Some infinite series

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January 4 , 2022

Abstract. We present a family of infinite series

Introduction: A general formula

Entry 1. For $p > q > 0$, $0 < \theta \leq \pi/2$, we have

$$\frac{2}{\sqrt{p^2-q^2}} \tan^{-1}\left(\sqrt{\frac{p-q}{p+q}} \tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{\sqrt{p^2-q^2}} + \frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{q}{2p}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\left((2n-2k)\theta\right)}{2n-2k}$$

$$-\frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{q}{2p}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\left((2n-2k+1)\theta\right)}{2n-2k+1}$$

Recall that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Examples

Entry 2.

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}(-1)^n\, 2^{-4n-2} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(-1)^k}{2n-2k+1}$$

$$\pi = \sum_{n=0}^{\infty}(-1)^n\, 2^{-3n} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(-1)^k}{2n-2k+1}$$

Entry 3.

$$\frac{(-1)^n 2^{4n+1}}{(2n+1)\binom{2n}{n}} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(-1)^k}{2n-2k+1}$$

Entry 4.

$$\frac{\pi}{12}\sqrt{11-4\sqrt{6}}$$
$$= \sum_{n=0}^{\infty} \left(\frac{\sqrt{14-4\sqrt{6}}}{10}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(\pi/4)\big)}{2n-2k+1}$$
$$- \sum_{n=1}^{\infty} \left(\frac{\sqrt{14-4\sqrt{6}}}{10}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(\pi/4)\big)}{2n-2k}$$

Entry 5.

$$\frac{\pi}{12}\sqrt{3-\frac{4\sqrt{2}}{3}} = \sum_{n=0}^{\infty} \left(\frac{3\sqrt{2}-2}{14}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(\pi/3)\big)}{2n-2k+1}$$
$$- \sum_{n=1}^{\infty} \left(\frac{3\sqrt{2}-2}{14}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(\pi/3)\big)}{2n-2k}$$

Entry 6.

$$\frac{\pi}{8}\left(2\sqrt{2+\sqrt{2}}-\sqrt{2}-1\right)$$
$$= \sum_{n=0}^{\infty} \left(\frac{\sqrt{100+62\sqrt{2}}-5\sqrt{2}+4}{34}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(\pi/4)\big)}{2n-2k+1}$$
$$- \sum_{n=1}^{\infty} \left(\frac{\sqrt{100+62\sqrt{2}}-5\sqrt{2}+4}{34}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(\pi/4)\big)}{2n-2k}$$

Entry 7.

$$\frac{5\pi}{9} = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(2\pi/3)\big)}{2n-2k+1}$$
$$- \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(2\pi/3)\big)}{2n-2k}$$

Entry 8.

$$\frac{5\pi}{9} = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(\pi/3)\big)}{2n-2k+1} + \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(\pi/3)\big)}{2n-2k}$$

Entry 9.

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} 2^{-4n-2} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(\pi/3)\big)}{2n-2k+1} + \sum_{n=1}^{\infty} 2^{-4n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(\pi/3)\big)}{2n-2k}$$

Entry 10.

$$\frac{2\pi}{3} = \sum_{n=0}^{\infty} \left(\frac{\sqrt{3}}{4}\right)^{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{\sin\big((2n-2k+1)(\pi/6)\big)}{2n-2k+1} + \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}}{4}\right)^{2n} \sum_{k=0,k\neq n}^{2n} \binom{2n}{k} \frac{\sin\big((2n-2k)(\pi/6)\big)}{2n-2k}$$

References

[1] Arndt, J., and Haenel, C. : π unleashed. Springer-Verlag , 2001.

[2] Beckmann, P. : A History of π . 2nd ed., Golem Press, Boulder, CO, 1971.

[3] Berggren, L., Borwein, J., and Borwein, P. : Pi: a Source Book. Springer-Verlag, 1997.

[4] Gradshteyn, I.S., and Ryzhik, I.M. : Table of Integrals, Series and Products. 5th ed., ed. Alan Jeffrey. Academic Press , 1994.