# Fractal Spacetime from Nonequilibrium Thermodynamics

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Abstract

The motivation for the continuous dimensionality of spacetime near the Fermi scale stems from two premises, namely, 1) dimensional regularization of perturbative quantum field theory (QFT), 2) the existence of non-trivial fixed points of Renormalization Group equations. Here we discuss a third reason, rooted in the behavior of non-equilibrium phase-space flows.

**Key words**: fractal dimensionality, minimal fractal manifold, multifractals, phase-space flows, non-equilibrium thermodynamics, beyond the physics of the Standard Model.

## 1. Introduction

It is known that, both in principles and methodology, QFT is built as replica of classical equilibrium thermodynamics. However, it is also known that equilibrium thermodynamics is *not universal* and that there are many collective phenomena that fall outside of its realm of validity. By contrast, *non-equilibrium thermodynamics* (NET) is an evolving field of research whose methods apply to a far broader range of contexts [1-2]. NET may be linked to complex manifestations of nonlinear dynamics in condensed matter, materials science, chemistry, and fluid physics [5-8, 10-14]. Insofar as QFT is concerned, a key ingredient of NET is that open systems outside equilibrium undergo *decoherence*, leading to the transition from quantum to classical behavior in the presence of persistent and unbalanced perturbations [9].

In line with the ideas of complex dynamics, it can be argued that the continuous dimensionality of spacetime may be inferred from two premises, namely, 1) dimensional regularization of perturbative QFT, and 2) the existence of non-trivial fixed points of Renormalization Group equations [5]. In this brief report we suggest that NET offers an independent route to the minimal fractal geometry of spacetime in proximity to the Fermi scale.

The paper is organized in the following way: section two elaborates on the topic of generic phase-space flows and their Lyapunov exponents; section three covers the connection between fractal dimensionality and the rate of information loss, as well as the emergence of fractal attractors in phase-space driven by the onset of NET.

## 2. Generic flows in phase-space

Let $\vec{\Gamma}(t) = \{\vec{\pi}, \vec{\varphi}\}$ represent the phase-space state of a generic system of classical fields $\vec{\varphi}$ with momenta $\vec{\pi} = \dot{\vec{\varphi}}$. The phase-space is denoted by $M$ and its dimension by $D$. Omitting the vector notation for simplicity, phase-space trajectories are described by the first-order differential equation [3]

$$\dot{\Gamma} = F(\Gamma) \tag{1}$$

in which $F$ is a vector-valued function of dimension $D$. The solution of (1) represents a *phase-space trajectory (or flow)* in $M$ and maps a phase point $\Gamma(0)$ to another point $\Gamma(t)$ as in $\Gamma(t) = f^t \Gamma(0)$. An arbitrary perturbation vector evolves in time according to

$$\delta\dot{\Gamma} = J(\Gamma)\,\delta\Gamma \tag{2}$$

where the Jacobian matrix $J(\Gamma)$ determines either the growth or shrinking tendency of the phase-space point $\Gamma(t)$. If the perturbation is normalized to unity ($\|\delta\Gamma\| = 1$) and points in a particular phase-space direction, matrix elements built from the Jacobian quantify the local growth or decay of $\|\delta\Gamma(t)\|$ and fix the local Lyapunov exponent $\Lambda(\Gamma(t))$ at the phase-space point $\Gamma(t) = \{\varphi(t), \pi(t)\}$. The evolution of an elemental volume in phase-space is described by

$$\delta V^{(D)}(t) \approx \delta V^{(D)}(0) \exp(\sum_{l=1}^{D} \lambda_l t) \tag{3}$$

where $\lambda_l$ represent global Lyapunov exponents. Alternatively, the sum of global Lyapunov exponents may be computed from [3]

$$\sum_{l=1}^{D} \lambda_l = \frac{d(\ln \delta V^{(D)}(t))}{dt} = \frac{\partial \dot{\Gamma}}{\partial \Gamma} \tag{4}$$

## 3. Information loss and the emergence of fractal dimensions

With reference to fig. 1, consider a system of classical fields in a *stationary non-equilibrium state*. Because the system is driven away from equilibrium, on a time-average basis, there is an irreversible energy transfer ($Q$) to a reservoir (referred to as a "*bath*"). In our context, the "*thermostat*" is a regulator whose function is to maintain a constant energy balance either by absorbing the instantaneous surplus or releasing the instantaneous energy deficit to the system. As Fig. 1 indicates, the transfer between thermostat and system is controlled by the parameter $\Omega$. It can be shown that the equations describing the behavior of the overall ensemble take the form [3-4]

$$\dot{\varphi}_i = \pi_i + \Phi(\{\varphi\},\{\pi\})\,X(t) \tag{5a}$$

$$\dot{\pi}_i = -\frac{\partial V(\varphi_i)}{\partial \varphi_i} + \Pi(\{\varphi\},\{\pi\})\,X(t) - s_i \Omega \pi_i \tag{5b}$$

Here, $\Phi X$ and $\Pi X$ stand for the driving "forces", $V(\varphi_i)$ for the potential function, whereas $-\Omega \pi_i$ denotes the action of the thermostat on the field $i$ selected with the binary switch $s_i \in \{0,1\}$. Although $\Omega$ may fluctuate and assume either positive or negative values, its time average $\langle \Omega \rangle > 0$ stays positive as energy is extracted from the system in the long run ($t \to \infty$).


**Fig 1**: Thermodynamic model of the field system (5)

By (4), the following thermodynamic relations hold [3-4]

$$\left\langle \frac{d \ln \delta V^{(D)}}{dt} \right\rangle = \sum_{l=1}^{D} \lambda_l = -d \sum_{i=1}^{N} s_i \langle \Omega \rangle = -\left\langle \frac{dS_{irr}}{dt} \right\rangle < 0 \tag{6}$$

where Boltzmann constant is set to unity $(k_B = 1)$, $d$ is the dimension of the physical space and where

$$\frac{dQ}{T} = dS_{irr} > 0 \tag{7}$$

Let's introduce at this point two plausible hypotheses reflecting the natural tendency of non-equilibrium systems to favor *dissipation* versus *steady evolution*. Specifically, we assume that

**A1)** both $\langle \Omega \rangle$ and the sum $\sum_i s_i$ stay constant in the long run,

**A2)** the average rate of entropy production (or information loss) $\langle \dot{S}_{irr} \rangle$ flows continuously with time.

It follows from A1), A2) and (6) that the spacetime dimensionality $(d+1)$ must also be a *continuous function of time* $d = d(t)$. Moreover, a positive average entropy rate $(\langle \dot{S}_{irr} \rangle > 0)$ necessarily implies a monotonically increasing spacetime dimension $(\dot{d} > 0)$, from the early Universe to the present era of cosmological evolution. This conclusion is in line with the fundamental viewpoint of dimensional flow and *fractional field theory* [5].

Two asymptotic cases are of interest regarding (6):

a) an analog of the ***thermodynamic limit*** condition is given by an unbounded entropy rate (or instantaneous thermalization) $(\langle \dot{S}_{irr} \rangle \to \infty)$ for systems having infinitely many regulated fields ($N \to \infty, s_i = 1, \forall i$),

b) an analog of a ***vacuum condition*** is given by a vanishing entropy rate (or infinite thermalization time) $(\langle \dot{S}_{irr} \rangle = 0)$ and a vanishing number of regulated fields ($N < \infty$, $s_i = 0, \forall i$).

Following [3], a key observation is that the stationary measure of the system resides on a *fractal attractor* with a vanishing phase-space volume. Also, the Gibbs entropy can be shown to diverge to minus infinity, which indicates that the thermodynamic entropy is *undefined* for the stationary non-equilibrium states described by (5) and (6).

It would be instructive to take the limit $d \to 3$ and evaluate the potential impact of these findings on the path-integral formulation of QFT and on the standard computation of transition amplitudes using propagators.

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