Model of Collision DART with Dimorphos

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Abstract – A digital model of the collision of DART with Dimorphos, the 'moon' of the asteroid Didymos, learns that the published long-term prediction of the orbit of this 'moon' is incorrect.

I Introduction

Reference [1] provides sufficient background information on the planned collision of satellite DART with the 'moon' Dimorphos of the asteroid Didymos to build a digital model of this collision and to judge the predicted orbits of that 'moon' by the relevant scientists after such a collision. The model, described in this article, has been built in Excel.

II Description of the applied model

The centre of Didydos (Did) has been located in the origin of a plane with the axes x and y.
Dimporhos (Dim) its orbit is described by the coordinates d_x and d_y, its velocities by v_x resp. v_y and its accelerations by a_x resp. a_y.
The distance between Did and Dim is given by the variable r, with $r = \sqrt{(d_x^2 + d_y^2)}$.
The mass of Did is represented by M ($2 \cdot 10^{30}$ kg) and the mass of Dim by m.
G is the gravitational constant: $6.7 \cdot 10^{-11}$ Nm^2kg^{-2}.

The gravitational force between Dim and Did thus is GMm/r^2.
This gravitational force on Dim results in an acceleration a, thus $m \cdot a$.
So $m \cdot a = GMm/r^2$, resulting in the equation $a = GM/r^2$ and in the conclusion that the mass m does not play any role in the model, so neither in reality!

The following calculations are carried out.
Initially m is positioned at $d_x = d_{x0}$, $d_y = d_{y0}$, resulting in: $a_x = -a \cdot (d_x/r)$ and $a_y = -a \cdot (d_y/r)$.
Mathematically expressed, v_x and v_y are now calculated as:

$$v_x = v_{x0} + \int a_x(t)\, dt \qquad\qquad v_y = v_{y0} + \int a_y(t)\, dt$$

and d_x and d_y as:

$$d_x = d_{x0} + \int v_x(t)\, dt \qquad\qquad d_y = d_{y0} + \int v_y(t)\, dt$$

resulting in:

$$a_x = -a \cdot (d_x/r) \qquad\qquad a_y = -a \cdot (d_y/r).$$

with which the calculation circle is completed and can the orbit be created. Most likely it is possible to find the mathematical expression for the orbit, because the result is always an ellipse, being a perfect circle in the most extreme situation. The author has chosen for a digital simulation. The background of the following calculations has been presented in the appendix.

$$v_x(nT) = v_x(nT-T) + \{a_x(nT)+a_x(nT-T)\} \cdot T/2, \quad v_y(nT) = v_y(nT-T) + \{a_y(nT)+a_y(nT-T)\} \cdot T/2$$

$$d_x(nT) = d_x(nT-T) + \{v_x(nT)+v_x(nT-T)\} \cdot T/2, \quad d_y(nT) = d_y(nT-T) + \{v_y(nT)+v_y(nT-T)\} \cdot T/2$$

At time nT = 0 the variables $v_{x,y}(-T)$ and $d_{x,y}(-T)$ are equal to the respective mentioned initial values of these variables.

III Test of the applied model

III.1 Input parameters

The following values have been used, given in reference [1].

Mass M of Didymos	$5.28 \cdot 10^{11}$	kg
Mass m of Dimorphos	$4.8 \cdot 10^{9}$	kg

According to reference [2]:

Semi-major axis orbit Dim	1190 ± 300	m
Eccentricity	< 0.05	
Orbital period	11.93 ± 0.01	hr

Taking an eccentricity of 0.05 the semi-minor axis becomes 1189 m. So the orbit is assumed to be a perfect circle. Its radius is chosen to be (exactly) 1181 m. Such a radius fulfils the criterion that the orbital period has to be in between 11.92 and 11.94 hr: 11,9327 hr/42957.8 s.

III.2 Test

If not mentioned otherwise, the applied dimensions are based on the m, kg, s units.

The initial test is aimed at the creation of an orbit as a perfect circle with r = 1181 m. The launch will be at the coordinates: d_x = -1181 m, d_y = 0. The initial velocity v_{x0} is 0. v_{y0} is calculated as follows.

In case the orbit is a perfect circle, it fulfils the criterion that centrifugal force equals gravitational force *at a constant v and r*: $mv^2/r = GMm/r^2$, so $v^2 = GM/r$.

Given the values above and $G = 6.7 \cdot 10^{-11}$ Nm^2kg^{-2}, v = 0,17274 m/s. This value is applied for v_{y0}.

Yet one parameter of which the value has to be determined is left: the sample time T.

As already mentioned in the appendix, a digital integration is an approximation of the analog integration (as happens in reality). The accuracy of a digital integrator can infinitely be increased, as long as the sample time can infinitely be decreased. Initially, the accuracy is sufficient when the calculated trajectory closes so closely equal to the launch coordinates that a difference is not visible in the corresponding graph. However such a criterion doesn't work, given the fact that the collision to be investigated has an extreme small effect on the orbit. The final criterion resulted in such a small sample time that the Excel program is just still operable. The result is a sheet with 200 thousand rows.

The result is shown in figure 1. The deviation at the launch coordinates is 0.44 meter towards the left on the x-axis. The orbital period is 42969.4 instead of the theoretical 42957.8 s.

If the launch direction would be chosen 89.97° an ellipse like the present orbit of Dimorphos, with a semi-minor axis of 1.7 m shorter than the semi-major axis, would be the result.

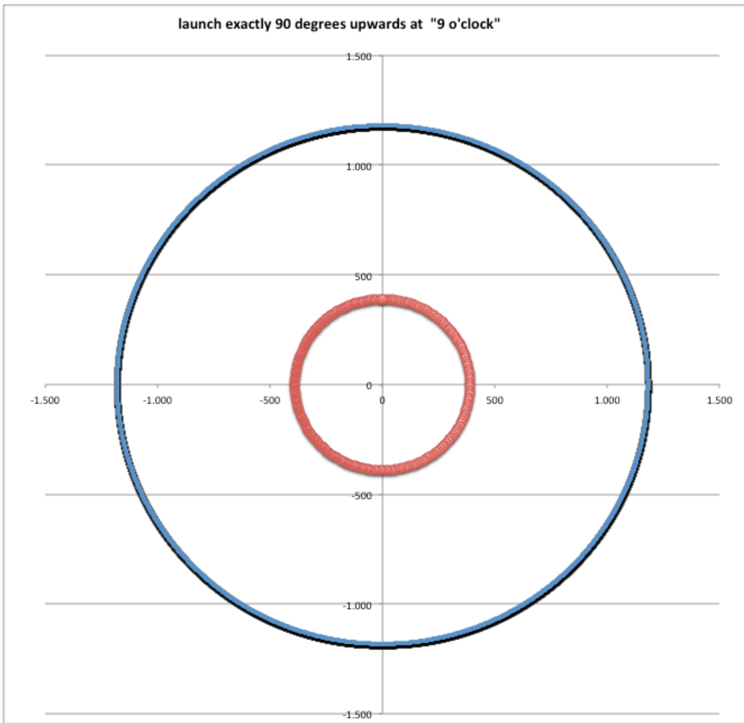

Figure 1 Orbit in case of a perfect circle

The final test is a launch at the same position, but with a launch direction of 45°.

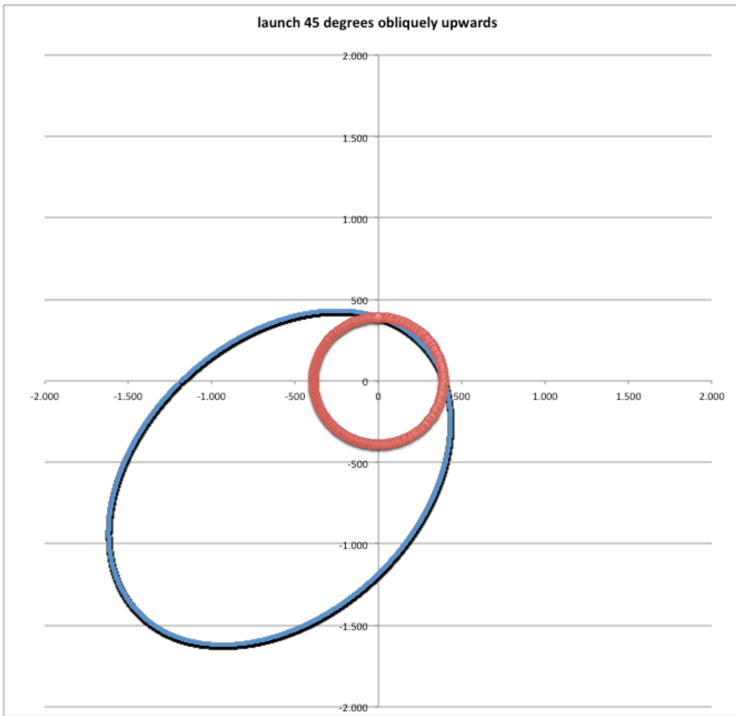

Figure 2 Orbit in case of a strong ellipse

Remarkable properties:

The orbit closes perfectly at the position of the launch, at least in the graph.
The direction of the major axis of the ellipse equals the launch direction.
This orbit is not possible in reality, due to the size of the central mass.

IV Repeatability of the circular orbit

Reference [2] says, under "Mission/impact":
"It is estimated that the impact of the 500 kg DART at 6.6 km/s will produce a velocity change on the order of 0.4 mm/s, which leads to a small change in trajectory of the asteroid system, *but over time, it leads to a large shift of path. Over a span of years, the cumulative trajectory change from such a small change in velocity could mitigate the risk of a hypothetical Earth-bound asteroid hitting Earth. The impact will target the center of figure of Dimorphos and should decrease the orbital period, currently 11.92 hours, by roughly 10 minutes."*

The text, italicized by the author, speaks clearly: the prediction is that the orbit period will increase for years to come after the collision.
It will be shown below that the results of the modelling by the scientists concerned have most likely been misleading as to the impact of the collision.

As mentioned already, the digital integrators applied in this model have a restricted accuracy that, in principle can be controlled by the value of the sample time. Up to now a sample time of 0.2 sec. has been used, leading the ~40000/0.2 rows in the Excel sheet.

The inaccuracy of this configuration will be shown in the following experiments.
After each completed orbit the values of the variables, at the time the orbit most closely approximates the start of the previous orbit, are copied and used as the starting values for the next orbit. The reason to apply this method is that an Excel sheet of significantly more than 200000 rows is not operable anymore.

The criterion applied for the best fit of the initial variables is concentrated only on the value of d_y. The moment this value is as close to zero as possible, the values at this time are copied. Two variables are chosen to present the accuracy: the value of d_x and of the orbital period O_p at the end of the orbit.

				T_s	0.2			
orbit	**r**	v_x	v_y	d_x	d_y	O_p	ΔO_p	Δd_x
0	1181,00	1,058E-17	1,727E-01	-1181,00	0,0000	42957,8	theoretical	
1	1181,43	-1,170E-06	1,727E-01	-1181,43	-0,0081	42969,4	11,6	-0,43
2	1181,87	5,118E-07	1,727E-01	-1181,87	0,0034	42993,2	23,8	-0,43
3	1182,30	-3,713E-09	1,726E-01	-1182,30	-0,0002	43016,8	23,6	-0,43
4	1182,74	2,327E-06	1,726E-01	-1182,74	0,0157	43040,6	23,8	-0,43
5	1183,17	2,463E-06	1,726E-01	-1183,17	0,0166	43064,2	23,6	-0,43
6	1183,60	4,087E-07	1,725E-01	-1183,60	0,0025	43087,8	23,6	-0,43
7	1184,04	1,197E-06	1,725E-01	-1184,04	0,0078	43111,6	23,8	-0,43

Table I

				T_s	0.4			
orbit	**r**	**vx**	**vy**	**dx**	**dy**	**Op**	**ΔOp**	**Δdx**
0	1181,00	1,058E-17	1,727E-01	-1181,00	0,0E+00	42957,8	theoretical	
1	1181,87	2,699E-06	1,727E-01	-1181,87	1,8E-02	42981	23,4	-0,87
2	1182,74	1,005E-06	1,726E-01	-1182,74	6,4E-03	43029	47,6	-0,87
3	1183,60	4,997E-06	1,725E-01	-1183,60	3,4E-02	43076	47,2	-0,87
4	1184,47	4,605E-06	1,725E-01	-1184,47	3,1E-02	43124	47,6	-0,87
5	1185,34	-1,524E-07	1,724E-01	-1185,34	-2,2E-03	43170	46,8	-0,87
6	1186,20	7,614E-07	1,724E-01	-1186,20	3,9E-03	43218	48,0	-0,87
7	1187,07	-2,681E-06	1,723E-01	-1187,07	-2,0E-02	43265	46,8	-0,87
8	1187,94	-4,733E-07	1,722E-01	-1187,94	-5,1E-03	43313	48,0	-0,87

Table II

The results clearly show that the 'large shift' in: "*....over time, it leads to a large shift of path"* is only caused by the inaccuracy of the applied integrators. In whatever digital model!

V The impact of the collision on the velocity of Dimorphos.

At collision the kinetic energy of DART is $1.1 \cdot 10^{10}$ J and of Dimorphos $7.2 \cdot 10^{7}$ J.
The momentums of these objects are: $M_1 = 3.3 \cdot 10^6$ respectively $M_2 = 8.8 \cdot 10^8$ kg·m/s

These values show that the impact of the collision on the velocity of Dimorphos can certainly not be calculated using de variable 'kinetic energy'.

Taking the momentums it has to be concluded that mathematically the influence of the collision on the velocity of Dimorphos can, in first instance be expressed by: $m_2 \cdot v_2 - m_1 \cdot v_1 = m_2 \cdot v_2'$, with v_2' the velocity of Dimorphos after the collision. This leading to: $v_2' = v_2 - m_1/m_2 \cdot v_1$.

$m_1 = 500$ kg $m_2 = 4.8 \cdot 10^9$ kg $v_1 = 6600$ m/s $v_2 = 0.17274$ m/s

As a result: $v_2' = 0.17205$ m/s so $\Delta v_2 = -0{,}0007$ m/s = -0,7 mm/s.

That fact that reference [2] shows -0.4 mm/s can most likely be explained by the argumentation that during the collision DART is destroyed. So effectively its mass reduces during the collision. Given the result, apparently half of DART's mass has been taken in such a calculation.

VI Modelling of the collision

If the launch direction would be chosen 89.97° an ellipse like the present orbit of Dimorphos would be the result, with a semi-minor axis of 1.7 m shorter than the semi-major axis.

As has been shown already the orientation of the long axis of the ellipse coincides with the launch direction. In this case rather accurate along the y-axis.

The collision is simulated by starting with this direction, but with a velocity decreased by 0.4 mm/s. It will cause the following changes:

$v_{mean} = 0.1727$ m/s	–>	$v_{mean} = 0.1723$ m/s
orbital period = 42969 s	–>	orbital period = 42673 s
orbital length = 7421 m	–>	orbital length = 7353 m

Conclusions

The modelling of the collision of DART with Dimorphos have learned that orbits in general have the following three remarkable properties:

1. An orbits' mean velocity equals the initial velocity.
2. Its shape and orientation is fixed at the time of the launch / collision, as these parameters are already determined by the initial position of the orbiting mass, relative to the centre mass, and by the direction and magnitude of the initial velocity.
3. Orbits will, after launch / collision, exactly end at the start position.

The orbital period of Dimorphos decreases after collision from 42969 to 42673 s, being ~5 minutes shorter. This decrease will be maintained forever. The predicted 10 minutes, *after many years*, is not correct, given the above mentioned properties.

References

[1] https://en.wikipedia.org/wiki/Dimorphos
[2] https://en.wikipedia.org/wiki/Double_Asteroid_Redirection_Test

Appendix **Formula for digital integration**

If

$F(s)$ is the Laplace transformation of $f(t)$

then

$F(s)/s$ is the Laplace transformation of $\int f(t)\,dt$

and

$e^{-sT} F(s)$ is the Laplace transformation of $f(t-T)$

The expression $e^{-sT} F(s)$ can be written as $z^{-1} \cdot F(z)$ by applying $z^{-1} = e^{-sT}$.

$F(z)$ is now the sequence of (digital) numbers representing $f(t)$ at time intervals of T.

T is the so-called sample time.

The function e^{-sT} can be approximated by $(1-sT/2)/(1+sT/2)$.

e^{-sT} can also be approximated by $1-sT$, but that would lead to a less accurate digital integrator.

Given the relation $z^{-1} \approx (1-sT/2)/(1+sT/2)$, it follows that $s \approx 2(1-z^{-1}) / (1+z^{-1})\,T$

Applying this approximation of s in the Laplace transformation $G(s) = F(s)/s$ results in the digital integrator $G(z) = F(z) \cdot (1+z^{-1})\,T / 2(1-z^{-1})$, leading to:

$$2(G-Gz^{-1}) = (f+fz^{-1})\,T$$

$$G = Gz^{-1} + (f + fz^{-1}) \cdot T/2$$

$$G = G(-T) + \{f + f(-T)\} \cdot T/2$$

So in the model the following expressions have been programmed:

$v_x(nT) = v_x(nT-T) + \{a_x(nT)+a_x(nT-T)\} \cdot T/2$, $v_y(nT) = v_y(nT-T) + \{a_y(nT)+a_y(nT-T)\} \cdot T/2$

$d_x(nT) = d_x(nT-T) + \{v_x(nT)+v_x(nT-T)\} \cdot T/2$, $d_y(nT) = d_y(nT-T) + \{v_y(nT)+v_y(nT-T)\} \cdot T/2$

At time $nT = 0$ the initial values are used:

$v_x(-T) = v_{x0}$ $v_y(-T) = v_{x0}$

$d_x(-T) = d_{x0}$ $d_y(-T) = d_{y0}$