TENTATIVES FOR OBTAINING A PROOF OF THE RIEMANN HYPOTHESIS - VERSION 1.0, DECEMBER 2021 -

Abdelmajid BEN HADJ SALEM

Abstract. — This report presents some tentatives to obtain a final proof of the Riemann Hypothesis.

The last paper of the report is submitted to a mathematical journal for review.

Résumé. — Ce rapport présente quelques tentatives pour obtenir une démonstration de l'hypothèse de Riemann.

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2000 ***Mathematics Subject Classification.*** — Primary:11AXX; Secondary:11M26.

Key words and phrases. — Zeta function, non trivial zeros of Riemann zeta function, zeros of Dirichlet eta function inside the critical strip, definition of limits of real sequences.

To the memory of my Parents, to my wife Wahida, my daughter Sinda and my son Mohamed Mazen

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CHAPTER 1

UNE SOLUTION DE L'HYPOTHÈSE DE RIEMANN - A SOLUTION OF THE RIEMANN HYPOTHESIS -

Contents

Abstract. — In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},\ \textit{for} \quad \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$.

We give proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann Hypothesis.

Résumé. - En 1859, Georg Friedrich Bernhard Riemann avait annoncé la conjecture suivante, dite Hypothèse de Riemann: *Les zéros non triviaux $s = \sigma + it$ de la fonction zeta définie par:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},\ \textit{pour} \quad \Re(s) > 1$$

ont comme parties réelles $\sigma = \frac{1}{2}$.

On donne une démonstration que $\sigma = \frac{1}{2}$ en utilisant une proposition équivalente de l'Hypothèse de Riemann.

1.1. Introduction

En 1859, G.F.B. Riemann avait annoncé la conjecture suivante [**1**]:

Conjecture 1.1. — *Soit $\zeta(s)$ la fonction complexe de la variable complexe $s = \sigma + it$ définie par le prolongement analytique de la fonction:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},\ \textit{pour}\ \Re(s) = \sigma > 1$$

sur tout le plan complexe sauf au point $s = 1$. Alors les zéros non triviaux de $\zeta(s) = 0$ sont de la forme:

$$s = \frac{1}{2} + it$$

Dans cette communication, nous donnons une démonstration que $\sigma = \frac{1}{2}$. Notre idée est de partir d'une proposition équivalente de l'Hypothèse de Riemann et en utilisant la définition de la limite des suites réelles.

1.1.1. La fonction ζ. — Notons par $s = \sigma + it$ la variable complexe de $\mathbb{C}$. Pour $\Re(s) = \sigma > 1$, appelons ζ_1 la fonction définie par :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ avec } \Re(s) = \sigma > 1$$

Nous savons qu'avec la définition précédente, la fonction ζ_1 est une fonction analytique de s. Notons par $\zeta(s)$ la fonction obtenue par prolongement analytique de $\zeta_1(s)$, alors nous rappelons le théorème suivant [**2**]:

Theorem 1.2. — *Les zéros de $\zeta(s)$ satisfont :*

1. $\zeta(s)$ n'a pas de zéros pour $\Re(s) > 1$;

2. le seul pôle de $\zeta(s)$ est au point $s = 1$; son résidu vaut 1 et il est simple;

3. les zéros triviaux de $\zeta(s)$ sont déterminés pour les valeurs $s = -2, -4, \ldots$;

4. les zéros non triviaux se situent dans la région $0 \leq \Re(s) \leq 1$ dite bande critique et ils sont symétriques respectivement par rapport à l'axe vertical $\Re(s) = \dfrac{1}{2}$ et l'axe des réels $\Im(s) = 0$.

On sait aussi que les zéros de $\zeta(s)$ dans la bande critique sont tous des nombres complexes $\neq 0$ (voir page 30 de [**3**]).

La conjecture relative à l'Hypothèse de Riemann est exprimée comme suit:

Conjecture 1.3. — *(Hypothèse de Riemann,[**2**]) Tous les zéros non triviaux de $\zeta(s)$ sont sur la droite critique $\Re(s) = \dfrac{1}{2}$.*

En plus des propriétes citées par le théorème cité ci-dessus, la fonction $\zeta(s)$ vérifie la relation fonctionnelle [**2**] pour $s \neq 1$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1.1}$$

où $\Gamma(s)$ est la fonction définie sur le demi-plan $\Re(s) > 0$ par:

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

Alors, au lieu d'utiliser la fonctionnelle donnée par (4.1), nous allons utiliser celle présentée par G.H. Hardy [**3**] à savoir la fonction eta de Dirichlet [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

La fonction eta est convergente pour tout $s \in \mathbb{C}$ avec $\Re(s) > 0$ [**2**].

1.1.2. Une Proposition équivalente à l'Hypothèse de Riemann. — Parmi les propositions équivalentes à l'Hypothèse de Riemann celle de la fonction eta de Dirichlet qui s'énnonce comme suit [**2**]:

Equivalence 1.4. — *L'Hypothèse de Riemann est équivalente à l'énnoncé que tous les zéros de la fonction eta de Dirichlet:*

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

qui se situent dans la bande critique $0<\Re(s)<1$*, sont sur la droite critique* $\Re(s)=\frac{1}{2}$.

1.2. Démonstration que les zéros de $\eta(s)$ vérifient $\Re(s)=\frac{1}{2}$

Proof. — Notons par $s=\sigma+it$ avec $0<\sigma<1$. Considérons maintenant un zéro de $\eta(s)$ qui se trouve dans la bande critique et appelons $s=\sigma+it$ ce zéro, nous avons donc $0<\sigma<1$ et $\eta(s)=0\Longrightarrow(1-2^{1-s})\zeta(s)=0$. Notons $\zeta(s)=A+iB$, et $\theta=tLog2$, alors:

$$(1-2^{1-s})\zeta(s)=\left[A(1-2^{1-\sigma}cos\theta)-2^{1-\sigma}Bsin\theta\right]+i\left[B(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta\right]$$

$(1-2^{1-s})\zeta(s)=0$ donne le système:

$$A(1-2^{1-\sigma}cos\theta)-2^{1-\sigma}Bsin\theta=0$$
$$B(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta=0$$

Comme les fonctions *sin* et *cos* ne s'annulent pas simultanément, supposons par exemple que $sin\theta\neq0$, la première équation du système donne $B=\frac{A(1-2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}$, la deuxième équation s'écrit:

$$\frac{A(1-2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta=0\Longrightarrow A=0$$

Par suite, $B=0\Longrightarrow\zeta(s)=0$, il s'ensuit que:

(1.2)

$$\boxed{s\ est\ un\ zéro\ de\,\eta(s)\ dans\ la\ bande\ critique\ est\ aussi\ un\ zéro\ de\,\zeta(s)}$$

Reciproquement, si s est un zéro de $\zeta(s)$ dans la bande critique, soit $\zeta(s)=A+iB=0\Longrightarrow\eta(s)=(1-2^{1-s})\zeta(s)=0$, donc s est aussi un zéro de $\eta(s)$ dans la bande critique. Nous pouvons écrire:

(1.3)

$$\boxed{s\ est\ un\ zéro\ de\,\zeta(s)\ dans\ la\ bande\ critique\ est\ aussi\ un\ zéro\ de\,\eta(s)}$$

Ecrivons la fonction η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn} . e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn} (cos(tLogn) - isin(tLogn))$$

Remarquons que la fonction η est convergente pour tout $s \in \mathbb{C}$ avec $\Re(s) > 0$, mais non absolument convergente. Comme $\eta(s) = 0$, c'est-à-dire:

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

ou encore:

$$\forall \tau > 0 \quad \exists n_0, \forall \mathscr{N} > n_0, \ \left| \sum_{n=1}^{\mathscr{N}} \frac{(-1)^{n-1}}{n^s} \right| < \tau$$

Définissons la suite de fonctions $((\eta_n)_{n \in \mathbb{N}^*}(s))$, par:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

avec $s = \sigma + it$ et $t \neq 0$.

Soit s un zéro de η dans la bande critique, soit $\eta(s) = 0$, avec $0 < \sigma < 1$. Par suite, on peut écrire $lim_{n \longrightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. Ce qui donne:

$$lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Utilisons la définition de la limite d'une suite, on peut écrire:

$$(1.4) \forall \epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2 < \epsilon_1^2$$

$$(1.5) \forall \epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2 < \epsilon_2^2$$

Ce qui donne:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

En prenant $\epsilon = \epsilon_1 = \epsilon_2$ et $N > max(n_r, n_i)$, on obtient en faisant la somme membre à membre des deux dernières inégalités, on obtient:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{1.6}$$

1.2.1. Cas $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. — On suppose que $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$.

Commençons par rappeler le théorème de Hardy (1914) [**2**],[**3**]:

Theorem 1.5. — *Il y'a une infinité de zéros de $\zeta(s)$ sur la droite critique.*

Des propositions (4.5-4.6), nous déduisons la proposition suivante:

Proposition 1.6. — Il y'a une infinité de zéros de $\eta(s)$ sur la droite critique.

Soit $s_j = \frac{1}{2} + it_j$ un des zéros de la fonction $\eta(s)$ sur la droite critique, soit $\eta(s_j) = 0$. L'équation (4.9) s'écrit pour s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

ou encore:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Si on fait tendre N vers $+\infty$, la série $\sum_{k=1}^{N} \frac{1}{k}$ est divergente et devient infinie. Soit:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Par suite, nous obtenons le résultat suivant:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{1.7}$$

sinon, nous aurons une contradiction avec le fait que :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ est convergente pour } s_j = \frac{1}{2} + it_j$$

Comme $t_j \neq 0$, et qu'il y'a une infinité de zéros sur la droite critique, alors le résultat de la formule donnée par (4.11) est indépendant de t_j. Revenons maintenant à $s = \sigma + it$ un zéro de $\eta(s)$ dans la bande critique, soit $\eta(s) = 0$.

Prenons $\sigma = \dfrac{1}{2}$. En partant de la définition de la limite des suites, appliquée ci-dessus, nous obtenons:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

avec sans aucune contradiction. De la proposition (4.5) il s'ensuit que $\zeta(s) = \zeta(\frac{1}{2}+it) = 0$. Il existe donc des zéros de $\zeta(s)$ sur la droite critique $\Re(s) = \dfrac{1}{2}$.

1.2.2. Cas $0 < \sigma < \dfrac{1}{2}$. —

1.2.2.1. *Cas où il n'existe pas de zéros de $\eta(s)$ avec $s = \sigma + it$ et $0 < \sigma < \dfrac{1}{2}$.* — En utilisant, pour ce cas, le point 4 du théorème (4.2), nous déduisons que la fonction $\eta(s)$ n'a pas de zéros avec $s = \sigma + it$ et $\dfrac{1}{2} < \sigma < 1$. Par suite, d'après la proposition (4.5), il s'ensuit que la fonction $\zeta(s)$ a ses zéros seulement sur la droite critique $\Re(s) = \sigma = \dfrac{1}{2}$ et **l'Hypothèse de Riemann est vraie**.

1.2.2.2. *Cas où il existe des zéros de $\eta(s)$ avec $s = \sigma + it$ et $0 < \sigma < \dfrac{1}{2}$.* — Supposons qu'il existe $s = \sigma + it$ un zéro de $\eta(s)$ soit $\eta(s) = 0$ avec $0 < \sigma < \frac{1}{2} \Longrightarrow s \in$ à la bande critique. Nous écrivons l'équation (4.9),:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

ou:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

Or $2\sigma < 1$, il s'ensuit que $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}}$ tende vers $+\infty$ et nous obtenons par suite:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty \tag{1.8}$$

Là aussi, le résultat ci-dessus est indépendant de t.

1.2.3. Cas $\dfrac{1}{2} < \Re(s) < 1$. — Soit $s = \sigma + it$ le zéro de $\eta(s)$ dans $0 < \Re(s) < \dfrac{1}{2}$, objet du paragraphe précédent. Suivant le point 4 du théorème 4.2, le nombre complexe $s' = 1 - \sigma + it = \sigma' + it'$ avec $\sigma' = 1 - \sigma$ et $t' = t$ est aussi un zéro de la fonction $\eta(s)$ dans la bande $\frac{1}{2}<\Re(s)<1$. En appliquant (4.9), nous obtenons:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}<2\epsilon^2 \tag{1.9}$$

Comme $\sigma<\frac{1}{2}$, d'où $2\sigma'=2(1-\sigma)>1$, alors la série $\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}$ est convergente vers une constante positive non nulle $C(\sigma')$. De l'équation (4.13), nous déduisons que :

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}>-\infty$$

Considérons maintenant la fonction $F_N(u,t)$, $N\in\mathbb{N}^*\geq 2$, définie par:

$$F_N(u,t)=\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^u k'^u}=$$

$$=\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}cos(tLog(k/k'))e^{-uLog(kk')},\quad u\in]0,1[,\ t\in]0,+\infty[$$

La fonction $F_N(u,t)$ est continue pour $\forall N\geq 2$ et $(u,t)\in]0,1[\times]0,+\infty[$, et nous avons obtenu précédement que $\forall t>0$, pour $N\longrightarrow+\infty$:

$$\begin{cases}\displaystyle\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2} & \text{pour } u=\sigma'=1-\sigma>\frac{1}{2}\\ \displaystyle\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}=-\infty & \text{pour } u=\frac{1}{2}\\ \displaystyle\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}=-\infty & \text{pour } u=\sigma<\frac{1}{2}\end{cases}$$

Fixons $t=t_0>0$ une valeur arbitraire et écrivons que $F_N(u,t_0)$ est continue au point $u=1/2$, on peut écrire:

$$\forall\epsilon>0,\exists\delta \text{ tel que } \forall\,u\,/\,|u-1/2|<\delta\Longrightarrow|F_N(u,t_0)-F_N(1/2,t_0)|<\epsilon$$

Soit $u = \sigma' \in]0,1[$ avec $\sigma' > \frac{1}{2}$ vérifiant $\sigma' - \frac{1}{2} < \delta$, on a alors l'équation:

$$|F_N(\sigma', t_0) - F_N(1/2, t_0)| < \epsilon \Longrightarrow$$

$$-\epsilon + F_N(\sigma', t_0) < \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_0 Log(k/k'))}{\sqrt{k}\sqrt{k'}} < \epsilon + F_N(\sigma', t_0)$$

$$\Longrightarrow -\epsilon + \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_0 Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_0 Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Comme pour t, u fixés, la fonction F_N est définie pour tout entier $N \geq 2$, faisons alors tendre N vers $+\infty$, nous obtenons:

$$-\epsilon + \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma'} k'^{\sigma'}} \leq \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

$$\Longrightarrow -\epsilon - \frac{C(\sigma')}{2} \leq \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty$$

D'où la contradiction avec $C(\sigma')$ bornée. Par suite, l'hypothèse qu'il existe des zéros de $\eta(s)$ dans l'intervalle $\frac{1}{2} < \Re(s) < 1$ étudiée au début de cette section est fausse. Il s'ensuit que la fonction $\eta(s)$ ne s'annule pas dans les intervalles $0 < \Re(s) < \frac{1}{2}$ et $\frac{1}{2} < \Re(s) < 1$ et par suite la fonction $\eta(s)$ a ses zéros non triviaux sur la droite critique $\Re(s) = \frac{1}{2}$ de la bande critique.

□

1.3. Conclusion

En résumé: pour nos démonstrations, nous avons fait usage de la convergence simple de la fonction $\eta(s)$ de Dirichlet:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

dans la bande critique $0 < \Re(s) < 1$, en obtenant:

- $\eta(s)$ s'annule pour $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ ne s'annule pas pour $0 < \sigma = \Re(s) < \frac{1}{2}$ et $\frac{1}{2} < \sigma = \Re(s) < 1$.

Par suite, tous les zéros non triviaux de $\eta(s)$ dans la bande critique $0 < \Re(s) < 1$ s'annulent sur la droite critique $\Re(s) = \frac{1}{2}$. En appliquant la proposition équivalente à l'Hypothèse de Riemann 4.5, tous les zéros non triviaux de la fonction $\zeta(s)$ se trouvent sur la droite critique $\Re(s) = \frac{1}{2}$. La démonstration de l'Hypothèse de Riemann est ainsi achevée.

Nous annonçons donc le théorème important :

Theorem 1.7. — *Tous les zéros non triviaux de la fonction* $\zeta(s)$ *avec* $s = \sigma + it$ *se situent sur l'axe vertical* $\Re(s) = \dfrac{1}{2}$.

CHAPTER 2

IS THE RIEMANN HYPOTHESIS TRUE (V1)?

Contents

2.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 2.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the non-trivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$ except at most for a finite number of zeros.

2.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \ \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 2.2. — *. The function $\zeta(s)$ satisfies the following :*

1. *$\zeta(s)$ has no zero for $\Re(s) > 1$;*
2. *the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;*
3. *$\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;*
4. *the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.*

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line. We have also the theorem (see page 16, [**3**]):

Theorem 2.3. — *. For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]). Then, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

The Riemann Hypothesis is formulated as:

Conjecture 2.4. — *. (The Riemann Hypothesis,[**2**]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

In addition to the properties cited by the theorem 4.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1 - s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{2.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s)>0$, given by the formula :

$$\Gamma(s)=\int_0^{\infty} e^{-t}t^{s-1}dt,$$

So, instead of using the functional given by (4.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

The function eta is convergent for all $s\in\mathbb{C}$ with $\Re(s)>0$ [**2**].

2.1.2. A Equivalent statement to the Riemann Hypothesis. — Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 2.5. — . The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s),\quad \sigma>1 \tag{2.2}$$

that fall in the critical strip $0<\Re(s)<1$ lie on the critical line $\Re(s)=\frac{1}{2}$.

The series (4.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s)=\sigma>0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s),\quad \Re(s)=\sigma>0 \tag{2.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s)=\rho.e^{i\alpha}\Longrightarrow \rho^2=\eta(s).\overline{\eta(s)} \tag{2.4}$$

and $\eta(s)=0\Longleftrightarrow\rho=0$.

2.2. Proof that the zeros of $\eta(s)$ verify $\Re(s)=\frac{1}{2}$

Proof. — . We denote $s=\sigma+it$ with $0<\sigma<1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s=\sigma+it$, then we obtain $0<\sigma<1$ and $\eta(s)=0\Longrightarrow(1-2^{1-s})\zeta(s)=0$. Let us denote $\zeta(s)=A+iB$, and $\theta=tLog2$, then :

$$(1-2^{1-s})\zeta(s)=\Big[A(1-2^{1-\sigma}cos\theta)-2^{1-\sigma}Bsin\theta\Big]+i\Big[B(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta\Big]$$

$(1-2^{1-s})\zeta(s)=0$ gives the system:

$$A(1-2^{1-\sigma}cos\theta)-2^{1-\sigma}Bsin\theta=0$$
$$B(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta=0$$

As the functions *sin* and *cos* are not equal to 0 simultaneously, we suppose for example that $sin\theta \neq 0$, the first equation of the system gives $B = \dfrac{A(1-2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}$, the second equation is written as :

$$\frac{A(1-2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta = 0 \Longrightarrow A = 0$$

Then, $B = 0 \Longrightarrow \zeta(s) = 0$, it follows that:
(2.5)

s *is one zero of* $\eta(s)$ *that falls in the critical strip, is also one zero of* $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = A + iB = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:
(2.6)

s *is one zero of* $\zeta(s)$ *that falls in the critical strip, is also one zero of* $\eta(s)$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall\epsilon' > 0 \quad \exists n_0, \forall\mathcal{N} > n_0, \left|\sum_{n=1}^{\mathcal{N}}\frac{(-1)^{n-1}}{n^s}\right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We

obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}=0$$
$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}=0$$

Using the definition of the limit of a sequence, we can write:

(2.7) $\forall\epsilon_1>0\quad\exists n_r,\forall N>n_r\quad|\Re(\eta(s)_N)|<\epsilon_1\Longrightarrow|\Re(\eta(s)_N)|^2<\epsilon_1^2$

(2.8) $\forall\epsilon_2>0\quad\exists n_i,\forall N>n_i\quad|\Im(\eta(s)_N)|<\epsilon_2\Longrightarrow|\Im(\eta(s)_N)|^2<\epsilon_2^2$

Then:

$$0<\sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma}<\epsilon_1^2$$

$$0<\sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma}<\epsilon_2^2$$

Taking $\epsilon=\epsilon_1=\epsilon_2$ and $N>max(n_r,n_i)$, we get by making the sum member to member of the last two inequalities:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}<2\epsilon^2 \tag{2.9}$$

We can write the above equation as :

$$0<\ \rho_N^2\ <2\epsilon^2 \tag{2.10}$$

or $\rho(s)=0$.

2.2.1. Case $\sigma=\frac{1}{2}\Longrightarrow 2\sigma=1$. — We suppose that $\sigma=\frac{1}{2}\Longrightarrow 2\sigma=1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

Theorem 2.6. — . *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (4.5-4.6), it follows the proposition :

Proposition 2.7. — . There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j=\frac{1}{2}+it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j)=0$. The equation (4.9) is written for s_j:

$$0<\sum_{k=1}^{N}\frac{1}{k}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}<2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$(2.11) \qquad \boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

Let $s = \sigma + it$ one zero of $\eta(s)$ on the critical line $\Longrightarrow \eta(s) = 0$. We take $\sigma = \frac{1}{2}$. Starting from the definition of the limit of sequences, applied above, we obtain:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

with any contradiction. From the proposition (4.5), it follows that $\zeta(s) = \zeta(\frac{1}{2} + it) = 0$. There are therefore zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$.

2.2.2. Case $0 < \Re(s) < \frac{1}{2}$. —

2.2.2.1. *Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$.* — Using, for this case, point 4 of theorem (4.2), we deduce that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. Then, from the proposition (4.5), it follows that the function $\zeta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \frac{1}{2}$ and the **Riemann Hypothesis is true**.

2.2.2.2. *Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$.* — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (4.9):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty} \tag{2.12}$$

2.2.3. Case $\frac{1}{2} < \Re(s) < 1$. — Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. According to point 4 of theorem 4.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$, is also a zero of the function $\eta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, that is $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (4.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{2.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1 - \sigma) > 1$, then the series $\sum_{k=1}^{N} \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (4.13), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{2.14}$$

Then, we have the two following cases:

1)- There exists an infinity of complex numbers $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$. For each s'_l, the left member of the equation (4.14) above is finite and depends of σ'_l and t'_l, but the right member is a function only of σ'_l equal to $\zeta(2\sigma'_l)$. Hence the contradiction because for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of σ", therefore, the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l + it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the paragraph (2.2.2.2) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false.

Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (4.5), it follows that **the Riemann hypothesis is verified**.

2)- There is at most a single zero $s_0 = \sigma_0 + it_0$ of $\eta(s)$ with $\sigma_0 \in]0, 1/2[, t_0 > 0$ such that $\eta(s_0) = 0$. Let us call this zero *isolated zero* that we denote by (IZ). Therefore, the interval $]1/2, 1[$ contains a single zero $s'_0 = 1 - \sigma_0 + it_0$. Since the critical line contains an infinity of zeros of $\zeta(s) = 0$, it follows that all the nontrivial zeros of $\zeta(s)$ are on the critical line $\sigma = \dfrac{1}{2}$, except the 4 zeros relative to (IZ). Here too, we deduce that **the Riemann Hypothesis holds** except at most for the (IZ) in the critical band. □

2.3. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;

- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$ except at most for the (IZ) (with its symmetrical) inside the critical band.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ vanish on the critical line $\Re(s) = \dfrac{1}{2}$ except at most at (IZ) (with its symmetrical). Applying the equivalent proposition to the Riemann Hypothesis 4.5, all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$ except at most at (IZ) (with its symmetrical) inside the critical band. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 2.8. — . *All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$, except for at most four zeros of respective affixes $(\sigma_0, t_0), (1 - \sigma_0, t_0), (\sigma_0, -t_0), (1 - \sigma_0, -t_0)$, belonging to the critical band.*

CHAPTER 3

IS THE RIEMANN HYPOTHESIS TRUE (V2)?

Contents

3.1. Introduction.

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 3.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},\ \textit{for}\ \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. *Then the non-trivial zeros of* $\zeta(s) = 0$ *are written as :*

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \frac{1}{2}$.

3.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 3.2. — *The function* $\zeta(s)$ *satisfies the following :*

1. $\zeta(s)$ *has no zero for* $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ *is at* $s = 1$; *it has residue 1 and is simple;*

3. $\zeta(s)$ *has trivial zeros at* $s = -2, -4, \ldots$;

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ *(called the critical strip) and are symmetric about both the vertical line* $\Re(s) = \frac{1}{2}$ *and the real axis* $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 3.3. — *(The Riemann Hypothesis,*[**2**]*) All nontrivial zeros of* $\zeta(s)$ *lie on the critical line* $\Re(s) = \frac{1}{2}$.

In addition to the properties cited by the theorem 4.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{3.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (4.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 3.4. — *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]).

3.1.2. A Equivalent statement to the Riemann Hypothesis. —

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 3.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{3.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$. The series (4.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{3.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{3.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

3.2. Preliminaries of the proof

Proof. — . We denote $s=\sigma+it$ with $0<\sigma<1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s=\sigma+it$, then we obtain $0<\sigma<1$ and $\eta(s)=0 \Longleftrightarrow (1-2^{1-s})\zeta(s)=0$. We verifies easily the two propositions:
(3.5)

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s)=0 \Longrightarrow \eta(s)=(1-2^{1-s})\zeta(s)=0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:
(3.6)

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)}$$

Let us write the function η:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn}=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn}=$$
$$=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$
$$=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn)-isin(tLogn))$$

The function η is convergent for all $s\in\mathbb{C}$ with $\Re(s)>0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=0$$

or:

$$\forall\epsilon'>0 \quad \exists n_0, \forall\mathscr{N}>n_0, \ \left|\sum_{n=1}^{\mathscr{N}}\frac{(-1)^{n-1}}{n^s}\right|<\epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^s}=\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}-i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s=\sigma+it$ and $t\neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s)=0$, with $0<\sigma<1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s)=0=\eta(s)$. We

obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}=0$$
$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}=0$$

Using the definition of the limit of a sequence, we can write:

(3.7) $\forall\epsilon_1>0 \quad \exists n_r,\forall N>n_r \quad |\Re(\eta(s)_N)|<\epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2<\epsilon_1^2$

(3.8) $\forall\epsilon_2>0 \quad \exists n_i,\forall N>n_i \quad |\Im(\eta(s)_N)|<\epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2<\epsilon_2^2$

Then:

$$0<\sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma}<\epsilon_1^2$$

$$0<\sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma}<\epsilon_2^2$$

Taking $\epsilon=\epsilon_1=\epsilon_2$ and $N>max(n_r,n_i)$, we get by making the sum member to member of the last two inequalities:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}<2\epsilon^2 \tag{3.9}$$

We can write the above equation as :

$$0<\ \rho_N^2\ <2\epsilon^2 \tag{3.10}$$

or $\rho(s)=0$.

3.3. Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$.

We suppose that $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

Theorem 3.6. — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (4.5-4.6), it follows the proposition :

Proposition 3.7. — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (4.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{3.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

3.4. Case $0 < \Re(s) < \dfrac{1}{2}$.

3.4.1. Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$. — As there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$, it follows from the proposition (4.5) that $\zeta(s)$ has also no zeros with $0 < \sigma < \dfrac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s = \sigma + it$ and $\dfrac{1}{2} < \sigma < 1$. We deduce from the proposition (4.6) that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\dfrac{1}{2} < \sigma < 1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \dfrac{1}{2}$ and from the equivalent statement 4.5, we conclude that **the Riemann Hypothesis is true**.

3.4.2. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$. — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (4.9):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} \dfrac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty} \tag{3.12}$$

3.5. Case $\dfrac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (4.5), $\zeta(s) = 0$. According to point 4 of theorem 4.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (4.6) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (4.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{3.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (4.13), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{3.14}$$

Let $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$. For each s'_l, the left member of the equation (4.14) above is finite and depends of $\sigma'_l = 1 - \sigma_l$ and

$t_l' = t_l$, but the right member is a function only of σ_l' equal to $-\zeta(2\sigma_l')/2$. Hence the contradiction because for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of $\sigma"$, then in particular for all $\sigma"$ with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of $\sigma" \Longrightarrow$ the equation (4.14) is false, then, the function $\eta(s)$ has no zeros for all $s_l' = \sigma_l' + it_l'$ with $\sigma_l' \in]1/2, 1[$, it follows that the second case of the paragraph (4.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (4.5), it follows that **the Riemann hypothesis is verified**.

□

3.6. Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (4.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 3.8. — *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

CHAPTER 4

IS THE RIEMANN HYPOTHESIS TRUE? YES, IT IS.

Contents

4.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 4.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the non-trivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$.

4.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 4.2. — *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;

3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 4.3. — *(The Riemann Hypothesis,[**2**]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

In addition to the properties cited by the theorem 4.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{4.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (4.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 4.4. — *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]).

4.1.2. A Equivalent statement to the Riemann Hypothesis. — Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 4.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{4.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$. The series (4.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{4.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{4.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

4.2. Preliminaries of the proof

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(4.5)

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(4.6)

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \Big| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \Big| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n \longrightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. We

obtain:

$$lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(4.7) $\forall \epsilon_1 > 0 \, \exists n_r, \forall N > n_r, \, |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2$

(4.8) $\forall \epsilon_2 > 0 \, \exists n_i, \forall N > n_i, \, |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{4.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{4.10}$$

or $\rho(s) = 0$.

4.3. Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$.

We suppose that $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

Theorem 4.6. — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (4.5-4.6), it follows the proposition :

Proposition 4.7. — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (4.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{4.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4.4. Case $0 < \Re(s) < \frac{1}{2}$.

4.4.1. Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. — As there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$, it follows from the proposition (4.5) that $\zeta(s)$ has also no zeros with $0 < \sigma < \frac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. We deduce from the proposition (4.6) that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \frac{1}{2}$ and from the equivalent statement 4.5, we conclude that **the Riemann Hypothesis is true**.

4.4.2. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$. — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (4.9):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \dfrac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} = -\infty} \tag{4.12}$$

4.5. Case $\dfrac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (4.5), $\zeta(s) = 0$. According to point 4 of theorem 4.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (4.6) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (4.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{4.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (4.13), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{4.14}$$

Let $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$.

Firstly, we suppose that $t_l \neq 0$. For each s'_l, the left member of the equation (4.14) above is finite and depends of $\sigma'_l = 1 - \sigma_l$ and $t'_l = t_l$, but the right member is a function only of σ'_l equal to $-\zeta(2\sigma'_l)/2$. Hence the contradiction because for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of σ", then in particular for all σ" with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of $\sigma" \Longrightarrow$ the equation (4.14) is false.

Secondly, we suppose that $t_l = 0 \Longrightarrow t'_l = 0$. The equation (4.14) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'_l} k'^{\sigma'_l}} = -\frac{C(\sigma'_l)}{2} = -\frac{\zeta(2\sigma'_l)}{2} > -\infty \tag{4.15}$$

Then $s'_l = \sigma'_l > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s'_l) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'_l}} = 0 \tag{4.16}$$

Let us define the sequence S_m as:

$$S_m(s'_l) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'_l}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'_l}} = S_m(\sigma'_l) \tag{4.17}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) \tag{4.18}$$

We have also:

$$S_1(\sigma'_l) = 1 > 0 \tag{4.19}$$

$$S_2(\sigma'_l) = 1 - \frac{1}{2^{\sigma'_l}} > 0 \quad \textit{because } 2^{\sigma'_l} > 1 \tag{4.20}$$

$$S_3(\sigma'_l) = S_2(\sigma'_l) + \frac{1}{3^{\sigma'_l}} > 0 \tag{4.21}$$

We proceed by recurrence, we suppose that $S_m(\sigma'_l) > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma'_l) = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'_l}} = S_m(\sigma'_l) + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$, it gives:

$$S_{m+1}(\sigma'_l) = S_m(\sigma'_l) + \frac{(-1)^{2q}}{(m+1)^{\sigma'_l}} = S_m(\sigma'_l) + \frac{1}{(m+1)^{\sigma'_l}} > 0 \Rightarrow S_{m+1}(\sigma'_l) > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma'_l)$ as:

$$S_{m+1}(\sigma'_l) = S_{m-1}(\sigma'_l) + \frac{(-1)^{m-1}}{m^{\sigma'_l}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$$

We have $S_{m-1}(\sigma'_l) > 0$, let $T = \dfrac{(-1)^{m-1}}{m^{\sigma'_l}} + \dfrac{(-1)^m}{(m+1)^{\sigma'_l}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'_l}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'_l}} = \frac{1}{(2q+1)^{\sigma'_l}} - \frac{1}{(2q+2)^{\sigma'_l}} > 0 \tag{4.22}$$

and $S_{m+1}(\sigma'_l) > 0$.

Then all the terms $S_m(\sigma'_l)$ of the sequence S_m are great then 0, it follows that $lim_{m\longrightarrow+\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) > 0$ and $\eta(\sigma'_l) < +\infty$ because $\Re(s'_l) = \sigma'_l > 0$ and $\eta(s'_l)$ is convergent. We deduce the contradiction that s'_l is a zero of $\eta(s)$ and the equation (4.15) is false. Then, the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l + it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the second case of the paragraph (4.4) above concerning the case $0 < \Re(s) < \dfrac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (4.5), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following proposition:

Proposition 4.8. — For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

4.6. Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \dfrac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (4.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 4.9. — *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ *with* $s = \sigma + it$ *lie on the vertical line* $\Re(s) = \frac{1}{2}$.

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