Gamma function, Lambert W-function: “The integral”

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Abstract. We give an integral involving the Gamma function and the Lambert W-function.

Keywords: Gamma function, Lambert W-function, integrals

Introduction

Recall that

$$\left(\Gamma\left(\frac{1}{3}\right)\right)^3=\frac{16\sqrt[3]{2}}{\sqrt{3}}\int_0^\infty\int_0^\infty\frac{\sqrt[3]{\cosh x}}{\cosh(x+y)+\cosh(x-y)}\,dx\,dy$$

where $\Gamma(x)$ is the Gamma function.

The (real-valued) Lambert W-functions are solutions of the nonlinear equation

$$we^w=y\ \ ,y\in\mathbb{R}$$

If $y>0$, there is a unique real solution, $w(y)$, satisfying $0<w(y)<\infty$. If $-\frac{1}{e}\leq y<0$, there are exactly two real solutions, $w_0(y)$ and $w_{-1}(y)$, satisfying respectively $-1\leq w_0(y)<0$ and $-\infty<w_{-1}(y)\leq-1$. Clearly , $w(0+)=0, w(0-)=0,$ and $w_{-1}(0-)=-\infty$, while $w_0(-1/e)=w_{-1}(-1/e)=-1$. For $y<-1/e$, there are no real solutions of $we^w=y\ \ ,y\in\mathbb{R}$. For a discussion of the various branches of the Lambert W-functions , also in the complex plane , see [6] .

Integral for $\left(\Gamma(1/3)\right)^3$

For $\alpha=4\sqrt[3]{2}\,\ln 2$, we have

$$\left(\Gamma\left(\frac{1}{3}\right)\right)^3=\alpha+\int_\alpha^\infty\left(1-\sqrt{1-4\left(\frac{3}{2x}W\left(\frac{2x}{3}\right)\right)^{3/2}}\right)dx$$

where $W(x)$ is the Lambert W-function.

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