Bifurcations and the Gauge Structure of the Standard Model

Ervin Goldfain

Ronin Institute, Montclair, New Jersey 07043, USA

E-mail ervin.goldfain@ronininstitute.org

Abstract

As of today, the reason for the unique composition of the Standard Model (SM) gauge group $SU(3)_c \times SU(2)_L \times U(1)$ remains an open question. Taking complex-scalar field theory as benchmark model, we argue here that the SM group unfolds sequentially from the flow of the self-interaction coupling. Numerical estimates are found to be reasonably consistent with experimental data.

Key words: Bifurcations, Feigenbaum route to chaos, complex-scalar field theory, gauge symmetries, Standard Model, electroweak model.

1. Introduction

By construction, SM is a non-Abelian gauge field theory built from the symmetry group $SU(3)_c \times SU(2)_L \times U(1)$. This group has $8+3+1=12$

generators with a non-trivial commutator algebra and describes the electroweak (EW) coupling of leptons and quarks, as well as the strong interaction of quarks and gluons. [1-2]. SM contains an octet of gluons associated with the *SU* (3) color generators, and a quartet of EW gauge bosons W^+, W^-, Z^0, and γ. Gluons and the photon γ are massless because the symmetry induced by the other three generators is spontaneously broken. *Spontaneous symmetry breaking* (SSB) takes place in the EW sector and is characterized by the following attributes [2]

a) It occurs when there is a family of degenerate vacua transforming onto one another under the action of the gauge group. The symmetry is spontaneously broken as the system eventually settles in one of its vacua.

b) SSB is enabled in Quantum Field Theory because the latter has an unbounded number of degrees of freedom prone to undergo vacuum tunneling.

c) Since gauge symmetry is local and spontaneously broken, the associated Goldstone bosons morph into the third polarization of the W^+, W^-, Z^0 bosons, rendering them massive in the process.

The goal of this work is to offer an alternative scenario to the standard SSB interpretation, approximately matching its content and predictions. Our scenario stems from the theory of bifurcations applied to classical scalar field theory. We find that, unlike the SSB model, massive electroweak bosons do not arise from the absorption of Goldstone bosons, but from the *geometry of the bifurcation process*, along with *boson condensation* induced by the minimal fractality of spacetime above the EW scale. A prerequisite of this scenario is the mechanism of *decoherence*, which acts near the EW scale and drives the transition from quantum to classical behavior [3]. Appealing to the complex-scalar field theory as benchmark model, we speculate that the SM group unfolds sequentially from the flow of the self-interaction coupling. Numerical estimates are found to be reasonably consistent with experimental data.

The paper is partitioned in the following way: next section elaborates on the relationship between complex-scalar field theory and the global $U(1)$ symmetry; the cubic map representation of field dynamics is derived in section three, while the bifurcation analysis of the map is developed in the next couple of sections; Section six displays a comparison between numerical estimates and experimental data; Concluding remarks are detailed in the last section.

The reader is urged to keep in mind that this work is strictly introductory in nature. It is a sequel to our previous contributions and requires independent validation, rebuttal, or further refinements.

2. Free complex-scalar fields and the *U (1)* symmetry

Consider a classical complex-scalar field described by the pair of independent components

$$\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\,\Phi_2) \tag{1a}$$

$$\Phi^* = \frac{1}{\sqrt{2}}(\Phi_1 - i\,\Phi_2) \tag{1b}$$

One associates to (1) the massive Lagrangian

$$L = \partial_\mu \Phi\, \partial^\mu \Phi^* - m^2 \Phi^* \Phi \tag{2}$$

which is invariant under the global gauge transformation

$$\Phi \to \exp(-i\eta)\Phi \tag{3a}$$

$$\Phi^* \to \exp(i\eta)\,\Phi^* \tag{3b}$$

where η is a real constant. The conserved current induced by (3) is given by

$$J^\nu = i\,(\Phi^* \partial^\nu \Phi - \Phi\, \partial^\nu \Phi^*) \tag{4a}$$

with the vanishing four-divergence

$$\partial_\nu J^\nu = 0 \tag{4b}$$

and conserved charge

$$Q = \int J^0 d^3x = i \int (\Phi^* \frac{\partial \Phi}{\partial t} - \Phi \frac{\partial \Phi^*}{\partial t})\, d^3x\,; \quad \frac{dQ}{dt} = 0 \tag{5}$$

Since $\exp(i\eta)$ represents a unitary "matrix" in one dimension, that is,

$$\exp(i\eta)[\exp(i\eta)]^+ = 1 \tag{6a}$$

the gauge transformation (3) amounts to a rotation in field space

$$\Phi_1^r = \Phi_1 \cos\eta + \Phi_2 \sin\eta \tag{6b}$$

$$\Phi_2^r = -\Phi_1 \sin\eta + \Phi_2 \cos\eta \tag{6c}$$

which is representative for the symmetry group $U(1)$.

It follows from this analysis that gauge invariant complex-scalar field theory inherently carries a *global* $U(1)$ charge. Moreover, demanding invariance of (2) under a *local* gauge transformation, gives rise to the electromagnetic field and its conserved charge Q [4].

3. Self-interacting scalar fields as cubic maps

The goal of this section is to explore the dynamics of self-interacting field theory in connection to the flow of the Higgs scalar with the energy scale. To this end, we start from the Lagrangian

$$L_c = \partial^\mu \Phi \, \partial_\mu \Phi^* - V(\Phi\Phi^*) \tag{7}$$

in which the potential function assumes the form [1-2]

$$V(\varphi) = \lambda(|\varphi|^2 - \frac{1}{2}\mathrm{v}^2) \tag{8}$$

and where

$$|\varphi|^2 = \Phi\Phi^* \tag{9}$$

For simplicity, we omit below the modulus notation and write

$$|\varphi| \to \varphi \tag{10}$$

The flow of (10) with the energy scale μ is given by

$$\dot{\varphi} = \mu \frac{d\varphi}{d\mu} = \frac{d\varphi}{d(\log\mu)} = \frac{d\varphi}{d\tau} \tag{11}$$

which yields

$$\dot{\varphi} = -\frac{\partial V(\varphi)}{\partial \varphi} = 2\lambda\varphi(\mathrm{v}^2 - 2\varphi^3) \tag{12}$$

(12) may be rendered in a more familiar form through the substitution

$$y = \frac{\sqrt{2}}{\mathrm{v}}\varphi \tag{13}$$

Since φ and v have mass dimension $[\varphi]=[\mathrm{v}]=M$, dimensional consistency requires passing to the normalized control parameter

$$m=\frac{2\lambda \mathrm{v}^2}{m_0^2} \tag{14a}$$

where m_0 is an arbitrary reference mass. (14a) runs with the energy scale as in

$$m=m(\tau);\ \lambda=\lambda(\tau);\ \ m_0=m_0(\tau) \tag{14b}$$

One finds that (12) reduces to the equation of a *classical cubic oscillator*

$$\dot{y}=my(1-y^2) \tag{15}$$

The map analog of (15) around the origin $y=0$ may be presented as (Appendix A)

$$\boxed{y_{n+1}=f(m,y_n)=my_n(1-y_n^2)} \tag{16}$$

where the ranges of the control parameter and the y variable are set to, respectively,

$$0 \leq m \leq 3; \quad -1 \leq y \leq 1 \tag{17}$$

4. Working assumptions

A1) The flow (16) evolves in non-equilibrium conditions and (at least in principle) is incompatible with the perturbative Renormalization Group, where quantum fluctuations are present and the flow equations asymptotically settle on fixed points [8, 11, 14].

A2) Non-equilibrium regime near or above the EW scale implies *statistical behavior* in the classical sense, as well as the onset of *decoherence* triggered by chaotic mixing and diffusion [3]. Obviously, a more realistic setup must account for effects that are absent from (16), such as, a) random perturbations with short or long correlations, and b) the *self-interacting* nature of electroweak bosons.

A3) Following the ideas of [6-8, 12], *topological condensation* reflects the confining behavior of spacetime endowed with minimal fractality. This condensation mechanism is similar (but not identical) to the Anderson localization of quantum waves in random potentials. It leads to the formation of weakly coupled clusters of scalar or vector bosons.

A4) The bifurcation process is associated with broken symmetries defining critical phenomena [15-16]. In our context, symmetry breaking is a combined outcome of nonlinearity and fast bifurcations, commensurate in duration with the EW scale [10]. We posit below that the upper bifurcation branches contain exclusively boson condensates, whereas lower branches nearly free boson states.

A5) For simplicity, we assume that $\lambda(\tau)$ runs much faster than $m_0(\tau)$, that is, $\dot{\lambda} >> \dot{m}_0$. While this assumption is unwarranted in real life, it serves here as a convenient approximation.

5. Bifurcation analysis

The fixed points of (16) are determined by

$$y_n^* = f(m, y_n^*) \tag{18}$$

leading to a trivial and a pair of symmetric solutions, i.e.

$$y^* \Rightarrow 0, \pm\sqrt{1-\frac{1}{m}} \tag{19}$$

The fixed point $y^* = 0$ is attracting (stable) for $m < 1$ and repelling (unstable) for $m > 1$. It can be also shown that the fixed points $y^* = \pm\sqrt{1-\frac{1}{m}}$ are both stable for $1 < m < 2$ and unstable for $2 < m < 3$ [5]. The bifurcation diagram of (16) (pictured below) displays the progressive generation of its critical points $y^*(m)$ under the flow of $m = m(\tau)$.

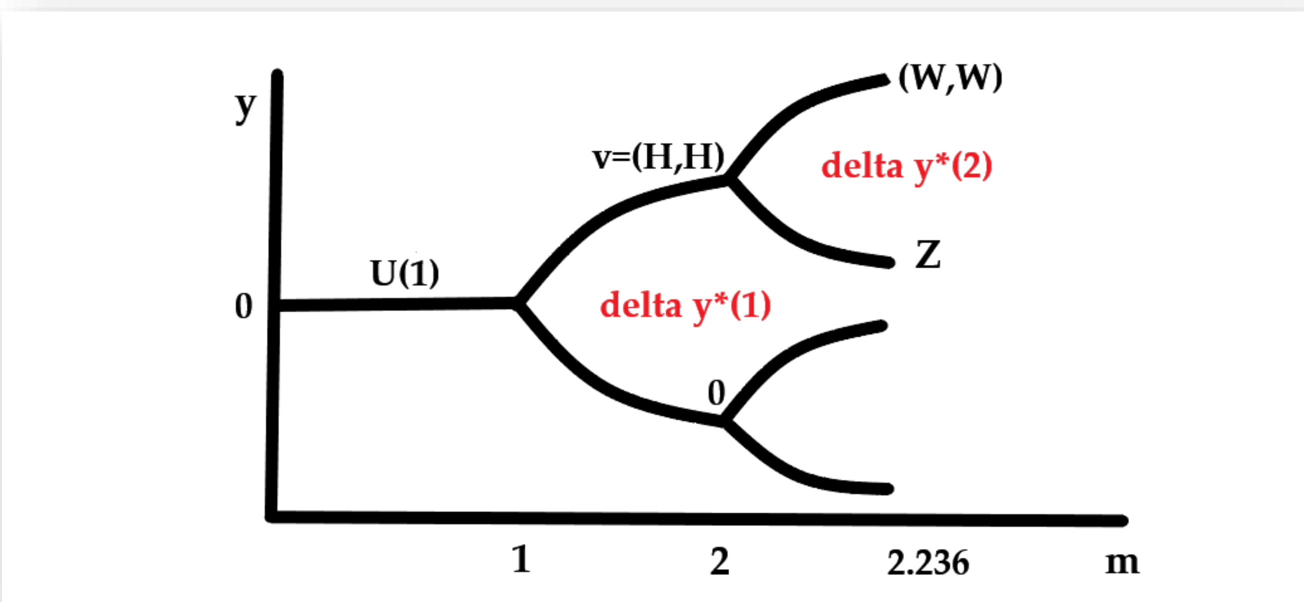

Fig. 1: Bifurcation diagram of the cubic map (16)

In what follows, bifurcation vertices are indexed in natural progression $V_i, (i=1,2,...)$. The lower and upper branches are denoted using different superscripts, respectively, as in $V_i^j,\ j=1,2,...$.

As the diagram indicates, the first bifurcation occurs at

$$m(V_1)=1;\ \ y^*(V_1)=0 \tag{20a}$$

It is seen that (20a) recovers the SM Higgs mass in the form [1-2]

$$\boxed{m_0^2 = m_H^2 = 2\lambda \mathrm{v}^2} \tag{20b}$$

The second bifurcation occurs at

$$m(V_2)=2;\ \ y^*(V_2^1)=\sqrt{1-\frac{1}{2}};\ \ y^*(V_2^2)=-\sqrt{1-\frac{1}{2}} \tag{21}$$

The separation between the two branches amounts to

$$\Delta y^*(1) = y^*(V_2^1)-y^*(V_2^2)=\sqrt{2} \tag{22}$$

By (13), the critical point at $m(V_2)=2$ expressed in terms of the scalar field is given by

$$\Delta\varphi^*(1) = \mathrm{v} \tag{23a}$$

Relation (23) is consistent with the standard SSB mechanism, whereby symmetry breaking implies picking a preferential direction in $SU(2)$ space corresponding to the Higgs vacuum [1-2, 7]. By A3) and A4), the vertex V_2^1 contains a scalar condensate that we choose to identify with a *weakly coupled Higgs doublet*. In symbolic form we write

$$\mathrm{v} \Rightarrow (H, H) \tag{23b}$$

The next bifurcation develops at [5]

$$m(V_3) = 2.236 \tag{24}$$

and generates a set of four critical points as in

$$y^*(V_3^1) = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{m^2}}}\,; \quad y^*(V_3^2) = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{m^2}}} \tag{25a}$$

$$y^*(V_3^3) = -y^*(V_3^2)\,; \quad y^*(V_3^4) = -y^*(V_3^1) \tag{25b}$$

Using again A3) and A4), we choose to identify the condensate at V_3^1 with the pair of vector bosons (W^+, W^-). The separation between the two branches at this vertex is

$$\Delta y^*(2) = y^*(V_3^1) - y^*(V_3^2) = 0.325 \tag{27}$$

The ratio between (22) and (27) amounts to

$$\chi = \frac{\Delta y^*(2)}{\Delta y^*(1)} = 0.22981 \tag{28}$$

which, by (23a), yields

$$\frac{\Delta\varphi^*(2)}{\mathrm{v}} = \chi \tag{29}$$

Summarizing the results of this section and on account of A3) and A4), we are led to suggest the following mass relationships

$$\mathrm{v} \approx 2m_H \tag{30a}$$

$$\mathrm{v} \approx 2m_W + m_Z \tag{30b}$$

$$\Delta\varphi^*(2) = \chi\,\mathrm{v} \approx 2m_W - m_Z \tag{30c}$$

6. Estimates versus existing data

The aim of this section is to compare (30a) - (30c) against existing theoretical and experimental data. To this end, we choose to define the following set of *mass errors* (E) and *normalized mass errors* (e) as in

$$E_H = \mathrm{v} - 2m_H \tag{31a}$$

$$E_{WZ} = \mathrm{v} - (2m_W + m_Z) \tag{31b}$$

$$E_\Delta = (2m_W - m_Z) - \chi \mathrm{v} \tag{31c}$$

$$e_i = \frac{E_i}{\mathrm{v}_{SM}}; \quad i = \{H, WZ, \Delta\} \tag{31d}$$

(31d) is built under the assumption that the SM vacuum (v_{SM}) sets the *natural scale* of SSB in the EW sector. Results are displayed below based on the following SM input parameters (in GeV):

$$\mathrm{v}_{SM} = 246$$

$$m_W = 80.385; \quad m_Z = 91.1876$$

$$m_H = 125.35$$

$e_H\,(\%)$	1.91
$e_{WZ}\,(\%)$	2.42
$e_\Delta(\%)$	5.28

Tab. 1: Mass errors normalized to the SM vacuum

7. Conclusions and follow-up challenges

…Text to follow…

APPENDIX A: Derivation of the cubic map

The map representation of the cubic oscillator equation

$$\dot{y} = my(1-y^2) \tag{A1}$$

is obtained from discretization of (A1), which gives

$$y_{n+1} = y_n + \tau_0\, m y_n (1-y_n^2) \tag{A2}$$

where τ_0 is the "time-like" step associated with (11). Carrying out the substitution

$$\tau_0 m = \frac{2\tau_0 \lambda v^2}{m_0^2} = \frac{2\lambda' v^2}{m_0^2} \tag{A3}$$

leads to the cubic map (16) under the following assumed constraints

$$\left| \frac{2\lambda' v^2}{m_0^2}(1 - y_n^2) \right| >> 1; \quad |y_n| << 1 \tag{A4}$$

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