23 \times 11 \cdots\cdots$. For the completeness of the formula, $221, 247, 253, \cdots\cdots$ are still retained, which only make the valuation further smaller.

The actual number of triple prime numbers in $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | 630 \le x < 1470\}$ is 35.

The estimation formula for the number of triple prime numbers in $\{(11 + 210 * x, 17 + 210 * x, 23 + 210 * x) | 630 \le x < 1470\}$ is $C(840)$:

$$840(1 - \frac{3}{11})(1 - \frac{3}{13}) \cdots (1 - \frac{3}{211})$$

$$(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{223})(1 - \frac{3}{433}) \cdots (1 - \frac{3}{421})(1 - \frac{3}{631})$$

$$(1 - \frac{3}{641})(1 - \frac{3}{851})(1 - \frac{3}{1061})(1 - \frac{3}{1271}) \cdots (1 - \frac{3}{841})(1 - \frac{3}{1051})(1 - \frac{3}{1261})(1 - \frac{3}{1471})$$

$$\approx 20.96$$

The effective range of the above estimation formula is: $1471^2 > 210 * 7 * 1470 > 210 * 3150$.

More generally,

$$(210 * 2^k - 210)^2 - 210 * (210 * 2^{k+1} - 210) = 210^2 * (2^{2k} - 2^{k+2} + 2) > 0$$

$$\Longrightarrow (210 * 2^k - 210)^2 > 210 * (210 * 2^{k+1} - 210)$$

where $k \ge 2$.

When $k = 2$, the inequality means $630^2 > 210 * 1470$.
When $k = 3$, the inequality means $1470^2 > 210 * 3150$.
When $k = 4$, the inequality means $3150^2 > 210 * 6510$.
$\cdots\cdots$

Therefore, the above estimation formula is applied to the front part of the effective range, which is the main reason that the above estimation formula is lower than the actual number of primes.

It is easy to obtain the following:

$$\frac{C(420)}{C(210)} = 2(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{223})(1 - \frac{3}{433})\cdots(1 - \frac{3}{421})(1 - \frac{3}{631})$$

$$\frac{C(840)}{C(420)} = 2(1 - \frac{3}{641})(1 - \frac{3}{851})(1 - \frac{3}{1061})(1 - \frac{3}{1271})\cdots(1 - \frac{3}{841})(1 - \frac{3}{1051})(1 - \frac{3}{1261})(1 - \frac{3}{1471})$$

$$\frac{C(1680)}{C(840)} = 2(1 - \frac{3}{1481})(1 - \frac{3}{1691})(1 - \frac{3}{1901})(1 - \frac{3}{2111})\cdots(1 - \frac{3}{2521})(1 - \frac{3}{2731})(1 - \frac{3}{2941})(1 - \frac{3}{3151})$$

$$\cdots\cdots$$

Similar to the proof of Lemma 3, it is easy to prove that,

$$(1 - \frac{d}{210 \times x + c}) < (1 - \frac{d}{210 \times (2x + e + 1) + c})(1 - \frac{d}{210 \times (2x + e + 2) + c})$$

$$x \geq 0, e \geq 0, 0 < c < 212, 0 < d < 210x.$$

Therefore,

$$(1 - \frac{3}{1481})(1 - \frac{3}{1691}) > (1 - \frac{3}{641}), (1 - \frac{3}{1901})(1 - \frac{3}{2111}) > (1 - \frac{3}{851}), \cdots\cdots,$$

Thus

$$\cdots\cdots > \frac{C(1680)}{C(840)} > \frac{C(840)}{C(420)} > \frac{C(420)}{C(210)}$$

Since

$$\frac{C(420)}{C(210)} = 2(1 - \frac{3}{221})(1 - \frac{3}{431})(1 - \frac{3}{223})(1 - \frac{3}{433})\cdots(1 - \frac{3}{421})(1 - \frac{3}{631})$$

$$= 2 \times 0.506 > 1$$

Therefore

$$\cdots\cdots > C(1680) > C(840) > C(420) > C(210) \approx 17.46$$

Similarly, we can always find another larger interval, which has more than 17.46 triple prime numbers. Therefore, there are infinite triple primes.

Very similarly, we can further consider possible form of triple primes $\{(17 + 210 * x, 23 + 210 * x, 29 + 210 * x)|x \geq 0\}$, $\{(31 + 210 * x, 37 + 210 * x, 43 + 210 * x)|x \geq 0\}$,

$\{(47 + 210 * x, 53 + 210 * x, 59 + 210 * x) | x \geq 0\}, \cdots\cdots$, and prove that there are infinite triple primes.

To sum up, we get the following theorem.

Theorem 3. For any triple primes $p > 7$, $q > 7$ and $r > 7$, there are infinite triple primes (p_i, q_i, r_i), such that $p_i - q_i = p - q$, $p_i - r_i = p - r$, where $i > 0$.

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Jianqin Zhou (1963-), Ph.D., Professor. He graduated from the Department of mathematics of East China Normal University in 1983 and Fudan University in 1989 with a master degree in mathematics; In 2017, he obtained Ph.D. from Department of computing in Curtin University, Australia. His main research fields are theoretical computer science, combinatorial mathematics and algorithms.

In 1989, he proved a conjecture in combinatorics proposed by the famous mathematician *Paul Erdös* (Please see the paper "A proof of Alavi conjecture on integer partition", *Acta Mathematicae Sinica*, 1995, **38(5)**636-641). Since 1989, he has published more than 120 papers in *Acta Mathematicae Sinica, Designs, Codes and Cryptography, Acta Mathematicae Applicatae Sinica, Combinatorica* and other academic journals at home and abroad.