An Easy Proof of the Triangle Inequality

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Abstract

High school and undergraduate algebra and calculus textbooks don't provide a fast and easy proof of the triangle inequality. Here is a proof that seems relatively easy. It does require a little bit of logic, but that can be a plus.

Introduction

A perusal of typical high school, say College Algebra level textbooks [1], as well as undergraduate Calculus textbooks [2, 4] reveals that these books don't give a proof of the triangle inequality. The calculus texts do give a proof of the inequality in their chapters on vectors. These proofs do imply the simpler $\mathbb{R}$ case. The proofs are involved, requiring the Schwarz inequality. In Thomas's book he actually, one could say, apologizes to the reader for the length and complexity of his $\mathbb{R}^k$ proof. Curiously, the law of cosines does imply the triangle inequality, but this is not mentioned in any of these textbooks. Wikipedia doesn't give a simple proof either.

Rudin in his analysis text [3] does give a proof for the complex, hence the real, case; it is relatively brief and simple. College algebra texts covering complex numbers could include such a proof. It's a nice application of the conjugate of a complex number.

The following proof seems to make concrete the intuitions of why the triangle inequality holds in the specific real case: if x and y have opposite signs, their sum is reduced from the sum of their absolute values.

Proof

Theorem 1. *For real numbers x and y*

$$|x+y| \leq |x| + |y|. \tag{1}$$

Proof. If and only if x and y are both positive or both negative then

$$\frac{|x+y|}{|x|} > 1 \text{ and } \frac{|x+y|}{|y|} > 1. \tag{2}$$

This follows as

$$|x+y| = |x| + |y| \tag{3}$$

in these cases and in turn this implies

$$\frac{|x+y|}{|x|} = 1 + \frac{|y|}{|x|}$$

and this is true if and only if

$$\frac{|x+y|}{|x|} < 1.$$

By symmetry, the RHS of (2) also holds.

So one case is resolved; (3) implies (1).

Take the negation of (2); this is

$$\frac{|x+y|}{|x|} \leq 1 \text{ or } \frac{|x+y|}{|y|} \leq 1. \tag{4}$$

Say the former, then

$$|x+y| \leq |x| \text{ implies } |x+y| \leq |x| + |y|$$

and that's (1) as well. Mutatis mutandis, the right hand side of (4) also implies (1).

If one or more of x and y are 0, the theorem holds. So all cases have been addressed. □

References

[1] Blitzer, R. (2010). *Algebra and Trigonometry*, 4th ed., New York: Prentice-Hall.

[2] Larson, R. and Edwards, B.H. (2010). *Calculus*, 9th ed., New York: Brooks/Cole.

[3] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed., New York: McGraw-Hill.

[4] Thomas, G. B. (1976). *Calculus and Analytic Geometry*, 4th ed., New York: Addison-Wesley.