Orbit precession in classical mechanics

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Abstract

As you know, precessing ellipses appear as solutions to the equations of the general theory of relativity. At the same time, it is generally accepted that in classical mechanics there are only the following equations of orbits: circles, ellipses, parabolas and hyperbolas. However, precessing ellipses also appear in classical mechanics. As you know, orbital precession is observed not only when the planets move in the Solar System. The precession of the periastron of the orbit is also observed in close binary systems, the components of which have evolved into pulsars ([1],[2],[3],[4],[5]). In such systems, the masses of the components – neutron stars – are of the same order of magnitude. Consequently, they will move in similar orbits around the center of mass. The orbits will be uniformly precessing ellipses. We write down the equation of such an orbit and derive from it an expression for the force of attraction acting between bodies. As a result, it turns out that, in addition to the Newtonian force, which is inversely proportional to the square of the distance between the bodies, a term appears in the expression for the force that is inversely proportional to the cube of the distance.

Before obtaining an expression for the force acting between two bodies moving in precessing orbits, it is necessary to write the equation of the orbit itself in polar coordinates. The equation for the precessing ellipse will be different from the equation for an ordinary ellipse. To describe the precession, it is necessary to introduce a coefficient in the equation of an ordinary ellipse under the cosine of the polar angle.

Consider the motion of two bodies along similar ellipses that are uniformly precessing in the direction of motion of the bodies. The equation of the relative trajectory of bodies:

$$\rho = \frac{p}{1 + e \cos k\varphi}, \quad p = a\left(1 - e^2\right), \quad k < 1.$$

Here ρ is the distance between the bodies, φ is the polar angle measured from the periastron, p is the focal parameter of the ellipse, a is the semi-major axis of the ellipse, e is the eccentricity, k is the precession parameter. Here, the precession parameter is $k < 1$, which corresponds to the displacement of the periastron in the direction of motion of the bodies. According to Kepler's second law the product of the square of the distance between the bodies and the angular velocity is constant. Based on this, in the periastron we can write:

$$\rho^2 \dot{\varphi} = v_p a (1 - e) = h,$$

where h is a constant, v_p is the relative velocity of bodies in the periastron.

Let's calculate the relative acceleration of bodies. It is equal to:

$$-w_\rho = \rho \dot{\varphi}^2 - \ddot{\rho} = \frac{h^2 k^2}{p \rho^2} + \frac{h^2 (1 - k^2)}{\rho^3},$$
$$w_\varphi = \frac{1}{\rho} \frac{d}{dt} \left(\rho^2 \dot{\varphi} \right) = 0.$$

For strength, you can write:

$$F(\rho) = -\frac{w_\rho}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{A}{\rho^2} + \frac{B}{\rho^3},$$
$$A = \frac{\mu h^2 k^2}{p}, \quad B = \mu h^2 \left(1 - k^2\right), \quad \mu = \frac{m_1 m_2}{m_1 + m_2},$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of two bodies. Thus, we have obtained that the force of attraction between two bodies moving in precessing orbits consists of two terms. The first term is Newtonian force, inversely proportional to ρ^2, and the second term is inversely proportional to ρ^3. Considering that

$$A = \frac{\mu h^2 k^2}{p} = G m_1 m_2,$$

we get:

$$B = \frac{1 - k^2}{k^2} p A = \frac{1 - k^2}{k^2} G p m_1 m_2,$$
$$h^2 k^2 = G p (m_1 + m_2) = G a \left(1 - e^2\right) (m_1 + m_2).$$

Let us derive Kepler's third law for precessing orbits. Integral

$$\int_0^{2\pi} \frac{d\varphi}{(1 + e \cos \varphi)^2}$$

can be calculated by the methods of complex analysis. However, we will consider an ordinary ellipse to calculate it. Let's write for it:

$$\int_0^{2\pi} \rho^2 d\varphi = 2\pi a b.$$

Substituting the expressions here:

$$\rho = \frac{a\left(1-e^2\right)}{1+e\cos\varphi}, \quad b = a\sqrt{1-e^2},$$

we get:

$$\int_0^{2\pi} \frac{d\varphi}{\left(1+e\cos\varphi\right)^2} = \frac{2\pi}{\left(1-e^2\right)^{3/2}}.$$

We apply the resulting integral to the problem:

$$\int_0^{2\pi/k} \rho^2 d\varphi = \frac{2\pi a^2\sqrt{1-e^2}}{k} = hT.$$

Thus,

$$T = \frac{2\pi a^2\sqrt{1-e^2}}{hk} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{Ga\left(1-e^2\right)\left(m_1+m_2\right)}} = \frac{2\pi a^{3/2}}{\sqrt{G\left(m_1+m_2\right)}}.$$

Hence,

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G\left(m_1+m_2\right)}.$$

Thus, we have obtained Kepler's third law, which, as can be seen, is also valid in the case of precessing orbits. However, it should be noted that the period T in this formula means the time elapsed between two periastres, i.e. time during which the polar angle changes by

$$\Delta\varphi = \frac{2\pi}{k}.$$

Considering that

$$\Delta\varphi = 2\pi + \dot{\omega}T,$$

where $\dot{\omega}$ is the rate of change in the longitude of the periastron of the orbit, we find the relationship between the precession parameter k and $\dot{\omega}$:

$$k = \left(1+\frac{\dot{\omega}T}{2\pi}\right)^{-1}.$$

In conclusion, we obtain an expression for the energy integral. The equations of motion of two interacting bodies under the action of the force of attraction are as follows:

$$m_1\vec{w}_1 = -\vec{F}, \quad m_2\vec{w}_2 = \vec{F}.$$

Here m_1 and m_2 are the masses of the bodies, $\vec{w}_1$ and $\vec{w}_2$ are their accelerations, $\vec{F}$ is the force with which the first body acts on the second. Let's assume for definiteness that $m_1 \geqslant m_2$.

We scalarly multiply the first equation by $d\vec{\rho}_1$, and the second by $d\vec{\rho}_2$, where $\vec{\rho}_1$ and $\vec{\rho}_2$ are the radius vectors of the bodies. Then we add the resulting equations term by term. As a result, we get:

$$m_1\left(\vec{w}_1 d\vec{\rho}_1\right) + m_2\left(\vec{w}_2 d\vec{\rho}_2\right) = \left(\vec{F} d\left(\vec{\rho}_2 - \vec{\rho}_1\right)\right) = \left(\vec{F} d\vec{\rho}\right),$$

where $\vec{\rho} = \vec{\rho}_2 - \vec{\rho}_1$ is the relative radius vector (radius vector of the second body relative to the first). Because

$$m_1\left(\vec{w}_1 d\vec{\rho}_1\right) + m_2\left(\vec{w}_2 d\vec{\rho}_2\right) = m_1\left(\vec{v}_1 d\vec{v}_1\right) + m_2\left(\vec{v}_2 d\vec{v}_2\right) = d\left(\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}\right) = d\frac{\mu v^2}{2},$$

where $\vec{v} = \vec{v}_2 - \vec{v}_1$ is the relative speed of two bodies, and

$$\vec{F} = -\left(\frac{A}{\rho^2} + \frac{B}{\rho^3}\right)\vec{e}_\rho,$$

where $\vec{e}_\rho = \frac{\vec{\rho}}{|\vec{\rho}|}$ is a unit vector drawn in the direction from the first body to the second, then

$$d\frac{\mu v^2}{2} = -\left(\frac{A}{\rho^2} + \frac{B}{\rho^3}\right)d\rho = d\left(\frac{A}{\rho} + \frac{B}{2\rho^2}\right).$$

Hence:

$$\frac{\mu v^2}{2} - \frac{A}{\rho} - \frac{B}{2\rho^2} = C,$$

where C is a constant. Thus, $T = \frac{\mu v^2}{2}$ is the kinetic energy of the bodies, $U = -\frac{A}{\rho} - \frac{B}{2\rho^2}$ is the potential energy of their interaction. Let us find the constant C. To do this, we substitute in the expression for the square of the relative speed of bodies:

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2$$

relative orbit equation:

$$\rho = \frac{p}{1 + e\cos k\varphi}, \quad \rho^2\dot{\varphi} = h.$$

After the transformations, we get:

$$\frac{\mu v^2}{2} - \frac{A}{\rho} - \frac{B}{2\rho^2} = -\frac{A}{2a}.$$

Thus, the total energy of the system is:

$$E = T + U = -\frac{A}{2a}.$$

References

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