

# A New Theory on the Derivation of Metacentric Radius Governing the Stability of Ships

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## Abstract

In this paper, we develop a new theory on the derivation of the transverse metacentric radius which governs the stability of ships.

As a new development in its derivation process, it was shown that the direction of movement of the center of buoyancy due to lateral inclination of ship is the direction of the half angle of the heel angle  $\theta$ .

By finding it, we were able to derive a metacentric radius worthy of its name by showing that the metacentric radius correctly represents the radius centered on the metacenter, which is the center of inclination.

*Keywords : Metacentric Radius, Ship's Stability, Naval Architecture,  
Half Angle of the Heel Angle, Movement of the Center of Buoyancy*

## 1. Introduction

The transverse metacentric radius  $\overline{BM}$ , which governs the stability performance of ships, can be calculated as follows, where  $V$  is the volume of underwater portion and  $I_{cl}$  is the quadratic moment about the centerline of the water plane.

$$\overline{BM} = \frac{I_{cl}}{V} \dots\dots\dots(1)$$

Here, the above equation is a well-known basic formula in naval architecture.

Eq. (1) for this  $\overline{BM}$  was derived by Bouguer<sup>(1)</sup>, and Nowacki<sup>(2)</sup> & Ferreiro<sup>(3)</sup> have introduced the historical background. It is also described by Goldberg<sup>(4)</sup> in the US "Principles of Naval Architecture", the bible of naval architecture. More recently, it has been considered by Mège and Kliava<sup>(5)</sup>. In Japan and other countries, it has been described by Takagi<sup>(6)</sup>, Nishikawa<sup>(7)</sup>, Sugihara<sup>(8)</sup>, Ohgushi<sup>(9)</sup>, Ohta & Kuwahara *et al.*<sup>(10)</sup>, and Akedo<sup>(11)</sup> in the past, and recently by Nohara & Shoji<sup>(12)</sup>, Barrass & Derrett<sup>(13)</sup>, Ikeda & Furukawa *et al.*<sup>(14)</sup> and Shin<sup>(15)</sup> in many textbooks of naval architecture and nautical mechanics.

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Although the result itself does not change with respect to such a basic formula for  $\overline{BM}$  in Eq. (1), as a new development in its derivation process, it was shown that the direction of movement of the center of buoyancy due to the lateral inclination of ship is the direction of the half angle  $\frac{\theta}{2}$  of the heel angle  $\theta$ . By finding it, we were able to derive a metacentric radius  $\overline{BM}$  suitable for its name by showing that the metacentric radius correctly represents the radius centered on the metacenter  $M$ , which is the center of inclination. The process of new derivation<sup>(16)</sup> was published in the Journal “NAVIGATION” of Japan Institute of Navigation in 2017, with the preparedness of receiving criticism from distinguished scholars.

In this paper, a new development of the derivation process<sup>(16),(17)</sup> for metacentric radius  $\overline{BM}$  is described in detail.

## 2. New Derivation of Metacentric Radius $\overline{BM}$

Fig. 1 shows a three-dimensional view of the ship when it is inclined laterally by heel angle  $\theta$  to the starboard side from upright position. The water line is  $WL$  and the center of buoyancy is  $B$  in the upright state, and the water line is  $W'L'$  and the center of buoyancy is  $B'$ <sup>(18)-(23)</sup> after inclination. The intersection point of the center line perpendicular to  $WL$  extending from  $B$  in the upright state and the action line of the buoyancy vertically upward from  $B'$  in the inclined state is the so-called “transverse metacenter”,  $M$ .

Since both hull sides of the ship can generally be assumed to be perpendicular to the water surface near

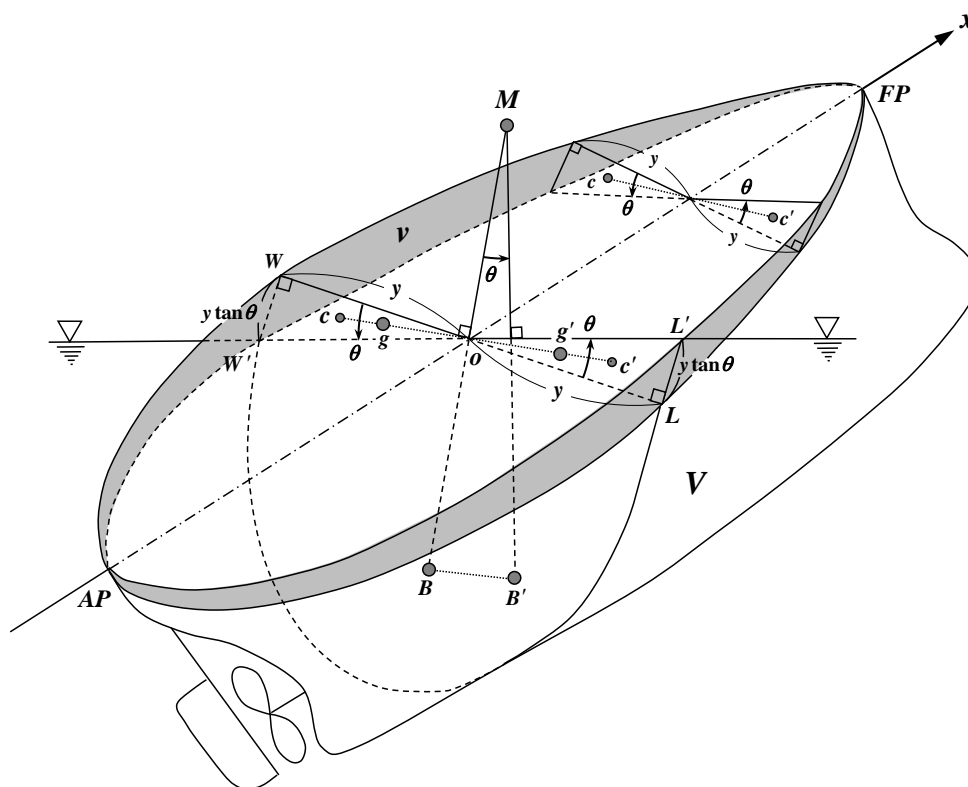


Fig. 1 Three-dimensional view of the wedge-shaped exposed and immersed portions of a laterally inclined ship.

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the water line, the exposed part  $\Delta WoW'$  and the submerged part  $\Delta LoL'$  are right triangles similar in all cross-sections from the stern  $AP$  to the bow  $FP$ , although the waterline width  $2y$  differs in the longitudinal direction  $x$ . Therefore,  $AP-WoW'-FP$  and  $AP-LoL'-FP$  are three-dimensionally wedge-shaped.

Since the volume  $V$  of ship's underwater portion remains the same after inclination, the volumes of the wedge-shaped  $AP-WoW'-FP$  in the exposed portion and the wedge-shaped  $AP-LoL'-FP$  in the immersed portion are equal. If the wedge-shaped volume is  $v$ , and the centroid of the exposed volume is  $g$  and the centroid of the immersed volume is  $g'$ , we can consider that a part of the underwater volume  $v$  has moved from  $g$  to  $g'$ .

Therefore, the direction and distance  $\overline{BB'}$  when the center of buoyancy, which is the centroid of the whole underwater volume  $V$ , moves from  $B$  to  $B'$  are determined as follows :

$$\left. \begin{aligned} \overline{BB'} &\parallel \overline{gg'} \\ \overline{BB'} &= \frac{v}{V} \cdot \overline{gg'} \end{aligned} \right\} \dots\dots\dots(2)$$

The result of Eq. (2) above is the dynamical law described in the textbooks<sup>(4),(6)-(15)</sup> of naval architecture and nautical mechanics, as in Eq.(1). In this paper, this law will be carefully explained in Appendix A-1. There, in Eq. (A-9) of its Appendix,  $A$  and  $a$  for area are replaced by  $v$  and  $V$  for volume.

**2.1 Consideration on the direction of movement  $\overline{BB'}$  of the center of buoyancy**

Fig. 2 depicts the cross-section of the laterally inclined ship shown in Fig. 1 at a certain ship's longitudinal ordinate  $x$ . Since the areas of the right triangles  $\Delta WoW'$  and  $\Delta LoL'$  in the exposed and immersed parts of the cross-section are equal, they are written as  $a$ , and their centroids of area are written as  $c$  and  $c'$  respectively. Since  $a$  and  $c, c'$  are functions of  $x$ , the volumes  $v$  of the wedge-shaped  $AP-WoW'-FP$  and  $AP-LoL'-FP$ , their moving moments  $v \cdot \overline{gg'}$ , and the direction of  $\overline{gg'}$  can be obtained by integrating from  $AP$  to  $FP$  in the longitudinal direction  $x$ , respectively, as follows :

$$\left. \begin{aligned} v &= \int_{AP}^{FP} a \, dx \\ v \cdot \overline{gg'} &= \int_{AP}^{FP} a \cdot \overline{cc'} \, dx \\ \overline{gg'} &\parallel \overline{cc'} \end{aligned} \right\} \dots\dots\dots(3)$$

Here, the line segment  $\overline{gg'}$  connecting  $g$  and  $g'$  coincides with the line segment  $\overline{cc'}$  connecting the areal centroid of the right triangles  $\Delta WoW'$  and  $\Delta LoL'$  in the cross-section, though the lengths are different, as shown in Fig. 1 and Fig. 2.

Hereafter, paying attention to the right triangle  $\Delta LoL'$  of the immersive part shown in Fig. 2, let's determine the direction of  $\overline{cc'}$  according to  $\overline{oc'}$  on starboard side. This is the core of the argument in this paper. Here, the heel angle due to lateral inclination is  $\angle LoL' = \theta$ , the angle formed by  $\overline{oc'}$  and the base  $\overline{oL}$  is  $\angle Loc' = \varphi$ , and the length of the triangular base  $\overline{oL}$  corresponding to the half width of the water line

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$WL$  is  $y$ . Here, the centroid  $c'$  of triangle  $\Delta oLL'$  is located at two-thirds of  $\overline{oL} = y$  in the base direction and one-third of  $\overline{LL'} = y \tan \theta$  in the height direction, so the tangent of  $\varphi$  is obtained as :

$$\left. \begin{aligned} \tan \varphi &= \frac{\frac{1}{3} \overline{LL'}}{\frac{2}{3} \overline{oL}} = \frac{\frac{1}{2} \overline{LL'}}{\overline{oL}} = \frac{\frac{1}{2} y \tan \theta}{y} = \frac{1}{2} \tan \theta \\ \therefore \varphi &= \tan^{-1} \left( \frac{1}{2} \tan \theta \right) \end{aligned} \right\} \dots\dots\dots(4)$$

This result of the former in the above equation means that if we extend  $\overline{oc'}$  through the centroid  $c'$  of the triangle  $\Delta oLL'$ , it will pass through the midpoint of the opposite side  $\overline{LL'} = y \tan \theta$ , which confirms what geometry teaches.

Now, if we assume that  $|\varphi| \ll 1$  and  $|\theta| \ll 1$  in the 2<sup>nd</sup>. line of Eq. (4), the angle  $\varphi$  can be Taylor-expanded with respect to  $\theta$  as follows :

$$\begin{aligned} \varphi &= \tan^{-1} \left( \frac{1}{2} \tan \theta \right) \\ &= \frac{1}{2} \tan \theta - \frac{1}{3} \left( \frac{1}{2} \tan \theta \right)^3 + \dots \\ &= \frac{1}{2} \left( \theta + \frac{\theta^3}{3} + \dots \right) - \frac{1}{24} \left( \theta + \frac{\theta^3}{3} + \dots \right)^3 + \dots \\ &= \frac{\theta}{2} + \frac{\theta^3}{8} + \dots \dots\dots(5) \end{aligned}$$

Strictly speaking,  $\varphi$  is slightly larger than  $\frac{\theta}{2}$  according to the above equation, but the following relational expression is obtained when the heel angle  $\theta$  is small to some extent, actually up to about 20°, in the range where  $W$  and  $L'$  in Fig. 2 are on both hull sides. Therefore, we find that  $\varphi$  is a half angle of  $\theta$  as follows :

$$\varphi = \angle Loc' = \frac{\theta}{2} \dots\dots\dots(6)$$

By doing so, the direction of movement of  $\overline{oc'}$ , i.e.  $\overline{cc'}$ , could be correctly determined within the range of linear theory regarding the heel angle  $\theta$  in the cross-section at longitudinal ordinate  $x$ .

Therefore, it is found from Eqs. (2), (3) and (6) that  $\overline{BB'}$  in underwater volume moves in the same direction as  $\overline{gg'}$  in wedge shape and  $\overline{cc'}$  in cross-section, as follows :

$$\angle L''BB' (= \angle Log' = \angle Loc') = \varphi = \frac{\theta}{2} \dots\dots\dots(7)$$

The conclusion of this section is that the direction  $\angle L''BB'$  of movement  $\overline{BB'}$  from the upright center of buoyancy  $B$  to the inclined center of buoyancy  $B'$  is the direction of the half angle of the heel angle  $\theta$ .

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2.2 Metacentric radius  $\overline{BM}$  in the true physical sense

Let's apply Eq. (7), which is a consequence of the previous section, to  $\triangle MBB'$  in the cross-section of the inclined ship shown in Fig. 2. Since  $\angle L''BM$  is a right angle, the angle  $\angle MBB'$  can be obtained as :

$$\angle MBB' = \angle L''BM - \angle L''BB' = \frac{\pi}{2} - \varphi = \frac{\pi}{2} - \frac{\theta}{2} \dots\dots\dots(8)$$

On the other hand, since the sum of the interior angles of a triangle is  $\pi$ , it can be written as follows :

$$\angle MBB' + \theta + \angle MB'B = \pi \dots\dots\dots(9)$$

Now, by using Eq. (8) in Eq. (9), the angle  $\angle MB'B$  can be calculated as :

$$\angle MB'B = \pi - \theta - \angle MBB' = \pi - \theta - \left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\pi}{2} - \frac{\theta}{2} \dots\dots\dots(10)$$

Therefore, since the right-hand sides of Eqs. (8) and (10) are equal, the following equality relation is obtained.

$$\angle MBB' = \angle MB'B \left( = \frac{\pi}{2} - \frac{\theta}{2} \right) \dots\dots\dots(11)$$

From this relationship, we can find that  $\triangle MBB'$  is an isosceles triangle with transverse metacenter  $M$  as its vertex. As a result, we were able to show the following relation.

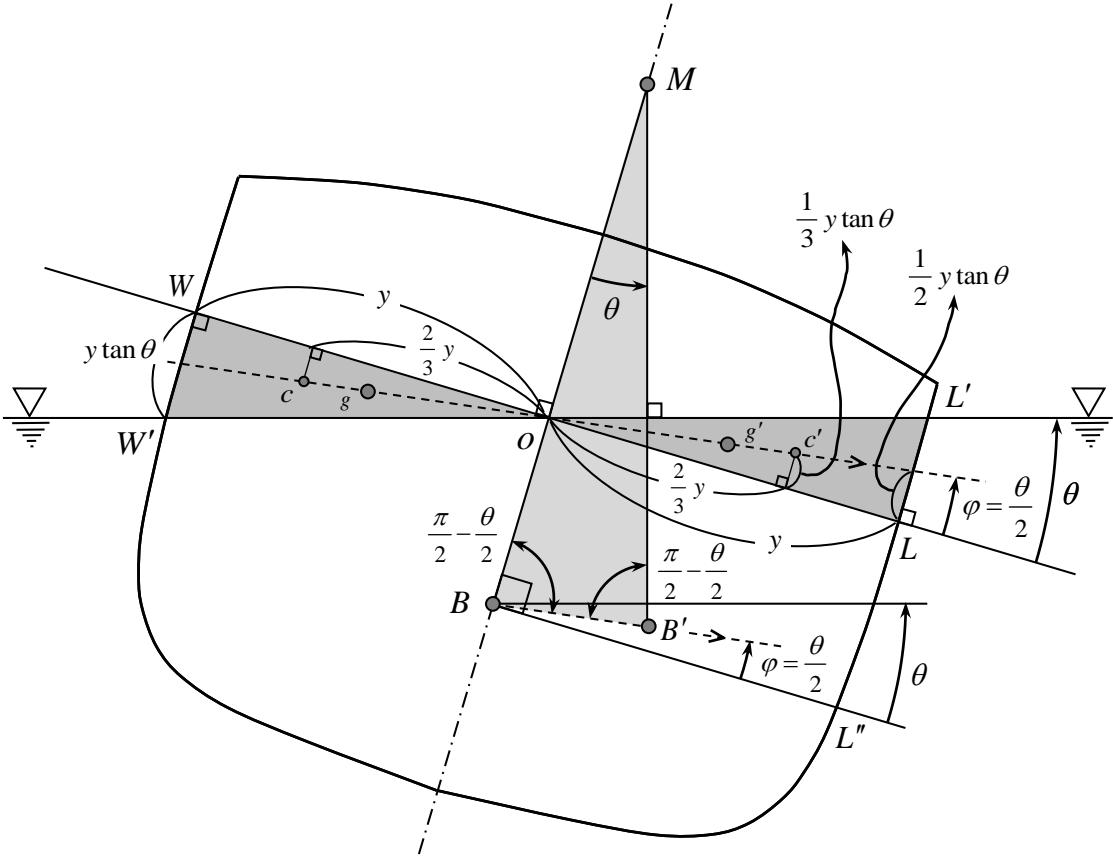


Fig. 2 Metacenter and movement of the center of buoyancy in the cross-section of a laterally inclined ship.

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$$\overline{BM} = \overline{B'M} \dots\dots\dots(12)$$

From this equality relation, it can be seen that both  $\overline{BM}$  and  $\overline{B'M}$  are geometrically the radii of the circle centered on  $M$ . In this way, we have been able to derive a metacentric radius  $\overline{BM}$  worthy of the name. We wouldn't like to think that it is self-righteousness of the authors to claim so.

**2.3 Relationship between  $\overline{BM}$  and  $\overline{BB'}$**

Let's find the moving distance  $\overline{BB'}$  of the center of buoyancy according to the explanation in the previous section. Applying the cosine theorem to the triangle  $\triangle MBB'$  shown in Fig. 2, the square of  $\overline{BB'}$  can be obtained by using Eq. (12), as follows :

$$\begin{aligned} \overline{BB'}^2 &= \overline{BM}^2 + \overline{B'M}^2 - 2 \overline{BM} \cdot \overline{B'M} \cos \theta \\ &= 2 \overline{BM}^2 (1 - \cos \theta) = 4 \overline{BM}^2 \sin^2 \frac{\theta}{2} \\ &= \overline{BM}^2 \left( \theta^2 - \frac{\theta^4}{12} + \dots \right) \dots\dots\dots(13) \end{aligned}$$

Then, by taking the square root of the above equation,  $\overline{BB'}$  can be calculated as twice the sine component of the half vertex angle  $\frac{\theta}{2}$  for the side length  $\overline{BM}$  of isosceles triangle  $\triangle MBB'$ , as follows :

$$\begin{aligned} \overline{BB'} &= \overline{BM} \sqrt{2(1 - \cos \theta)} = 2 \overline{BM} \sin \frac{\theta}{2} \\ &= \overline{BM} \left( \theta - \frac{\theta^3}{24} + \dots \right) \dots\dots\dots(14) \end{aligned}$$

Here, the bottom line of both Eqs. (13) and (14) above are the results by means of the Taylor expansion of  $\cos \theta$  or  $\sin \frac{\theta}{2}$  with respect to  $\theta$ , assuming  $|\theta| \ll 1$ .

Therefore, when the heel angle  $\theta$  is somewhat small, the moving distance  $\overline{BB'}$  of the center of buoyancy can be obtained in a simple form by using only the 1<sup>st</sup>. term in the 2<sup>nd</sup>. line of Eq. (14), as follows :

$$\overline{BB'} = \overline{BM} \theta \quad (= \widehat{BB'}) \dots\dots\dots(15)$$

Hence, the result of the above Eq. (15) shows that the line segment  $\overline{BB'}$  is equal to the arc length  $\widehat{BB'}$  with  $\overline{BM}$  as its radius, when  $\theta$  is small to some extent.

Therefore, the metacentric radius  $\overline{BM}$  can be calculated by solving Eq. (15) as follows :

$$\overline{BM} = \frac{\overline{BB'}}{\theta} \dots\dots\dots(16)$$

The above Eq. (16) shows that  $\overline{BM}$  can be determined by dividing the moving distance  $\overline{BB'}$  of the center of buoyancy by heel angle  $\theta$ .

**2.4 Moving distance  $\overline{BB'}$  of center of buoyancy**

In this section, let us consider the determination of  $\overline{BB'}$  by using the dynamical law of Eq. (2). The area  $a$  of each of the right triangles  $\triangle WoW'$  and  $\triangle LoL'$  in the cross-section shown in Fig. 2 and the line segment

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$\overline{cc'}$  connecting their centroid can be written as follows, using the important Eq. (6), where  $\varphi = \frac{\theta}{2}$ .

$$\left. \begin{aligned} a &= \frac{1}{2} y^2 \tan \theta = \frac{1}{2} y^2 \left( \theta + \frac{\theta^3}{3} + \dots \right) \\ \overline{cc'} &= 2 \overline{oc'} = \frac{4}{3} y \sec \frac{\theta}{2} = \frac{4}{3} y \left( 1 + \frac{\theta^2}{8} + \dots \right) \end{aligned} \right\} \dots\dots\dots(17)$$

Here, in the above equation, the Taylor-expanded form for  $\theta$  is also given. The moving moment  $a \cdot \overline{cc'}$  is then the product of the two in Eq. (17), and is calculated as follows :

$$\begin{aligned} a \cdot \overline{cc'} &= \frac{2}{3} y^3 \tan \theta \sec \frac{\theta}{2} \\ &= \frac{2}{3} y^3 \left( \theta + \frac{\theta^3}{3} + \dots \right) \cdot \left( 1 + \frac{\theta^2}{8} + \dots \right) = \frac{2}{3} y^3 \left( \theta + \frac{11}{24} \theta^3 + \dots \right) \dots\dots\dots(18) \end{aligned}$$

Hence, when the heel angle  $\theta$  is somewhat small, the moving moment  $a \cdot \overline{cc'}$  can be obtained by using the 1<sup>st</sup>. order term with respect to  $\theta$  in the above Eq. (18), as follows :

$$a \cdot \overline{cc'} = \frac{2}{3} y^3 \theta \dots\dots\dots(19)$$

Now, by integrating the above Eq. (19) from the stern  $AP$  to the bow  $FP$  in the longitudinal direction  $x$ , as shown in Eq. (3), the moving moment  $v \cdot \overline{gg'}$  of the wedge-shaped volume  $v$  can be calculated as follows :

$$\begin{aligned} v \cdot \overline{gg'} &= \int_{AP}^{FP} a \cdot \overline{cc'} dx = \frac{2}{3} \theta \int_{AP}^{FP} y^3 dx \\ &= \theta \int_{AP}^{FP} \frac{(2y)^3}{12} dx = I_{CL} \cdot \theta \dots\dots\dots(20) \end{aligned}$$

Here, since the integral in the above Eq. (20) corresponds to the quadratic moment of the rectangle of height  $2y$  and width  $dx$ , it represents the quadratic moment  $I_{CL}$  with respect to the center line of the water plane, as shown by the single-dotted line in Fig. 1. Therefore, the moving distance  $\overline{BB'}$  of the center of buoyancy can be determined by the latter part of the dynamical law in Eq. (2), as follows :

$$\overline{BB'} = \frac{v \cdot \overline{gg'}}{V} = \frac{I_{CL} \cdot \theta}{V} \dots\dots\dots(21)$$

The above Eq. (21) shows that the moving distance  $\overline{BB'}$  can be calculated by dividing the product of the quadratic moment  $I_{CL}$  and the heel angle  $\theta$  shown in Eq. (20) by the underwater volume  $V$  of a ship.

**2.5 Calculation formula for metacentric radius  $\overline{BM}$**

According to the results of Sections 2.3 and 2.4, the transverse metacentric radius  $\overline{BM}$  can be determined by substituting the moving distance  $\overline{BB'}$  obtained in Eq. (21) for the numerator of the right-hand side in Eq.(16), as follows :

$$\overline{BM} = \frac{\overline{BB'}}{\theta} = \frac{\frac{I_{CL} \cdot \theta}{V}}{\theta} = \frac{I_{CL}}{V} \dots\dots\dots(22)$$

The metacentric radius  $\overline{BM}$  of the above Eq. (22) can be calculated only by the geometric shape of the ship under the water plane, regardless of the heel angle  $\theta$  which cancels out the numerator and denominator. Therefore,  $\overline{BM}$  has a meaning as a parameter which governs the stability performance of a ship. The result is a well-known formula that can be found in any textbook<sup>(4),(6)-(15)</sup> of naval architecture and nautical mechanics.

**3. Some Considerations**

In this chapter, we will consider the explanations given in the textbooks so far.

In most textbooks<sup>(4),(6),(7),(9),(10),(11)</sup>, the moving direction of the center of buoyancy due to lateral inclination is approximated as follows, by assuming that heel angle  $\theta$  in Fig. 2 is tends to zero.

$$\left. \begin{aligned} \overline{BB'} \parallel \overline{WL} \\ \angle MBB' = \frac{\pi}{2} \end{aligned} \right\} \dots\dots\dots(23)$$

As a result, the moving distance  $\overline{BB'}$  of the center of buoyancy is often described as :

$$\overline{BB'} = \overline{BM} \tan \theta \dots\dots\dots(24)$$

Here, Goldberg<sup>(4)</sup>, Nishikawa<sup>(7)</sup>, Ohgushi<sup>(9)</sup>, and Akedo<sup>(11)</sup> specify the Eq. (23).

In addition, Sugihara<sup>(8)</sup>, Nohara & Shoji<sup>(12)</sup>, Barrass & Derrett<sup>(13)</sup>, and Shin<sup>(15)</sup> do not specify the direction of movement  $\overline{BB'}$ , but they write its moving distance as well as Eq. (15) in Section 2.3, as follows :

$$\overline{BB'} = \overline{BM} \theta \dots\dots\dots(25)$$

On the other hand, recent work by Ikeda & Furukawa *et al.*<sup>(14)</sup> accurately calculated the moving component parallel to  $\overline{WL}$ , not the moving distance  $\overline{BB'}$ . If we use the results of Eqs. (11) and (12) and write it in the notation of this paper, then it coincides with Eq. (14) in Section 2.3, as follows :

$$\left. \begin{aligned} \overline{BB'} \cos \frac{\theta}{2} &= \overline{B'M} \sin \theta \\ \therefore \overline{BB'} &= 2 \overline{B'M} \sin \frac{\theta}{2} = 2 \overline{BM} \sin \frac{\theta}{2} \end{aligned} \right\} \dots\dots\dots(26)$$

After all, the correct direction of movement of  $\overline{BB'}$  is still not mentioned, and the above researchers, other than the authors, derive the result by avoiding it.

**4. Summary of the Results Obtained**

It is claimed in this paper that the direction  $\angle L''BB'$  of movement  $\overline{BB'}$  from the upright center of buoyancy  $B$  to the inclined center of buoyancy  $B'$  is the direction of the half angle of the heel angle  $\theta$  due to lateral inclination as follows :



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$$\angle L''BB' (= \angle Log' = \angle Loc') = \varphi = \frac{\theta}{2} \dots\dots\dots \text{previously written (7)}$$

Here, the above equation is obtained by the moving direction  $\angle Loc'$  of a partial area from the exposed to the immersed portion, as given in Eq. (6).

As a result, we obtained the following relationship using by Eq. (7) of Section 2.2.

$$\angle MBB' = \angle MB'B \left( = \frac{\pi}{2} - \frac{\theta}{2} \right) \dots\dots\dots \text{previously written (11)}$$

By doing so, since we were able to show that  $\triangle MBB'$  shown in Fig. 2 is an isosceles triangle with metacenter  $M$  as its vertex, the following Eq. (12) was found as the radii centered on the metacenter  $M$ .

$$\overline{BM} = \overline{B'M} \dots\dots\dots \text{previously written (12)}$$

In this way, it is considered that the metacentric radius  $\overline{BM}$  suitable for the name could be derived geometrically.

As mentioned above, the conclusions of this paper can be summarized in the above Eqs. (7), (11) and (12). Subsequently, in Section 2.3 onwards, the well-known formula (22) for the metacentric radius  $\overline{BM}$  is described within the framework of the linear theory for the heel angle  $\theta$ , according to the usual method.

**5. Concluding Remarks**

One of the authors<sup>(24)</sup> has been teaching “Hydrostatics of Floating Bodies” as a compulsory subject in the Department of Naval Architecture (currently the Naval Architecture Course<sup>(25),(26)</sup>) at the *Nagasaki Institute of Applied Science* for more than ten years. Every year, especially in the last few years, I have been guilty of somewhat misrepresenting the moving direction of the center of buoyancy  $\overline{BB'}$  due to lateral inclination when explaining the theory of metacenter, which is the title of this paper. I have been lecturing on it, telling myself that it is an approximation by a minimal angle of inclination. I was always going to the lecture with reluctant heart because I was afraid of being questioned by the excellent students.

By summarizing this paper, we felt relieved from this worry. But we thought that it should not be self-righteous, so we submitted it. We are prepared to receive criticism from the great scholars who already know the theory developed in this paper and are lecturing as such. In addition, if the contents of this paper have already been published in textbooks or papers, please forgive it as a lack of searching related literature by an illiterate author.

**Acknowledgments**

In closing this paper, we would like to say the following words of thanks from the 1<sup>st</sup>. author<sup>(24)</sup>. I would like to communicate my deepest gratitude to my late teacher, *Pr. Masato KURIHARA*<sup>(27),(28),(29)</sup>, who cordially taught me the theory of “Hydrostatics of Ships” with detailed figures and formulas on the blackboard when I was a first-year undergraduate student and learned my first specialized subject of naval architecture in

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*the College of Naval Architecture of Nagasaki*. Therefore, I am following the appearance of my teacher at that time from more than 40 years ago as an exemplary example, when the 1<sup>st</sup>. author gives lectures to students on Hydrostatics of Floating Bodies<sup>(30),(31)</sup> and Theory of Ship Stability<sup>(32),(33)</sup>.

Finally, we would like to express my heartfelt gratitude to *Dr. Yoshihiro KOBAYASHI*, former professor of Sojo University and current president of Como-Techno Ltd., who always gave warm encouragement to the authors' research and recommended that this new theory should be published in English. We are greatly inspired by the vigorous academic spirit with which he writes about the results of his research in books<sup>(34),(35)</sup>.

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‡ Bold text in the list means that there is a HyperLink.

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## Appendix

### A-1. Movement of the centroid of whole area when a partial area moves

Fig.A-1 shows the case that a square  $\square ABDC$  (area  $A$ , centroid  $G$ ) transforms into an isosceles triangle  $\triangle CBE$  (area  $A$ , centroid  $G'$ ), when a right triangle  $\triangle ABC$  (gray-filled area  $a$ , centroid  $g$ ) is rotated  $90^\circ$  counterclockwise around point  $C$  and moved to a right triangle  $\triangle CDE$  (gray-filled area  $a$ , centroid  $g'$ ).

In this Appendix A-1, let's consider the distance and direction of movement of the centroid of the whole area, *i.e.*, from  $G$  of the square  $\square ABDC$  to  $G'$  of the isosceles triangle  $\triangle CBE$ . The right triangle  $\triangle CBD$  (white-filled area  $A - a$ , centroid  $o$ ) in Fig.A-1 is a fixed and common area before and after the movement. Here, the centroid  $G$  of the whole area is located geometrically on the line segment  $\overline{og}$  connecting the respective centroids  $o$  and  $g$ , and  $G'$  is located on the line segment  $\overline{og'}$  connecting  $o$  and  $g'$ .

### § 1. General theory

Firstly, we will develop the general theory without setting a specific area *etc.* .

For the square  $\square ABDC$  before the move, the following equation holds from the equilibrium of the area

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moments of  $a$  and  $A$  around point  $o$ , which is the centroid of a fixed triangle  $\triangle CBD$ .

$$\left. \begin{aligned} a \cdot \overline{og} &= A \cdot \overline{oG} \\ \rightarrow a \cdot \ell_g &= A \cdot \ell_G \end{aligned} \right\} \dots\dots\dots(A-1)$$

Here, for simplicity's sake, we have written  $\overline{og} = \ell_g$ ,  $\overline{oG} = \ell_G$ . By the above equation, the following relation is obtained as :

$$\frac{\ell_G}{\ell_g} = \frac{a}{A} \dots\dots\dots(A-2)$$

Next, for the isosceles triangle  $\triangle CBE$  after the move, the following equation holds from the equilibrium of the area moments of  $a$  and  $A$  around the point  $o$  as well.

$$\left. \begin{aligned} a \cdot \overline{og'} &= A \cdot \overline{oG'} \\ \rightarrow a \cdot \ell'_g &= A \cdot \ell'_G \end{aligned} \right\} \dots\dots\dots(A-3)$$

Here, we have abbreviated  $\overline{og'} = \ell'_g$ ,  $\overline{oG'} = \ell'_G$  in the same way. By the above equation, the following relation is obtained as well.

$$\frac{\ell'_G}{\ell'_g} = \frac{a}{A} \dots\dots\dots(A-4)$$

Let us now consider the trapezoid  $\square ABEC$ , which combines three right triangles, two before and after the move and one fixed. By Eqs. (A-2) and (A-4), the following relationship can be easily derived as :

$$\frac{\ell_G}{\ell_g} = \frac{\ell'_G}{\ell'_g} \left( = \frac{a}{A} \right) \dots\dots\dots(A-5)$$

This indicates that the scale ratio on the left side of the two small  $\triangle GoG'$  and large  $\triangle gog'$  triangles is equal to that on the right side. By transforming the above equation, we can obtain the relational equation as follows :

$$\frac{\ell_G}{\ell'_G} = \frac{\ell_g}{\ell'_g} \dots\dots\dots(A-6)$$

It shows that the ratio of the left side to the right side is the same in the two small  $\triangle GoG'$  and large  $\triangle gog'$  triangles. Furthermore, the apex angles of both small and large triangles are clearly common as follows :

$$\angle GoG' = \angle gog' \dots\dots\dots(A-7)$$

Therefore, according to Eqs. (A-6) and (A-7) above, we can see that both small and large triangles are similar as follows :

$$\triangle GoG' \sim \triangle gog' \dots\dots\dots(A-8)$$

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As a result of the above discussion, it can be seen that the ratio of  $\overline{GG'}$  to  $\overline{gg'}$ , which corresponds to the base of both triangles, is also the same as that in Eq. (A-5), and the two are parallel. It can be written as follows :

$$\left. \begin{aligned} \frac{\overline{GG'}}{\overline{gg'}} &= \frac{a}{A} \quad (<1) \rightarrow \therefore \overline{GG'} = \frac{a}{A} \cdot \overline{gg'} \\ \overline{GG'} &\parallel \overline{gg'} \end{aligned} \right\} \dots\dots\dots (A-9)$$

The above equation is the law of dynamics as described in textbooks<sup>(4),(6)-(15)</sup> on naval architecture and nautical mechanics. There is no restriction on the size of the area ratio  $a/A$  in the 1st. equation above, except that it is less than one. In this appendix, we have discussed the case where the area moves, which is the easiest to understand, but it can be applied by replacing  $a$  and  $A$  in the above Eq. (A-9) with  $v$  and  $V$  for volume and  $w$  and  $W$  for weight.

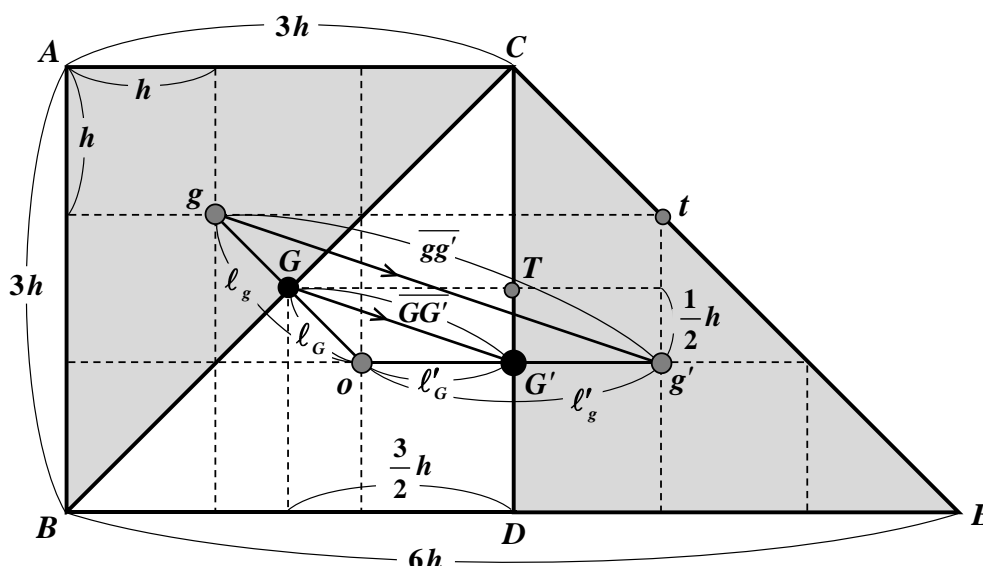


Fig. A-1 Movement of the centroid of whole area when a partial area moves.

§ 2. Numerical calculations for the verification of § 1

In this section, let's set numerical values for the area *etc.* and do some calculations. In that sense, the state of Fig. A-1 can be verified by the theory of § 1, because the position of the centroid  $G$  and  $G'$  before and after the move is geometrically known.

As shown in Fig. A-1, the square  $\square ABDC$  has a side of  $3h$  before the move and the isosceles triangle  $\triangle CBE$  has a base of  $6h$  and a height of  $3h$  after the move, the two moving right triangles  $\triangle ABC$  and  $\triangle CDE$  have a base and a height of  $3h$ . Therefore, the whole area  $A$ , the moving area  $a$  and their ratio are written as follows :

$$\left. \begin{aligned} A &= 9h^2 \\ a &= \frac{9}{2}h^2 \end{aligned} \right\} \rightarrow \frac{a}{A} = \frac{1}{2} \dots\dots\dots (A-10)$$

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Now, since the distance and direction of the movement of centroid of the whole area  $A$  due to the movement of a partial area  $a$  are shown in Eq. (A-9), we will consider the moving distance by breaking it down into its horizontal and vertical components.

As shown in Fig. A-1, each component in the moving distance of centroid of a partial area  $a$  is geometrically measured via point  $t$ , as follows :

$$\left. \begin{array}{l} \text{Horizontal} : \overline{gt} = 3h \\ \text{Vertical} : \overline{tg'} = h \end{array} \right\} \dots\dots\dots(\text{A-11})$$

Here, by the 2<sup>nd</sup>. line of Eq. (A-9), line segments  $\overline{GG'}$  and  $\overline{gg'}$  are parallel, so if we place point  $T$  corresponding to point  $t$ , both right triangles  $\Delta GTG'$  and  $\Delta gtg'$  are similar as follows :

$$\Delta GTG' \sim \Delta gtg' \dots\dots\dots(\text{A-12})$$

Therefore, the moving distance of centroid of the whole area  $A$  can be determined for horizontal and vertical direction via point  $T$  respectively, by adopting the value of Eqs. (A-10) and (A-11) into the 1<sup>st</sup>. line of Eq. (A-9), as follows :

$$\left. \begin{array}{l} \text{Horizontal} : \overline{GT} = \frac{1}{2} \overline{gt} = \frac{3}{2} h \\ \text{Vertical} : \overline{TG'} = \frac{1}{2} \overline{tg'} = \frac{1}{2} h \end{array} \right\} \dots\dots\dots(\text{A-13})$$

Then, the result of the above equation places the point  $G'$  at one-third of the height  $\overline{DC}$  of the isosceles triangle  $\Delta CBE$ , just above the midpoint  $D$  of the base  $\overline{BE}$ . This point  $G'$  is correctly the centroid of the isosceles triangle  $\Delta CBE$ . Since this fact is consistent with what geometry teaches, we were able to verify that Eq. (A-9), which is derived in the general theory of §1, is correct.

**A-2. Lecture videos on a new derivation of metacentric radius  $\overline{BM}$**

The theme of this paper, in which a new derivation process for metacentric radius  $\overline{BM}$  is developed, is lectured to second-year students of the naval architecture course<sup>(25),(26)</sup> as a subject of “Theory of Ship Stability” at the university where the one of the authors<sup>(24)</sup> works.

With the recent trend of remote lectures, the situation above is filmed in two parts, the 1<sup>st</sup>. half<sup>(30)</sup> and the 2<sup>nd</sup>. half<sup>(31)</sup>, and on-demand teaching materials are created and uploaded as YouTube videos. The explanation is in Japanese, but if you are interested, please have a look.

**A-3. Introducing examples and videos on the stability theory of ships**

One of the authors<sup>(24)</sup> has published two typical examples and three videos on the stability theory of ships based on the metacentric radius, which is the main theme of this paper. We will introduce them in this Appendix A-3.



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First, the basic example of determining the conditions under which a columnar ship with rectangular cross-section, made of homogeneous squared timber with arbitrary breadth and arbitrary material, floats stably in the upright position, was explained in the journal<sup>(36)</sup> “*NAVIGATION*” of *JIN* (Japan Institute of Navigation) and in the pre-print repository<sup>(37)</sup> “*viXra.org*”.

As a concrete example, when the specific weight of a ship is about half that of water, the theory of finding a stable breadth condition in the upright state is lectured at the university<sup>(25),(26)</sup> where the author<sup>(24)</sup> works. The lecture video<sup>(32)</sup> and the shot of the confirmation experiment<sup>(38)</sup> in the small water tank have been uploaded to YouTube as on-demand teaching materials.

In addition, we have also uploaded a lecture video<sup>(33)</sup> explaining about materials (*i.e.*, lighter or heavier than the specific weight of a timber) which are stable in the upright state for the ship with square cross-section.

Next, the solution to the problem of finding the stable attitude of a similar columnar ship, which does not satisfy the above conditions and floats in the inclined position, is also explained in the same journal of *JIN* as an advanced example<sup>(39)</sup>.

Both the explanatory treatises and the lecture videos are explained in Japanese, but if you are interested, please take a look and read them.

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