Multiplication Tensor and Number of Rational Numbers

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Abstract In this paper, it is going to be explained how all natural numbers can be presented as a multiplication tensor or M-tensor. This comes as an extension of the number line (that is graphical presentation of the numbers that is suitable for graphical presentation of addition/subtraction) and facilitates reasoning related to sieves, multiplication, prime and composite numbers and so on. Here, the new representation will be used to calculate number of rational numbers.

1 Introduction

In elementary mathematics, a number line is a picture of the graduated straight line that serves as an abstraction to real numbers. The integers on that line are usually shown as specially marked points that are evenly spaced on the line. Here, we are interested only about the natural numbers, or integers that are bigger than zero. Idea of number line was first introduced by John Napier [1], without mentioning any connection of the line and mathematical operations. Later, John Wallis [2] used this graphical representation to explain operations of addition and subtraction in terms of moving backward and forward under the metaphor of a person walking. However, that type of graphical interpretation is not particularly suitable to other operations, especially those ones that are dealing with multiplication.

In order to obtain another useful representation of natural numbers we are going to introduce a multiplication tensor, or M_N-tensor. Idea comes from the fundamental theorem of arithmetic [3]. It means that we are going to create tensor of numbers in which every element is unique. Using that representation of natural numbers, implementation of particular sieves can be made easier for analysis.

Proposed representation of natural numbers can be useful in other contexts, too. As an example, all rational numbers could also be represented by a tensor that is named M_Q-tensor. Using that representation it will be explained how we can estimate a number of rational numbers if we assume that the number of natural and number of prime numbers are known.

2 Multiplication tensor

The fundamental theorem of arithmetic states that every integer greater than 1 can be uniquely represented by a product of powers of prime numbers, up to the order of the factors [3]. Having that in mind, an infinite dimensional tensor $\mathbf{M}_N$ that contains all natural numbers only once, is going to be constructed. In order to do that we are going to mark vector that contains all prime numbers with **p**. So, $p(1) = 2$, $p(2) = 3$, $p(3) = 5$, and so on. Tensor $\mathbf{M}_N$ with elements $m_{i1\ i2\ i3\ \ldots}$ is defined by the following equation (i_1, i_2, i_3, … are some natural numbers):

$$m_{i_1 i_2 i_3 \ldots} = p(1)^{i_1-1}\, p(2)^{i_2-1}\, p(3)^{i_3-1} \ldots \quad .$$

The alternative definition is also possible. Now, the following notation is going to be assumed for some infinite size vectors

$$\mathbf{2} = [2^0\ 2^1\ 2^2\ 2^3\ \ldots],\ \mathbf{3} = [3^0\ 3^1\ 3^2\ 3^3\ \ldots],\ \ \mathbf{5} = [5^0\ 5^1\ 5^2\ 5^3\ \ldots]\ \ldots$$

It is simple to be seen that every vector is marked by bold number that is equal to some prime number and that components of the vector are defined as powers of that prime number, including power zero. Now, the $\mathbf{M}_N$-tensor can be defined as

$$\mathbf{M}_N = \mathbf{2} \circ \mathbf{3} \circ \mathbf{5} \circ \mathbf{7} \circ \ldots,$$

where ○ stands for outer product.

The tensor $\mathbf{M}_N$ is of infinite dimension (equal to number of prime numbers) and size, and contains all natural numbers exactly ones. It is easy to understand why it is so, having in mind the fundamental theorem of arithmetic. We are going to call this type of infinite tensor a half infinite tensor (why it is so, will become clear when we introduce a $\mathbf{M}_Q$-tensor).

In order to get a visual idea about the $\mathbf{M}_N$ tensor, a 3D tensor of infinite size, that contains all natural numbers that could be defined by the first 3 primes – 2, 3 and 5 and their non-negative powers (any three primes could be chosen – it does not change the line of reasoning) is going to be analyzed. That tensor represents 3D sub-tensor of $\mathbf{M}_N$-tensor.

Here we are going to illustrate the frontal, lateral and top slice of this tensor. The frontal, top and lateral slices of the tensor are represented by infinite size 2D matrices.

1	2^1	2^2	2^3	...
3^1	$2\cdot 3$	$2^2\cdot 3$	$2^3\cdot 3$	...
3^2	$2\cdot 3^2$	$2^2\cdot 3^2$	$2^3\cdot 3^2$	...
3^3	$2\cdot 3^3$	$2^2\cdot 3^3$	$2^3\cdot 3^3$	…
...	...	...	...	...

Front slice

1	2^1	2^2	2^3	...
5^1	$2\cdot 5$	$2^2\cdot 5$	$2^3\cdot 5$	...
5^2	$2\cdot 5^2$	$2^2\cdot 5^2$	$2^3\cdot 5^2$	...
5^3	$2\cdot 5^3$	$2^2\cdot 5^3$	$2^3\cdot 5^3$	…
...	...	...	...	...

Top slice

1	5^1	5^2	5^3	...
3^1	$5\cdot 3$	$5^2\cdot 3$	$5^3\cdot 3$	...
3^2	$5\cdot 3^2$	$5^2\cdot 3^2$	$5^3\cdot 3^2$	...
3^3	$5\cdot 3^3$	$5^2\cdot 3^3$	$5^3\cdot 3^3$	…
...	...	...	...	...

Lateral slice

Now, we are going to illustrate a sieve process using M-tensor representation. In the following Figure 1, a three initial steps of the Sundaram sieve [4] are presented, using the mentioned 3D sub-tensor of $\mathbf{M}_N$ tensor.

From Fig1. it can be seen that removal of a single thread [5, 6] defined by first prime number will result in collapse of the tensor along the dimension that is defined by that prime number. The whole sub-tensor is going to collapse to one element that contains prime number that defines the corresponding thread and that dimension of the tensor. All other elements along that dimension are empty (the empty elements are introduced in order to have reasonable tensor representation). At the end of the process we are going to have 2x2x2 tensor that have only 4 elements that are non-empty.

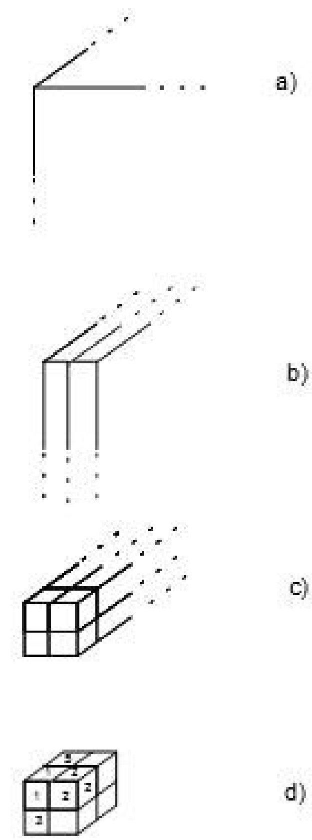

Figure 1. a) 3D infinite tensor; b) collapse of the tensor after removal of the thread defined by p(1); c) additional collapse of the tensor after removal of the thread defined by p(2); final collapse of the tensor after removal of the thread defined by p(3).

One is top, left corner that is filled by 1, and 3 of his neighbors along the orthogonal dimensions that are filled with first three prime numbers. All other elements (4 in total) are empty. In general case only those elements of the tensor that are going to be nonempty after implementation of full Sundaram sieve, are given by the following equation (tensor will have 2x2x2x … dimension)

$$m_{111...}=1\,;\, m_{12111...}=p(1)\,;\, m_{11211...}=p(2)\,;\, m_{11121...}=p(3)\,;\, ...\, m_{111...2(on\ position\, k+1)11...}=p(k)\,;\, ...\quad .$$

Here, one thing that requires additional attention is the following:

When the tensor is collapsed along the dimension defined by prime number 2 (for example), it is expected that the number of the numbers left is one half of all natural numbers. However, since the tensor represents the hyper-cuboid, the resulting volume (that can also be seen as the number of numbers in that hyper-cuboid) should be actually reduced by the factor $\log_2 N_\infty$, where N_∞ represents the number of natural numbers. Those two values cannot be the same, so some additional explanation is necessary. Here two possible explanations are offered (the third option is that both of them are valid)

- the tensor is mainly empty (so it preserves our intuition that number of even numbers is bigger than the number of numbers divisible by 3 and so on) or (and)
- the operation of multiplication and hyper-cuboid volume calculation do not hold for infinitely large numbers and dimensions (we have some kind of relativistic operators that are "bent" by infinitely large numbers).

3 Introduction of $\mathbf{M_Q}$-tensor and number of rational numbers

Now, a tensor $\mathbf{M_Q}$ that contains all positive rational numbers (that cannot be reduced) is going to be constructed by extending $\mathbf{M_N}$ tensor. That will happen if we generalize indexes i_1, i_2, i_3, … to be integers. In that cases individual elements $q_{i1,\, i2,\, i3,\, ...}$ of the $\mathbf{M_Q}$ tensor are defined in a same way as the elements of the $\mathbf{M_N}$ tensor, and are given by the following equation:

$$q_{i_1 i_2 i_3 ...}=p(1)^{i_1-1}\, p(2)^{i_2-1}\, p(3)^{i_3-1} ...\quad .$$

It is not difficult to be seen that this tensor contains all positive rational numbers only once (again, we are talking about irreducible rational numbers). Also it is not difficult to be seen that this tensor represents an infinite tensor (not half-infinite) since it has no beginning and no end. The central element of this tensor is in position 1111..., that contains number 1.

Now, we are going to calculate the number of positive rational numbers, if we assume that we know the number of natural numbers and number of prime numbers. The number of natural numbers is marked, as earlier, as N_∞. It is easy to understand that all dimensions of $\mathbf{M_Q}$ tensor are double the size of the dimensions of the $\mathbf{M_N}$ tensor. The number of dimensions of tensors $\mathbf{M_Q}$ and $\mathbf{M_N}$ is equal to the number of prime numbers (by construction) and we are going to mark that number as π_∞. In that case number of positive rational numbers, which we are going to be marked as Qp_∞, is given by the following equation

$$Qp_\infty = 2^{\pi_\infty} N_\infty \quad .$$

It is easy to understand that overall (including negative numbers and zero) number of rational numbers is $2Qp_\infty$+1. Further, recursive implementation of this process could lead to the proper estimation of the number of real numbers (for instance - creating hyper-tensor by allowing indexes i_1, i_2, i_3, … to take rational values, and then repeating the process again and again). However, this will not be further analyzed here.

References

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