An Elementary Proof of the Riemann Hypothesis

Jose Risomar Sousa

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Abstract

The Riemann hypothesis is true. In this paper I present a solution for it in a very short and condensed way, making use of one of its equivalent problems. But as Carl Sagan once famously said, extraordinary claims require extraordinary evidence. The evidence here is the newly discovered inversion formula for Dirichlet series.

1 Introduction

The Riemann hypothesis is the second of the 7 Millennium Problems proposed by the Clay Mathematics Institute back in year 2000, and it has a 1 million dollar prize for whoever solves it.

It was formulated by Bernhard Riemann in 1859, and has been unsolved for more than 150 years now. The Riemann hypothesis is about the zeta function, $\zeta(s)$, a special case of Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ if } \Re(s) > 1 \tag{1}$$

A Dirichlet series is any infinite sum of the type,

$$F_a(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where $a(n)$ is an arithmetic function and $F_a(s)$ is its generating function (hence, the zeta function has $a(n) = 1$).

The analytic continuation of the zeta function to $0 < \Re(s) < 1$ (the so-called critical strip) can be attained by means of the alternating zeta function (also known as the Dirichlet eta function). The zeta function can also be continued to $\Re(s) < 0$, by means of the Riemann functional equation, which pairs $\zeta(s)$ up with $\zeta(1-s)$, therefore enabling points with $\Re(s) > 1$ to complete the function for points with $\Re(s) < 0$. Finally, the Riemann hypothesis is the statement that all the non-trivial zeros of the function thus obtained have $\Re(s) = 1/2$.

This paper is not meant to be a definitive proof of the Riemann hypothesis, unless vetted by an expert, but it definitely adds novelties to the picture, besides being a really interesting exposition. At worst, the new facts presented here represent evidence in favor of the Riemann hypothesis, even if some details may be missing.

In this article, we try and make a distinction between the zeta function's Dirichlet series and the function itself.

2 The background

It stems from the Euler product that when $\Re(s) > 1$, $1/\sum 1/n^s$ is a Dirichlet series that has $a(n) = \mu(n)$, the so-called Möbius function. The Riemann hypothesis is then equivalent to a statement that can be made about this new Dirichlet series and the reciprocal of the zeta function in the upper half portion of the critical strip, $\Re(s) > 1/2$. But let's go over some of the details first.

2.1 The Euler product

In 1737, German mathematician Euler discovered an interesting relationship between the zeta function and the primes known as Euler product, valid when $\Re(s) > 1$:

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

When (2) is inverted, it reveals a relationship between the reciprocal of the zeta function and the square-free numbers:

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

The observation of this relation allows us to write $1/\zeta(s)$ as function of $\mu(n)$, a function introduced in 1832 by another German mathematician, August F. Möbius, which involves the concept of square-free numbers. That is, provided that $\Re(s) > 1$.

2.2 The Möbius function

A square-free number is a number that can't be divided by any squared prime. In other words, if n is square-free, $p_1 p_2 \cdots p_k$ is its unique prime decomposition. Hence, we can define a function $\mu(n)$ such that:

$$\mu(n) = \begin{cases} 1, & \text{if } n\text{=}1 \\ (-1)^k, & \text{if } n \text{ is square-free with } k \text{ prime factors} \\ 0, & \text{if } n \text{ is not square-free} \end{cases}$$

This is the Möbius function from the previous section.

Now it's easy to write the reciprocal of the zeta function's Dirichlet series, which also happens to be a Dirichlet series. By alternating the signs of the terms in the zeta series (and also eliminating most of them, the square-full), the result is guaranteed to converge even $s = 1$, probably for $\Re s > 1/2$ and possibly for $\Re s > 0$:

$$1/\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{3}$$

At this point, this expression is only valid if $\Re(s) > 1$. It can be proved through the Euler product, and can be rewritten simply as $1/\zeta(s)$.

However, unlike the zeta function's series, this new Dirichlet series should converge for $\Re(s) > 0$, and per the literature, a certain statement about it and the zeta function is equivalent to the Riemann hypothesis.

2.3 A restatement of the Riemann hypothesis

The Riemann hypothesis is equivalent to the statement that the equation,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{4}$$

is valid for every s with real part greater than $1/2$.

3 The solution

Let's prove that equation (4) holds for $\Re(s) > 1/2$, assuming that the right-hand side converges for $\Re(s) > 1/2$, so that at worst the Riemann hypothesis would be equivalent to the statement that the right-hand side of (4) converges for $\Re(s) > 1/2$.

In other words, what this proof really establishes is the validity of equation (4) for every s with $\Re(s) > 1/2$, even if $\zeta(s) = 0$ for some such s (which would mean the Riemann hypothesis is false). In that case, however, the equation would still hold in a sense, and the series on the right-hand side would diverge. The Riemann hypothesis therefore implies that the right-hand side of (4) never diverges for $\Re(s) > 1/2$.

Proving convergence of the series in (4) might not be trivial. For example, even though the zeta function's series converges for $\Re(s) > 1$, this new series should not converge when s is a non-trivial zero of the zeta function (it's a total mystery how a Dirichlet series can have two values with the same real part, $\Re(s) = 1/2$, but one converges and another doesn't). Besides, unlike the zeta's series, which always converges starting at a set of points, $\Re(s) > 1$, this one doesn't converge always starting at $\Re(s) > 0$ (assuming that it converges for $\Re(s) > 0$).

Intuitively we would think it's easy to see why the series should always converge: the sum of the square-free numbers with an odd number of primes (raised to s) should always be an infinity more or less equal to the sum of those with an even number of primes and these infinities should cancel out. As a matter of fact, these infinities will be exactly equal when $s = 1$ (which is a pole in the zeta function), causing (4) to zero out. However, this may not be as simple, since we know the series should not converge at the non-trivial zeta zeros.

3.1 Inversion formula for Dirichlet series

Before laying out the points of the proof, let's review the inversion formula for Dirichlet series, whose proof and ubiquitous evidence for were given in paper [1]. It's a proven theorem that relates a Dirichlet series to its arithmetic function, and can be stated as follows.

Theorem Suppose that $F_a(s)$ is a Dirichlet series and $a(n)$ is its associated arithmetic function. Then for any positive integer q such that $F_a(2q) < \infty$, $a(n)$ is given by:

$$a(n) = -2\sum_{i=q}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=q}^{i}\frac{(-1)^j(2\pi)^{-2j}F_a(2j)}{(2i+1-2j)!}$$

Proof Although not obvious, this is a very powerful result. The above power series converges for all n and is the analytic continuation of:

$$-\frac{\sin 2\pi n}{\pi n}\sum_{j=q}^{\infty} n^{2j}F_a(2j), \tag{5}$$

since they have the same Taylor series expansion and (5) only converges for $|n| < 1$. In some cases it's possible to find a closed-form for $a(n)$, though it can be challenging.

The proof is short and simple:

$$-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}F_a(2j)}{(2i+1-2j)!} \Rightarrow$$

$$-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\sum_{k=1}^{\infty}\frac{a(k)}{k^{2j}} \Rightarrow$$

$$\sum_{k=1}^{\infty}a(k)\left(-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}k^{-2j}}{(2i+1-2j)!}\right)$$

The theorem then follows from the following equation:

$$-2\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi k)^{-2j}}{(2i+1-2j)!} = \begin{cases}1, & \text{if } n = k\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

And the above equation is justified for being the convolution of $\mu_0(n)$, the unit function, and the arithmetic function of the series k^{-s}, $b(n)$, since from the convolution formula:

$$c(n) = (\mu_0 * b)(n) = \sum_{d|n} \mu_0(d) b\left(\frac{n}{d}\right) = b(n) = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{otherwise} \end{cases}$$

For more details on the proof, please refer to [1]. □

This is a real breakthrough and a surprising result. It says that every Dirichlet series has coefficients given by a Taylor series. One of the advantages of this formula is that if you know $F_a(s)$ at the even integers, you know the coefficients of its series expansion. Another advantage is that it extends $a(n)$ to the complex numbers. Perhaps this extended function even has some deeper connection to the Dirichlet series, however that's not currently known.

The inversion formula can also be used to check if a function is a Dirichlet series (if at the integers the $a(n)$ are finite and not all zero, the answer is yes).

3.2 Riemann hypothesis proof

With everything that's been said, we're now in a position to produce a short proof.

Suppose that for $\Re(s) > 0$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{7}$$

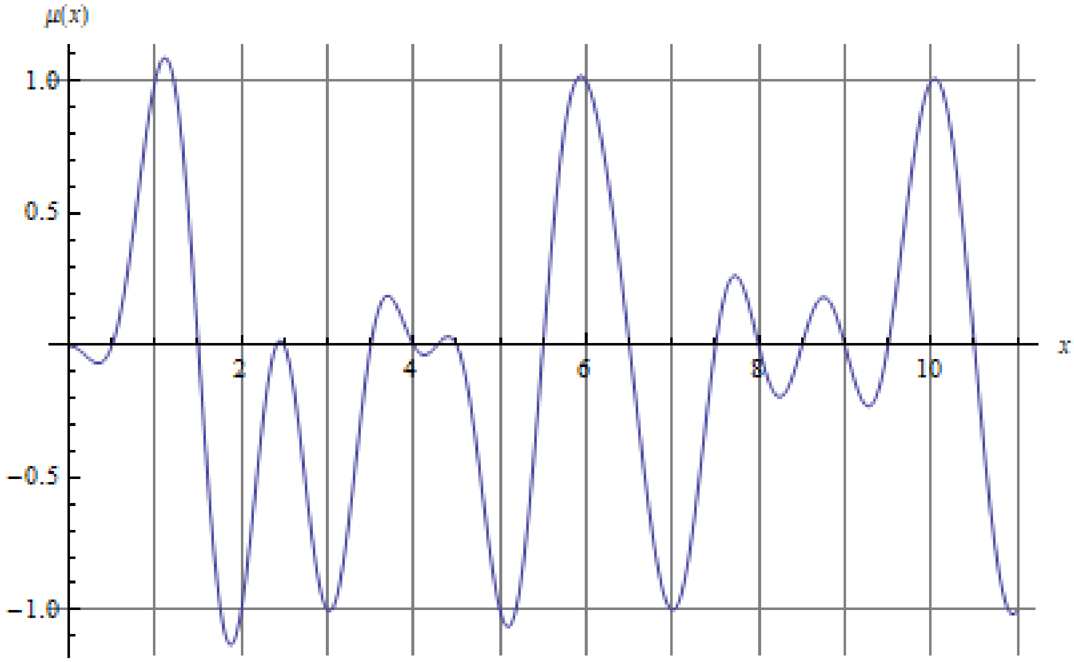
What is $a(n)$?

The inversion formula tells us that $a(n)$ is given by the below power series:

$$a(n) = -2 \sum_{i=1}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=1}^{i} \frac{(-1)^j (2\pi)^{-2j} \zeta(2j)^{-1}}{(2i+1-2j)!} \tag{8}$$

Despite the fact that this formula yields the coefficients of the Dirichlet series $1/\zeta(s)$ independently of its convergence domain, it has no known closed-form and hence in principle we don't know what values it assumes at the positive integers. However, since for $\Re(s) > 1$ we know from the Euler product that $a(n) = \mu(n)$, and (8) doesn't depend on s, we conclude that $a(n) = \mu(n)$ for every positive integer n.

Below the graph of $\mu(n)$ was plotted in the $(0, 11)$ interval for some insight into its shape and local minima and maxima (it crosses the x-axis at the square-full and half-integers):



Prior to this discovery nobody even knew that the Möbius function could be continuous or analytic, or what its shape could look like. Therefore, there is no question that equation (4) holds always, the only possible question is if its right-hand side converges for $\Re(s) > 0$. Intuitively we know it should, though convergence must be very slow as s approaches 0.

Notice formula (8) doesn't care about the analytically continued values of the zeta function, they're not used.

4 Conclusion

The equivalent problem shown in this paper is a result from the literature, some mathematician has found a way to prove that it implies the Riemann hypothesis. The missing piece of the puzzle was to come up with an alternative way to prove that,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \Rightarrow a(n) = \mu(n),$$

without the requirement that $\Re(s) > 1$, which the inversion formula for Dirichlet series does.

References

[1] Risomar Sousa, Jose *An exact formula for the prime counting function, eprint arXiv:1905.09818,* 2019.