Simple Definitions of The Division By Zero And The Division By Zero Calculus: $[a^x/\log a]_{a=1} = x + 1/2$

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Abstract: In this note, we will state the definitions of the division by zero and division by zero calculus for popular using for the sake of their generality and great applications to mathematical sciences and the universe containing our basic ideas. In particular, we consider the value of the function $f(x,a)/\log a$ at $a = 1$.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

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1 Introduction

Many people consider that the division by zero is mysteriously difficult and will be a nonsense world. However, for the division by zero there is a very simple and natural new definition with great impact and applications. Since the definition is very important, we will state the definitions of the division by zero and division by zero calculus for popular using for the sake of their generality and great applications to mathematical sciences and the universe containing our basic ideas. In particular, we consider the value of the function $f(x,a)/\log a$ at $a = 1$.

2 Definitions of division by zero and division by zero calculus

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will **define**

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

However, we can consider the general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [1, 2, 3, 4]. Of course, we have many definitions and introductions of the division by zero and division by zero calculus.

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

Note that

$$(x^n)' = nx^{n-1}$$

and so

$$\left(\frac{x^n}{n}\right)' = x^{n-1}.$$

Here, we obtain the right result for $n = 0$

$$(\log x)' = \frac{1}{x}$$

by the division by zero calculus.

3 Statement of results

For $a > 0, a \neq 1$, the derivative of the function

$$\frac{a^x}{\log a}$$

is a^x. Then, for $a = 1$, we have the right result that the derivative of $x + 1/2$ is 1, by the division by zero calculus.

Many applications and general impacts of the division by zero calculus, see [1, 2, 3, 4].

Remarks

Of course, we obtain the general result: for a function $f(x,a)$ that is analytic in a around $a=1$ we have

$$\frac{f(x,a)}{\log a}|_{a=1} = \frac{\partial f(x,a)}{\partial a}|_{a=1} + \frac{1}{2}f(x,1).$$

For example,

$$\frac{\sin(ax)}{\log a}|_{a=1} = x\cos x + \frac{1}{2}\sin x,$$
$$\frac{5^{ax}}{\log a}|_{a=1} = 5^x x\log 5 + \frac{1}{2}5^x,$$
$$\frac{x^a}{\log a}|_{a=1} = x\log x + \frac{1}{2}x,$$

and

$$\frac{(a-x)^n}{\log a}|_{a=1} = n(1-x)^{n-1} + \frac{1}{2}(1-x)^n.$$

By using Wolfram|Alpha, we shall state the typical general formulas:

- $$\frac{f(x,a)}{\sin^2 a}|_{a=0} = \frac{1}{6}\left(3f^{(0,2)}(x,0) + 2f(x,0)\right).$$
- $$\frac{f(x,a)}{\cos^2 a}|_{\pi/2} = \frac{1}{6}\left(3f^{(0,2)}(x,\pi/2) + 2f(x,\pi/2)\right).$$
- $$\frac{f(x,a)}{\tan^2 a}|_{a=0} = \frac{1}{6}\left(3f^{(0,2)}(x,0) - 4f(x,0)\right).$$
- $$\frac{f(x,a)}{(\log a)^2}|_{a=1} = f^{(0,1)}(x,1) + \frac{1}{2}f^{(0,2)}(x,1) + \frac{1}{12}f(x,1).$$
- $$f(x,a)\tan^2 a|_{a=\pi/2} = \frac{1}{6}\left(3f^{(0,2)}(x,\pi/2) - 4f(x,\pi/2)\right).$$
- $$\frac{f(x,a)}{a^2-n^2}|_{a=n} = -\frac{f(x,n) - 2nf^{(0,1)}(x,n)}{4n^2}.$$

- $$\frac{f(x)(ax+b)}{cx+d}|_{x=-d/c} = \frac{1}{c^2}\left((bc-ad)f'\left(-\frac{d}{c}\right) + acf\left(-\frac{d}{c}\right)\right).$$

We expect computers and mathematical softs can do the division by zero and division by zero calculus over the mysterious history of the division by zero. They are very applicable over mathematical sciences.

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References

[1] H. Okumura, *Geometry and division by zero calculus,* International Journal of Division by Zero Calculus, **1**(2021), 1-36. https://romanpub.com/dbzc.php

[2] S. Saitoh, *Introduction to the Division by Zero Calculus*, Scientific Research Publishing, Inc. (2021.2), 202 pages.

[3] S. Saitoh, *History of Division by Zero and Division by Zero Calculus*, International Journal of Division by Zero Calculus, **1** (2021), 1-38. https://romanpub.com/dbzc.php

[4] S. Saitoh, *Division by Zero Calculus - History and Development*, Scientific Research Publishing, Inc. (2021.11), 332 pages.