Eliminate the Irrelevant to the Subject and Prove Equations and Inequalities related to Beal's Conjecture

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Abstract

The subject of this article is exactly to analyze Beal's conjecture and prove it.

First, we classify mathematical expressions which consist of A^X, B^Y and C^Z, according to the parity of A, B and C, then get rid of two combinations of A^X, B^Y and C^Z, for they have nothing to do with the conjecture.

After that, we exemplify $A^X+B^Y=C^Z$ under the necessary constraints, where A, B, and C have at least one common prime factor.

Secondly, divide $A^X+B^Y\neq C^Z$ under the necessary constraints into two kinds, and prove one kind thereof in which any two terms have a common prime factor while another term has not it. Next, under known constraints, divide another kind of $A^X+B^Y\neq C^Z$ into four inequalities.

Furthermore, we derive four conclusions from the interrelation between an even number as the symmetric center and a sum of two odd numbers. This is just a preparation for proving the first two inequalities.

Then, the first two inequalities are proved by the mathematical induction, fundamental theorem of arithmetic and the binomial theorem. Then again, other two inequalities are proved by the reduction to absurdity.

Finally, after comparing $A^X+B^Y=C^Z$ and $A^X+B^Y\neq C^Z$ under the necessary constraints, we came to the conclusion that Beal's conjecture is true.

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1. Introduction

Beal's conjecture states that if $A^X+B^Y=C^Z$, where A, B, C, X, Y and Z are positive integers, and X, Y and Z are all greater than 2, then A, B and C must have a common prime factor.

The conjecture was discovered by Andrew Beal in 1993. Later, it was announced in December 1997 issue of the Notices of the American Mathematical Society, [1]. However, it remains a conjecture that has neither been proved nor disproved.

The conjecture indicates that whoever wants to prove it, must both exemplify $A^X+B^Y=C^Z$ in which A, B and C have a common prime factor, and prove $A^X+B^Y\neq C^Z$ in which A, B and C have no a common prime factor.

First of all, we consider the scope of values of each of A, B, C, X, Y and Z in $A^X+B^Y=C^Z$ as the necessary constraints.

2. On Mathematical Expressions which Consist of A^X, B^Y and C^Z

We classify mathematical expressions which consist of A^X, B^Y and C^Z, according to the parity of A, B and C, and from this get rid of following two kinds of $A^X+B^Y\neq C^Z$:

1) A, B and C are all odd numbers;

2) A, B and C are two even numbers and an odd number.

Then, we continue to have following two kinds which contain $A^X+B^Y=C^Z$ and $A^X+B^Y\neq C^Z$ under the necessary constraints:

1) A, B and C are all positive even numbers;

2) A, B and C are two positive odd numbers and one positive even number.

3. Exemplify $A^X+B^Y=C^Z$ Under the Necessary Constraints

For two retained indefinite equations at above, each of them has many sets of solution as positive integers actually, as shown in the following examples.

When A, B and C are all positive even numbers, let A=B=C=2, X=Y $\geq$3 and Z=X+1, then $A^X+B^Y=C^Z$ is changed to $2^X+2^X=2^{X+1}$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 2, 2 and 2, also A, B and C have one common prime factor 2.

In addition to the above example, let A=B=162, C=54, X=Y=3 and Z=4, then $A^X+B^Y=C^Z$ is changed to $162^3+162^3=54^4$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 162, 162 and 54, also A, B and C have two common prime factors 2 and 3.

When A, B and C are two positive odd numbers and one positive even number, let A=C=3, B=6, X=Y=3 and Z=5, then $A^X+B^Y=C^Z$ is changed to $3^3+6^3= 3^5$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 3, 6 and 3, also A, B and C have one common prime factor 3.

In addition to the above example, let A=B=7, C=98, X=6, Y=7 and Z=3, then $A^X+B^Y=C^Z$ is changed to $7^6+7^7=98^3$, so $A^X+B^Y=C^Z$ after the assignment of these values has one set of solution with A, B and C as 7, 7 and 98, also A, B and C have one common prime factor 7.

It follows that there are surely $A^X+B^Y=C^Z$ under the necessary constraints, but A, B and C must have at least one common prime factor.

Then again, according to the need that proves the conjecture, if we can further prove $A^X+B^Y\neq C^Z$ under the necessary constraints, where A, B and C have not a common prime factor, then the conjecture is tenable surely.

4. Divide $A^X+B^Y\neq C^Z$ into Two Kinds and Prove One Kind thereof

When A, B and C are all positive even numbers, they have at least one common prime factor 2, so A, B and C without common prime factor can only be two positive odd numbers and one positive even number.

If A, B, and C have not a common prime factor, then A^X, B^Y and C^Z have not a common prime factor either.

If A^X, B^Y and C^Z have not a common prime, then we can divide $A^X+B^Y\neq C^Z$ into following two kinds, and first prove one kind thereof.

(1) Any two terms of $A^X+B^Y\neq C^Z$ have a common prime factor, yet another term has not the common prime factor.

Proof· when any two of A^X, B^Y and C^Z have a common prime factor, we can extract this common prime factor from these two terms to become a prime factor of their sum or difference, yet another term has not this common prime factor, accordingly, it can only lead up to $A^X+B^Y \neq C^Z$ or $C^Z-A^X \neq B^Y$ or $C^Z-B^Y \neq A^X$, according to the unique factorization theorem of natural number [or call it "the fundamental theorem of arithmetic"], [2].

For $C^Z-A^X \neq B^Y$ and $C^Z-B^Y \neq A^X$, after you transpose a term of each of them, you get $A^X+B^Y \neq C^Z$ too.

Therefore, there is only $A^X+B^Y \neq C^Z$ under the necessary constraints, where any two terms have a common prime factor, yet another term have not it.

(2) No two terms of $A^X+B^Y \neq C^Z$ have a common prime factor.

The proof of $A^X+B^Y \neq C^Z$ in which no two terms have a common prime factor is the difficult point of this article, so we will elaborate on relevant every section hereinafter.

5. Divide Another Kind of $A^X+B^Y \neq C^Z$ into Four Inequalities

The inequality $A^X+B^Y \neq C^Z$ in which no two terms have a common prime factor is able to be divided into following two inequalities:

1) $A^X+B^Y \neq (2W)^Z$, i.e. $A^X+B^Y \neq 2^Z W^Z$;

2) $A^X+(2W)^Y \neq C^Z$, i.e. $A^X+2^Y W^Y \neq C^Z$.

In above-listed two inequalities, newly emerging W is an odd number ≥ 1.

Then, we continue to divide $A^X+B^Y \neq 2^Z W^Z$ into following two inequalities:

(1) $A^X+B^Y \neq 2^Z$;

(2) $A^X+B^Y \neq 2^Z O^Z$, where O is an odd number>1, the same below;

Then again, continue to divide $A^X+2^Y W^Y \neq C^Z$ into following two inequalities:

(3) $A^X+2^Y \neq C^Z$;

(4) $A^X+2^Y O^Y \neq C^Z$.

We regard which no two terms of each inequality at above have a common prime factor as a qualification, and further regard the qualification plus the necessary constraints as the known constraints for each such inequality.
So, under the known constraints, the proof of $A^X+B^Y \neq C^Z$ is turned to prove conclusive four inequalities as listed above.

6. Several Grounds which Prove the First Two Inequalities

Whether on positive half line of number axis or in the sequence of natural numbers, take an even point on the right of 3 or any even number≥4 as a symmetric center, [3], then we can draw following four conclusions from the interrelation between the even number and a sum of two odd numbers.

Conclusion 1˙ The sum of two bilateral symmetric odd numbers is equal to the double of the even number as the symmetric center.

Conclusion 2˙ The sum of two asymmetric odd numbers is not equal to the double of the even number as the symmetric center.

Conclusion 3˙ If the sum of two odd numbers is equal to the double of an even number, then these two odd numbers are symmetric with the even number as the symmetric center.

Conclusion 4˙ If the sum of two odd numbers is not equal to the double of an even number, then these two odd numbers are not symmetric with the even number as the symmetric center.

After this, we will cite these conclusions in the process that proves the first two inequalities.

On the whole, if regard $2^{Z-1}/2^{Z-1}O^Z$ as the symmetric center, a sum of two asymmetric odd numbers A^X and B^y is not equal to $2^Z/2^ZO^Z$, even if two of them have not a common prime factor, according to the conclusion 2.

Thus, when continue to prove the first two inequalities, we only need to consider symmetric odd numbers on two sides of a certain symmetric center.

7. Prove $A^X+B^Y\neq 2^Z$ under Known Constraints

Let us regard 2^{Z-1} as a symmetric center to prove $A^X+B^Y\neq 2^Z$ under known constraints by the mathematical induction, [4], *ut infra*.

(1) When Z-1=2, 3, 4, 5 and 6, symmetric odd numbers on two sides of each symmetric center are successively listed below.

1^6, 3, (2^2), 5, 7, (2^3), 3^2, 11, 13, 15, (2^4), 17, 19, 21, 23, 5^2, 3^3, 29, 31, (2^5), 33, 35, 37, 39, 41, 43, 45, 47, 7^2, 51, 53, 55, 57, 59, 61, 63, (2^6), 65, 67, 69, 71, 73, 75, 77, 79, 3^4, 83, 85, 87, 89, 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119, 11^2, 123, 5^3, 127

As listed above, it can be seen that there are no two of O^V with $V\geq 3$ on two places of each pair of bilateral symmetric odd numbers with 2^{Z-1} as a symmetric center, where Z-1=2, 3, 4, 5 and 6.

So, there are $A^X+B^Y\neq 2^3$, $A^X+B^Y\neq 2^4$, $A^X+B^Y\neq 2^5$, $A^X+B^Y\neq 2^6$ and $A^X+B^Y\neq 2^7$ under known constraints, according to the conclusion 2 in section 6. And that any two terms of each inequality have not a common prime factor.

In addition, we also found that there is no B^2 on the symmetric place of A^X, and there is no A^2 on the symmetric place of B^X.

(2) When Z-1=K with K≥6, suppose that there is only $A^X+B^Y\neq 2^{K+1}$ under known constraints.

(3) When Z-1=K+1, prove that there is only $A^X+B^Y\neq 2^{K+2}$ under the known constraints.

Proof· Since there is $A^X+B^Y\neq 2^{K+1}$ under the known constraints, according to second step of the mathematical induction.

So, there is $A^X+(A^X+2B^Y)\neq 2^{K+2}$ under the known constraints.

Next, let $A^X+2B^Y=O_2{}^M$, then there is $A^X+O_2{}^M\neq 2^{K+2}$, where O_2 express positive odd numbers, M is the exponent, and similarly hereinafter.

Since there is $A^X+B^Y=2^{K+1}$ under the known constraints except for Y, and Y=1, such as $3^3+37^1=2^6$ and $5^3+131^1=2^8$.

So, there is $A^X+(A^X+2B^Y)=2^{K+2}$ under the known constraints except for Y, and Y=1.

Next, let $A^X+2B^1=O_1{}^L$, then there is $A^X+O_1{}^L=2^{K+2}$ where O_1 express positive odd numbers, L is the exponent, and similarly hereinafter.

In $A^X+2B^Y=O_2{}^M$ and $A^X+2B^1=O_1{}^L$, since A^X is one and the same, B is one and the same, and Y≥3, so there is $A^X+2B^Y>A^X+2B^1$, i.e. $O_2{}^M>O_1{}^L$.

Since there is $A^X+O_1{}^L=2^{K+2}$, this indicates that A^X and $O_1{}^L$ are two bilateral symmetric odd numbers with 2^{K+1} as the symmetric center, according to the conclusion 3 in the section 6. That is to say, take 2^{K+1} as the symmetric center, then O_1^L lies on the symmetric place of A^X.

However, due to $O_2^M > O_1^L$, when take 2^{K+1} as the symmetric center, O_2^M does not lie on the symmetric place of A^X, therefore, there is $A^X+O_2^M \neq 2^{K+2}$, according to the conclusion 2 in the section 6.

Furthermore, let us analyze the only five cases of $O_2^M > O_1^L$ to confirm different limits of values of the exponent of O_2^M in these five cases.

(1) $O_2 > O_1$ and $M > L$;

(2) $O_2 > O_1$ and $M = L$;

(3) $O_2 > O_1$ and $M < L$;

(4) $O_2 = O_1$ and $M > L$;

(5) $O_2 < O_1$ and $M > L$.

Since three cases of five cases at above contain $M > L$, so for three such cases, even if let $L=1$, there is also $M \geq 2$. Without doubt, $M \geq 3$ is included in $M \geq 2$, and we substitute $Y \geq 3$ for $M \geq 3$, then there is $A^X+O_2^Y \neq 2^{K+2}$.

Since B and O_2 in $A^X+O_2^Y \neq 2^{K+2}$ can express same odd numbers, therefore, after substitute B for O_2, there is $A^X+B^Y \neq 2^{K+2}$ under the known constraints.

For other two case, when $L=1$: from $M < L$ to get $M=0$ and from $M=L$ to get $M=1$, both are useless here. .

Apply the preceding way of doing thing, we can continue to prove that when $Z-1=K+2$, $K+3$…up to every integer $\geq K+2$, there are likewise $A^X+B^Y \neq 2^{K+3}$, $A^X+B^Y \neq 2^{K+4}$ … up to general $A^X+B^Y \neq 2^Z$ under the known constraints.

8. Prove $A^X+B^Y \neq 2^Z O^Z$ Under the Known Constraints

For the proof of $A^X+B^Y\neq 2^Z O^Z$ under the known constraints, let us do it with a proof and an explanation.

Firstly, take $2^{Z-1}O^Z$ as a symmetric center to prove that O in $A^X+B^Y\neq 2^Z O^Z$ expresses every positive odd number by the mathematical induction, *ut infra*.

(1) When O=1, $2^{Z-1}O^Z$ i.e. 2^{Z-1}, as has been proved, there is only $A^X+B^Y\neq 2^Z$ under the known constraints, in the section 7.

(2) When O=J and J is an odd number ≥ 1, $2^{Z-1}O^Z$ i.e. $2^{Z-1}J^Z$, and we suppose that there is only $A^X+B^Y\neq 2^Z J^Z$ under the known constraints.

(3) When O=J+2, $2^{Z-1}O^Z$ i.e. $2^{Z-1}(J+2)^Z$, we will go to prove that there is only $A^X+B^Y\neq 2^Z(J+2)^Z$ under the known constraints.

***Proof*·** Since there is $(J+2)^Z = J^Z + 2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + \ldots + 2^n C_Z^n J^{Z-n} + \ldots + 2^Z C_Z^Z$, according to the binomial theorem, [5].

Thus, there is $2^Z(J+2)^Z = 2^Z(J^Z + 2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + \ldots + 2^n C_Z^n J^{Z-n} + \ldots + 2^Z C_Z^Z)$, namely $2^Z(J+2)^Z = 2^Z J^Z + 2^Z(2C_Z^1 J^{Z-1} + 2^2 C_Z^2 J^{Z-2} + \ldots + 2^n C_Z^n J^{Z-n} + \ldots + 2^Z C_Z^Z)$, and its part will be used in the following an equation and an inequality.

Due to $A^X+B^Y=2^Z J^Z$, A^X and B^Y are two bilateral symmetric odd numbers with $2^{Z-1}J^Z$ as the symmetric center, according to the conclusion 3 in the section 6.

Next, on the premise of $Z\geq 3$, if either of X and Y in $A^X+B^Y=2^Z J^Z$ is more than or equal to 3, then we can exemplify that another is surely equal to 1, for example $13^3+547^1=2^3 7^3$ and $(7\times 131)^1+19^3=2^5\times 3^5$. Since A^X and B^Y can switch places, so only let Y=1, then there is $A^X+B^Y=2^Z J^Z$ under the known constraints except for Y, or there is $A^X+B^1=2^ZJ^Z$ under the known constraints.

So there is $A^X+B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)=2^Z(J+2)^Z$.

Evidently, $B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ is an odd number, and let $B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)=O_1^S$.

As a consequence, there is $A^X+O_1^S=2^Z(J+2)^Z$ derived from participation of B^1.

While, there is $A^X+B^Y\neq 2^ZJ^Z$ under the known constraints, according to second step of the mathematical induction.

So, there is $A^X+B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)\neq 2^Z(J+2)^Z$

Likewise $B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ is an odd number, and let $B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)=O_2^G$.

As another consequence, there is $A^X+O_2^G\neq 2^Z(J+2)^Z$ derived from participation of B^Y with $Y\geq 3$.

Due to $Y\geq 3$, so $B^Y+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ is greater than $B^1+2^Z(2C_Z^1J^{Z-1}+2^2C_Z^2J^{Z-2}+...+2^nC_Z^nJ^{Z-n}+...+2^ZC_Z^Z)$ i.e. there is $O_2^G>O_1^S$.

Due to $A^X+O_1^S=2^Z(J+2)^Z$ as above a consequence, it indicates that A^X and O_1^S are two bilateral symmetric odd numbers with $2^{Z-1}(J+2)^Z$ as the symmetric center, according to the conclusion 3 in section 6. That is to say, take $2^{Z-1}(J+2)^Z$ as the symmetric center, O_1^S lies on the symmetric place of A^X.

However, due to $O_2^G>O_1^S$, when take $2^{Z-1}(J+2)^Z$ as the symmetric center, such that O_2^G does not lie on the symmetric place of A^X, so there is $A^X+O_2^G\neq 2^Z(J+2)^Z$, according to the conclusion 2 in the section 6.

Furthermore, let us analyze the only five cases of $O_2^G>O_1^S$ to confirm different limits of values of the exponent of O_2^G in these five cases.

(1) $O_2>O_1$ and $G>S$;

(2) $O_2=O_1$ and $G>S$;

(3) $O_2<O_1$ and $G>S$;

(4) $O_2>O_1$ and $G=S$;

(5) $O_2>O_1$ and $G<S$.

Since three cases of five cases at above contain $G>S$, so for the three cases, even if let $S=1$, there is also $G\geq 2$. Without doubt, $G\geq 3$ is included in $G\geq 2$, and we substitute $Y\geq 3$ for $G\geq 3$, then there is $A^X+O_2^Y\neq 2^Z(J+2)^Z$.

Since B and O_2 in $A^X+O_2^Y\neq 2^Z(J+2)^Z$ can express same odd numbers, thus, after substitute B for O_2, there is $A^X+B^Y\neq 2^Z(J+2)^Z$ under the known constraints. For other two case, when $S=1$: from $G<S$ to get $G=0$ and from $G=S$ to get $G=1$, both are useless here.

Apply the preceding way of doing thing, we can continue to prove that when $O=J+4$, $J+6$, …up to every odd number$\geq J+4$, there are likewise $A^X+B^Y\neq 2^Z(J+4)^Z$, $A^X+B^Y\neq 2^Z(J+6)^Z$ … up to general $A^X+B^Y\neq 2^ZO^Z$ under the known constraints.

Since O expresses all positive odd numbers at here, so it contains inevitably these odd numbers whose each and A^X or B^Y have at least one common prime factor. For inequalities in this case, we have proven them in the section 4. When A^X, B^Y and 2^ZO^Z have at least one common prime factor, we need to use each common prime factor to divide each and every term of the inequality, and then proceed as before.

Excepting the above, no two terms of each of remainder inequalities have a common prime factor, and that use the sign O_δ to express the odd number O in every such remainder inequality. As thus, we have proved $A^X+B^Y\neq 2^Z O_\delta^Z$.

Secondly, since there are many even numbers between $2^{Z-1}O_\delta^Z$ and $2^Z O_\delta^Z$, these even numbers can likewise become symmetric centers of A^X and B^Y, and it seem that being left out to use them. But, we have already proved $A^X+B^Y\neq 2^Z O_\delta^Z$, it can substitute for the proof of $A^X+B^Y\neq(2^{Z-1}+2h)O_\delta^Z$, where $h\geq 1$ and $2^Z>2^{Z-1}+2h$. Because when 2^{Z-1} rises to $2^{Z-1}+2h$, the exponent of 2 will decrease, and O_δ^Z will have also change accordingly, however, these changes only occur in positive odd numbers or/and their exponents.

9. Prove $A^X+2^Y\neq C^Z$ Under the Known Constraints

In this section, we are going to prove $A^X+2^Y\neq C^Z$ under the known constraints by the reduction to absurdity, [6].

***Proof*·** Based on exemplified $A^X+B^y=2^{K+1}$ under the known constraints except for Y, and Y=1, in the section 7, so there is $O_1^M+O_2^L=2^Y$ in which O_1 and O_2 are positive odd numbers, M and $Y\geq 3$, and L=1.

Assume that there is $A^X+2^Y=C^Z$ under the known constraints, then there is $A^X+O_1^M+O_2^L=C^Z$, i.e. $A^X+O_1^M=C^Z-O_2^L$.

Since there is $A^X+O_1^M\neq 2^G$ when X, M and $G\geq 3$, according to proven $A^X+B^Y\neq 2^Z$ under the known constraints in the section 7.

So, there is $C^Z-O_2^L\neq 2^G$, and after transpose a term of it, we get $O_2^L+2^G\neq C^Z$.

It is obvious that such an inequality $O_2^L+2^G\neq C^Z$ does not hold water, because there is surely $O_2^L+2^G=C^Z$ in which O_2 and C are positive odd numbers, G and Z≥3, and L is equal to 1 or even 2, such as $87^1+2^8=7^3$ and $7^2+2^5=3^4$.

Now that we deduce a false inequality derived from such an assumption, which means that such an assumption is wrong. That is to say, $A^X+2^Y=C^Z$ under the known constraints is wrong either.

Therefore, there is only $A^X+2^Y\neq C^Z$ under the known constraints.

10. Prove $A^X+2^YO^Y\neq C^Z$ Under the Known Constraints

We apply also the reduction to absurdity to prove $A^X+2^YO^Y\neq C^Z$ under the known constraints, in this section.

***Proof*·** Based on exemplified $A^X+B^Y=2^ZJ^Z$ under the known constraints except for Y, and Y=1, in the section 8, so there is $O_3^M+O_4^L=2^YO^Y$ in which O_3, O_4 and O are positive odd numbers, M and Y≥3, and L=1.

Assume that there is $A^X+2^YO^Y=C^Z$ under the known constraints, then there is $A^X+O_3^M+O_4^L=C^Z$, i.e. $C^Z-O_4^L=A^X+O_3^M$.

Since there is $C^Z-O_4^L=2^GO_n^G$ in which C, O_4 and O_n are positive odd numbers, Z and G≥3, and L is equal to 1 or even 2, such as $11^3-35^1=2^4\times3^4$; $3^4-7^2=2^5\times1^5$ and $1419857^5-1747866711689283^2=2^3\times6975757441^3$.

So, there is $A^X+O_3^M=2^GO_n^G$.

It is obvious that the equality $A^X+O_3^M=2^GO_n^G$ when X, M and G≥3 does not hold water, according to proven $A^X+B^Y\neq 2^ZO^Z$ under the known constraints in the section 8.

Now that we deduce a false equality derived from such an assumption, which means that such an assumption is wrong. That is to say, $A^X+2^YO^Y=C^Z$ under the known constraints is wrong either.

Therefore, there is only $A^X+2^YO^Y \neq C^Z$ under the known constraints.

11. Make A Summary and Reach the Conclusion

To sum up, on the one hand, we give examples to have exemplified $A^X+B^Y=C^Z$ under the necessary constraints, where A, B and C have at least one common prime factor, in the section 3.

On the other hand, we have already proved every kind of $A^X+B^Y \neq C^Z$ under the necessary constraints, where A, B and C have not a common prime factor, in the sections 4, 7, 8, 9 and 10.

Now that we have already proved each kind of equations and each kind of inequalities relating to the conjecture, then we continue to make a comparison between $A^X+B^Y=C^Z$ and $A^X+B^Y \neq C^Z$ under the necessary constraints, so it reaches surely the conclusion that an indispensable prerequisite of $A^X+B^Y=C^Z$ under the necessary constraints is exactly that A, B and C must have a common prime factor.

The proof was thus brought to a close. As a consequence, Beal's conjecture is tenable.

P.S. Prove Fermat's Last Theorem from Proven Beal's Conjecture

Since Fermat's last theorem is a special case of Beal's conjecture, [7], so we let X=Y=Z, then $A^X+B^Y=C^Z$ is changed to $A^X+B^X=C^X$.

After Beal's conjecture is proved to be true, we divide each term of

$A^X+B^X=C^X$ by the greatest common divisor of three terms of the equation itself, and get a set of solution with A, B and C as positive integers without common prime factor.

It is obvious that such a conclusion is in contradiction with proven Beal's conjecture. As thus, we have proved Fermat's last theorem by the reduction to absurdity, as easy as pie.

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