On Maximal Acceleration, Strings with Dynamical Tension, and Rindler Worldsheets

Carlos Castro Perelman
Ronin Institute, 127 Haddon Place, Montclair, N.J. 07043, USA
Quantum Gravity Research
Los Angeles, CA. 90290, USA
perelmanc@hotmail.com

November 2021, Revised January 2022

Abstract

Starting with a different action and following a different procedure than the construction of strings with dynamical tensions described by Guendelman [1], a variational procedure of our action leads to a coupled nonlinear system of $D+4$ partial differential equations for the D string coordinates X^μ and the *quartet* of scalar fields $\varphi^1, \varphi^2, \phi, T$, including the dilaton $\phi(\sigma)$ and the tension $T(\sigma)$ field. Trivial solutions to this system of complicated equations lead to a constant tension and to the standard string equations of motion. One of the most relevant features of our findings is that the Weyl invariance of the traditional Polyakov string is traded for the invariance under area-preserving diffeomorphisms. The final section is devoted to the physics of maximal proper forces (acceleration), minimal length within the context of Born's Reciprocal Relativity theory [6] and to the Rindler world sheet description of accelerated *open* and *closed* strings from a very different approach and perspective than the one undertaken by [7].

Keywords : Strings; Gravity; Rindler Spacetimes; Dynamical Tension; Born Reciprocal Relativity.

PACS : 04.50.Kd; 11.10.Kk; 14.80.-j

1 Introduction

Guendelman [1] (and references therein) over the years developed the modified measure formalism which allowed him, among other things, to show that the string tension does not have to be put in by hand, but it can be dynamically generated. The string tension appears, but as an additional dynamical degree of freedom . Recently, Guendelman [1] has shown that the string tension is not universal, but rather each string generates its own string tension, with its own different value. He also defined a new Tension scalar background field which can change locally the value of the string tension along the world sheets of the strings. When there are many strings with different string tensions this Tension field can be determined from the requirement of world sheet conformal invariance. For two types of string tensions, and depending on the relative sign of the tensions, he obtained non singular cosmologies and warp space scenarios and also scenarios where the Hagedorn temperature is avoided in the early universe or in regions of warped spacetime where the string tensions become very big.

In this first part of this work we shall use a very different action and follow a very different procedure than in [1]. A variational procedure of our action leads to a coupled nonlinear system of $D + 4$ partial differential equations for the D string coordinates X^μ and the *quartet* of scalar fields $\varphi^1, \varphi^2, \phi, T$, including the dilaton $\phi(\sigma)$ and the tension $T(\sigma)$ field. Trivial solutions to this system of complicated equations lead to a constant tension and to the standard string equations of motion of the Nambu-Goto string in the orthonormal gauge, which are the same as the equations of motion of the Polyakov string in the conformal gauge $\gamma_{ab} = e^{\phi(\sigma)}\eta_{ab}$.

One of the most relevant features of our findings is that the Weyl invariance of the traditional Polyakov string is traded for the invariance under area-preserving diffeomorphisms, which very recently has been an extensive area of research in Celestial Conformal Field Theories (CCFTs) which are based in introducing conformal correlation functions living on the celestial sphere and which capture the soft graviton and gluon scattering amplitudes (infrared physics) in asymptotically flat spacetimes (null infinity). It was found that the algebra $w_{1+\infty}$ plays a fundamental role [5]. Earlier work showing the relevance of w_∞ algebras in Quantum Gravity can be found in [4]. To our knowledge, the first person to realize how the w_∞ algebra emerges from a dimensional reduction mechanism of pure gravity to two dimensions was Yoon [3].

We will show that the current formulation of string theory based on the Polyakov action is just a very special case of a more complex action involving a dynamical tension field and which is part of the modified measure formalism. The final section is devoted to the physics of maximal proper forces (acceleration), minimal length within the context of Born's Reciprocal Relativity theory [6] and to the Rindler world sheet description of accelerated *open* and *closed* strings from a very different approach and perspective than the one undertaken by [7].

2 Modified String Actions, a Dynamical Tension, and Composite Worldsheet Vector Fields

Let us begin with the modified string action

$$S = - \int d^2\sigma \ \Phi(\varphi^1, \varphi^2) \ \left(\frac{1}{2} \ \gamma^{ab} \ \partial_a X^\mu \ \partial_b X^\nu \ g_{\mu\nu} \ - \ \frac{1}{2T_o \sqrt{|\gamma|}} \ \Phi(\varphi^1, \varphi^2) \right) \tag{1}$$

where T_o is a constant tension parameter; γ^{ab} is an auxiliary world sheet metric; $|\gamma| \equiv |det(\gamma_{ab})|$, and $X^\mu(\sigma^1, \sigma^2)$ are the string coordinates. We adopt the $c = 1$ units.

The measure field density $\Phi(\varphi^1, \varphi^2)$ in eq-(1) is a function of two dimensionless worldsheet scalars $\varphi^i(\sigma^1, \sigma^2), i = 1, 2$, and is defined in terms of the Poisson bracket $\{\varphi^1, \varphi^2\}$ taken with respect to the worldsheet coordinates. It is a mathematical curiosity that the Jacobian $\{\varphi^1, \varphi^2\}$ can also be rewritten in terms of the field strength F_{ab} associated with a *composite* Abelian world sheet vector field defined in terms of the scalars $A_c = \epsilon_{ij}\varphi^i\partial_c\varphi^j$ as follows

$$F_{cd} = \partial_c A_d - \partial_d A_c \Rightarrow \epsilon^{cd} F_{cd} \ = \ \Phi(\varphi^1, \varphi^2) \ = \ \epsilon_{ij} \ \epsilon^{cd} \ \partial_c \varphi^i \ \partial_d \varphi^j \tag{2}$$

The effective tension in (1) is no longer a constant but instead is a scalar field defined as the ratio of two scalar densities

$$T(\sigma^1, \sigma^2) \ \equiv \ \frac{\Phi(\varphi^1, \varphi^2)}{\sqrt{|\gamma|}} \tag{3}$$

The tension field defined by eq-(3) is given in terms of the Jacobian $\{\varphi^1, \varphi^2\}$ which describes the area-measure which in general is considered to be a positive quantity. The $\sqrt{|\gamma||}$ in the denominator is chosen with the positive sign. A negative tension would then correspond to having worldsheet locations with a negative area-measure. If one wishes to avoid regions with a negative area-measure then one should take the absolute values of the Jacobian $|\{\varphi^1, \varphi^2\}|$ ruling out negative tension values. When one performs a change of coordinates in multi-dimensional integrals one must take the absolute values of the Jacobian, meaning that one has to choose the correct sign in those regions where the Jacobian is negative. In other words, one must break the domain of integration into different regions and choose the appropriate sign in the Jacobian to ensure positivity of the measure in all of those regions. A trivial example is $|x| = x$ for $x > 0$; but $|x| = -x$ for $x < 0$.

A dynamical tension field $T \sim (l_s)^{-2}$ is correlated to a dynamical (effective) string length l_s. The gravitational constant G in $3 + 1$-dim is given in terms of the string coupling g_s and the string length l_s by $G = (g_s l_s)^2$. Therefore one would expect a dynamical G, not unlike it occurs in the Renormalization Group program. Nevertheless, since the string coupling g_s is determined in terms of the background dilaton field, in curved backgrounds one could envision the possibility that both g_s and l_s vary in such a way as to leave their product (G) invariant. However, in general, one would expect to have a dynamical G. [1]

The above action *differs* from the one in [1]

$$S \ = - \int d^2\sigma \ \Phi(\varphi^1, \varphi^2) \ \left(\frac{1}{2} \ \gamma^{ab} \ \partial_a X^\mu \ \partial_b X^\nu \ g_{\mu\nu} \ - \ \frac{1}{2\sqrt{|\gamma|}} \ \epsilon^{ab} \mathcal{F}_{ab} \right) \quad (4)$$

in the definition of the field strength $\mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a$ given in terms of an auxiliary Abelian (non-composite) gauge field $\mathcal{A}_a$ [2]. This will be a key difference between this work and that of [1].

It is known that the Polyakov p-brane actions admit a world volume cosmological constant except in the string case $p = 1$

$$S_p \ = -\frac{T_p}{2} \int d^{p+1}\sigma \ \sqrt{|\gamma|} \ \left(\gamma^{ab} \ \partial_a X^\mu \ \partial_b X^\nu \ g_{\mu\nu} \ - \ (p-1) \right) \quad (5)$$

because the world sheet cosmological constant vanishes in the string case $\Lambda_{p=1} = \frac{1}{2}(p-1)T_{p=1} = 0$ when $p = 1$. This finding results from an inconsistency in the field equations if a cosmological constant is introduced in the string case [2].

One of the key differences between the actions in eqs-(1,5) is that the tension in (3) is *no* longer *constant*, and as a result, the modified string action (1) admits a non-vanishing extra term of the form

$$\frac{\Phi(\varphi^1, \varphi^2)}{2 \ T_o \ \sqrt{|\gamma|}} \ \Phi(\varphi^1, \varphi^2) \ = \ \frac{1}{2} \ \frac{T^2(\sigma^1, \sigma^2)}{T_o} \ \sqrt{|\gamma|} \quad (6)$$

Therefore, one can see that the second term in eq-(6) will now play an analogous role of a variable world-sheet cosmological "constant" associated with the modified string action (1) when one replaces the ordinary measure $\sqrt{|\gamma|}$ for $\Phi(\varphi^1, \varphi^2)$ and introduces a variable effective string tension.

A variation of the action (1) with respect to the scalar fields $\varphi^i, i = 1, 2$ yields

$$\epsilon^{cd} \ (\partial_d \varphi^i) \ \partial_c \left(\ \frac{1}{2} \gamma^{ab} (\partial_a X^\mu)(\partial_b X^\nu) g_{\mu\nu} \ - \ \frac{\Phi(\varphi^1, \varphi^2)}{T_o \sqrt{|\gamma|}} \ \right) \ = \ 0, \quad i = 1, 2 \quad (7)$$

A *trivial* solution to eq-(7) would be to set the terms inside the parenthesis of (7) to a constant but we shall *not* follow this path and instead look for nontrivial solutions as follows. After defining

$$F(\sigma^1, \sigma^2) \ \equiv \ \frac{1}{2} \gamma^{ab} (\partial_a X^\mu)(\partial_b X^\nu) g_{\mu\nu} \ - \ \frac{\Phi(\varphi^1, \varphi^2)}{T_o \sqrt{|\gamma|}} \quad (8)$$

[1] We thank the referee for bringing to our attention the negative tension case and the possibility of a dynamical gravitational coupling

[2] A dimensionful parameter that was set to unity needs to be introduced in the $\epsilon^{ab} \mathcal{F}_{ab}$ term to match units

Eq-(7) can be rewritten in terms of Poisson brackets $\{,\}$ as

$$\{\varphi^1, F(\sigma^1,\sigma^2)\} \ = \ 0, \quad \{\varphi^2, F(\sigma^1,\sigma^2)\} \ = \ 0 \tag{9}$$

A variation of the action (1) with respect to γ^{ab} yields the stress energy tensor

$$T_{ab} \ = \ (\partial_a X^\mu)\ (\partial_b X^\nu)\ g_{\mu\nu} \ - \frac{1}{2T_o}\ \gamma_{ab}\ \frac{\Phi(\varphi^1,\varphi^2)}{\sqrt{|\gamma|}} \ = 0 \tag{10}$$

Taking the trace of the stress energy tensor yields

$$\gamma^{ab}T_{ab} \ = \ \gamma^{ab}\ (\partial_a X^\mu)\ (\partial_b X^\nu)\ g_{\mu\nu} \ - \ \frac{\Phi(\varphi^1,\varphi^2)}{T_o\sqrt{|\gamma|}} \ = 0 \tag{11}$$

and from eqs-(8,11) one can infer that

$$F(\sigma^1,\sigma^2) \ = \ \frac{1}{2}\gamma^{ab}(\partial_a X^\mu)(\partial_b X^\nu)g_{\mu\nu} - \frac{\Phi(\varphi^1,\varphi^2)}{T_o\sqrt{|\gamma|}} \ = \ -\frac{1}{2}\ \gamma^{ab}(\partial_a X^\mu)(\partial_b X^\nu)g_{\mu\nu} \ =$$

$$-\frac{1}{2}\ \frac{\Phi(\varphi^1,\varphi^2)}{T_o\sqrt{|\gamma|}} \tag{12}$$

In order to proceed further we shall make the following ansatz relating the the auxiliary metric γ_{ab}, and the induced metric h_{ab} on the world sheet resulting from the embedding, as follows

$$h_{ab} \ \equiv \ (\partial_a X^\mu)(\partial_b X^\nu)g_{\mu\nu} \ = \ \lambda(\sigma^1,\sigma^2)\ \gamma_{ab} \ \Rightarrow \ \gamma_{ab} \ = \ \frac{h_{ab}}{\lambda(\sigma^1,\sigma^2)} \tag{13}$$

where $\lambda(\sigma^1,\sigma^2)$ is a judicious scaling function to be determined. Therefore, from eqs-(12,13) one arrives at $F(\sigma^1,\sigma^2) = -\lambda(\sigma^1,\sigma^2)$, and the ratio of the tension scalar field T to the constant tension parameter T_o becomes

$$\frac{T(\sigma^1,\sigma^2)}{T_o} \ = \ \frac{\Phi(\varphi^1,\varphi^2)}{T_o\ \sqrt{|\gamma|}} \ = \ \frac{2\ \{\varphi^1,\varphi^2\}}{T_o\ \sqrt{|\gamma|}} \ = \ 2\ \lambda(\sigma^1,\sigma^2) \ = \ -2\ F(\sigma^1,\sigma^2) \tag{14}$$

And one finds that the measure field can be rewritten in terms of the string embedding coordinates as

$$\Phi(\varphi^1,\varphi^2) \ = \ 2\ \{\varphi^1,\varphi^2\} \ = \ 2\ T_o\ \sqrt{|det\ (\partial_c X^\mu)\ (\partial_d X^\nu)\ g_{\mu\nu}\ |} \ =$$

$$2\ T_o\ \sqrt{|det\ h_{cd}|} \ = \ 2\ T_o\ \sqrt{|h|} \tag{15}$$

Finally, from eqs-(14,15) one obtains the following relations

$$\{\varphi^1,\varphi^2\} \ = \ \frac{T(\sigma)\sqrt{|\gamma|}}{2} \ = \ \frac{T(\sigma)\sqrt{|h|}}{2\ \lambda(\sigma)} \ = \ T_o\ \sqrt{|h|} \tag{16}$$

The reparametrization invariance of the action (1) allows to choose a world-sheet coordinate system and rewrite the metric in the form $\gamma_{ab} = e^{\phi(\sigma)}\eta_{ab} \Rightarrow \sqrt{|\gamma|} = e^{\phi(\sigma)}$ (if there are no global topological obstructions), since every metric in $2D$ is conformally flat, and the first two terms of (16) become

$$\{\varphi^1, \varphi^2\} \;=\; \frac{T(\sigma)\; e^{\phi(\sigma)}\; \sqrt{|det(\eta_{ab})|}}{2} \;=\; \frac{T(\sigma)\; e^{\phi(\sigma)}}{2} \tag{17}$$

after choosing the conformal gauge [3]

In the same fashion that the Polyakov string still has a *residual* symmetry (after choosing the conformal gauge $\gamma_{ab} = e^{\phi}(\sigma)\eta_{ab}$) provided by the so-called conformal reparametrizations associated with the Virasoro algebra which preserve the conformal gauge, we have now the area-preserving diffeomorphisms (diffs) : $\{\tilde\sigma^1, \tilde\sigma^2\}_{\sigma^a} = 1$, as the residual symmetry which leave eq-(17) invariant.

The area-preserving diffs transformations are defined by

$$\tilde\sigma^1 \;=\; f^1(\sigma^1, \sigma^2); \;\; \tilde\sigma^2 \;=\; f^2(\sigma^1, \sigma^2); \;\; \{\tilde\sigma^1,\; \tilde\sigma^2\}_{\sigma^i} \;=\; 1 \tag{18}$$

Because φ^i, T, ϕ are scalars one has

$$T(\sigma^a) \;=\; \tilde T(\tilde\sigma^a), \;\; \tilde\varphi^i(\tilde\sigma^a) \;=\; \varphi^i(\sigma^a), \;\; e^{\phi(\sigma)} \;=\; e^{\tilde\phi(\tilde\sigma)} \tag{19}$$

since the square root of a determinant is a scalar density of unit weight one has

$$\sqrt{|det(\eta_{ab})|} \;=\; 1 \;\rightarrow\; \sqrt{|det(\eta_{ab})|}\; \left(\{\tilde\sigma^1,\; \tilde\sigma^2\}_{\sigma^a}\right)^{-1} \;=\; \sqrt{|det(\eta_{ab})|} \;=\; 1 \tag{20}$$

Consequently, eq-(17) remains invariant under area-preserving diffs

$$\{\varphi^1,\; \varphi^2\}_{\sigma^a} \;=\; \frac{T(\sigma)\; e^{\phi(\sigma)}}{2} \;=\; \{\tilde\varphi^1,\; \tilde\varphi^2\}_{\tilde\sigma^a} \;=\; \frac{\tilde T(\tilde\sigma)\; e^{\tilde\phi(\tilde\sigma)}}{2} \tag{21}$$

due to the invariance of the Poisson brackets under area-preserving diffs

$$\{\varphi^1,\; \varphi^2\}_{\sigma^a} \;\;=\; \{\varphi^1,\; \varphi^2\}_{\tilde\sigma^a}\; \{\tilde\sigma^1,\; \tilde\sigma^2\}_{\sigma^a} \;=\; \{\varphi^1,\; \varphi^2\}_{\tilde\sigma^a} \;=\; \{\tilde\varphi^1,\; \tilde\varphi^2\}_{\tilde\sigma^a} \tag{22}$$

A variation of the action (1) with respect to the string coordinates X^μ leads to the equations of motions of the string

$$\frac{1}{\Phi}\partial_a(\Phi\; \gamma^{ab}\; \partial_b X^\mu) \;+\; \gamma^{ab}\; \partial_a X^\nu\; \partial_b X^\rho\; \Gamma^\mu_{\nu\rho} \;=\; 0, \;\; \Phi \neq 0 \tag{22}$$

where $\Gamma^\mu_{\nu\rho}$ is the affine connection of the background metric $g_{\mu\nu}(X)$. One can compare the string equations of motion eq-(22) with the ones obtained from the Polyakov string action [2]

$$\frac{1}{\sqrt{|\gamma|}}\partial_a(\sqrt{|\gamma|}\; \gamma^{ab}\; \partial_b X^\mu) \;+\; \gamma^{ab}\; \partial_a X^\nu\; \partial_b X^\rho\; \Gamma^\mu_{\nu\rho} \;=\; 0 \tag{23}$$

[3] Note that at first sight eq-(17) does not seem correct because the left-hand side is a scalar density, while the second term in right-hand side is a scalar. The reason is that the value of the density factor is unity $\sqrt{|det(\eta_{ab})|} = 1$

the only difference is in the replacement $\Phi = 2\{\varphi^1, \varphi^2\} \rightarrow \sqrt{|\gamma|}$.

After using the expression for the measure Φ given by eq-(17), the string equations of motion (22) simplify even further for strings moving in flat Minkowski target spacetime backgrounds $g_{\mu\nu} = \eta_{\mu\nu} = diag(-1, +1, +1, \ldots, +1), \Gamma^\mu_{\nu\rho} = 0$,

$$\partial_a(\sqrt{|\gamma|}\ T\ \gamma^{ab}\ \partial_b X^\mu\ ) \ = \ \partial_a(\ T\ \eta^{ab}\ \partial_b X^\mu\ ) \ = $$
$$- \ \partial_{\sigma^1}(T\ \partial_{\sigma^1} X^\mu) \ + \ \partial_{\sigma^2}(T\ \partial_{\sigma^2} X^\mu) \ = \ 0, \quad \mu = 0, 1, 2. \ldots, D-1 \qquad (30a)$$

Eq-(30a) is a direct result of choosing the conformal gauge for the auxiliary world sheet metric $\gamma_{ab} = e^\phi \eta_{ab}, \gamma^{ab} = e^{-\phi}\eta^{ab} \Rightarrow \sqrt{|\gamma|}\ T\ \gamma^{ab} = T\eta^{ab}; \eta_{ab} = diag(-1, +1)$. Dividing the first term of (30a) by $\sqrt{|\gamma|} \neq 0$ it gives

$$\frac{1}{\sqrt{|\gamma|}}\ \partial_a(\sqrt{|\gamma|}\ T\ \gamma^{ab}\ \partial_b X^\mu\ ) \ = \ \frac{T}{\sqrt{|\gamma|}}\ \partial_a(\sqrt{|\gamma|}\ \gamma^{ab}\ \partial_b X^\mu\ ) + \gamma^{ab}\ (\partial_a T)\ (\partial_b X^\mu) \ = \ 0 \qquad (30b)$$

Because T, X^μ are world-sheet scalars, $\partial_a T, \partial_b X^\mu$ are world-sheet vectors so the third term of (30b) is a scalar. The second term of (30b) $T\gamma^{ab}\nabla_a\nabla_b X^\mu$ given by T times the D'Alambertian is a scalar. Therefore, eq-(30b) is fully invariant under reparametrizations but eq-(30a) is only invariant under area-prerserving diffs since it differs from (30b) by a scalar density factor $\sqrt{|\gamma|} \neq 0$.

We are not finish yet. The Jacobi identities among the triplet of scalars φ^1, φ^2, T, in conjunction with eq-(17) and the following equations, stemming directly from eq-(9),

$$\{\varphi^1\ ,\ T(\sigma^1, \sigma^2)\} \ = \ 0, \quad \{\varphi^2\ ,\ T(\sigma^1, \sigma^2)\} \ = \ 0 \qquad (31)$$

requires introducing the *additional* equation involving the dilaton $\phi(\sigma)$

$$\{T(\sigma^1, \sigma^2),\ T(\sigma^1, \sigma^2)\ e^{\phi(\sigma^1, \sigma^2)}\} \ = \ 0 \ \Rightarrow \ \{T(\sigma^1, \sigma^2),\ e^{\phi(\sigma^1, \sigma^2)}\} \ = \ 0 \qquad (32)$$

To sum up, collecting eqs-(17,30,31,32) we arrive at a coupled nonlinear system of $D + 4$ partial differential equations for the D string coordinates X^μ and the *quartet* of scalars $\varphi^1, \varphi^2, \phi, T$. It is interesting that besides the original two auxiliary scalars $\varphi^i, i = 1, 2$ defining the measure scalar density field one ends up adding a dilaton ϕ and the tension scalar field T.

One may note that the system of complicated eqs-(17,30,31,32) can be easily solved when $\varphi^1 = \kappa\sigma^1; \varphi^2 = \kappa\sigma^2$, or $\varphi^1 = \kappa\sigma^2; \varphi^2 = -\kappa\sigma^1$, both solutions lead to $T = 2\kappa^2 = 2T_o =$ constant, and $\phi(\sigma) = 0, \lambda(\sigma) = 1$. In this very simple case, eqs-(30) turn out to be

$$- \ \partial^2_{\sigma^1} X^\mu \ + \ \partial^2_{\sigma^2} X^\mu \ = \ 0, \quad \mu = 0, 1, 2, \ldots, D-1 \qquad (33)$$

and are equivalent to the usual equations of motion of a Nambu-Goto string in the orthonormal gauge, and which are also the same as the equations of motion of the Polyakov string in the conformal gauge $\gamma_{ab} = e^{\phi(\sigma)}\eta_{ab}$.

One also finds in this simple case that when $\phi(\sigma) = 0; \lambda(\sigma) = 1; \gamma_{ab} = e^{\phi}\eta_{ab} \to \eta_{ab}$, that the induced metric h_{ab} in eq-(13) becomes

$$h_{ab} \equiv (\partial_a X^\mu)\ (\partial_b X^\nu)\ \eta_{\mu\nu} = \eta_{ab} \Rightarrow$$
$$h_{12} = h_{21} = (\partial_1 X^\mu)\ (\partial_2 X_\mu) = 0$$
$$h_{11} = (\partial_1 X^\mu)\ (\partial_1 X_\mu) = -1; \quad h_{22} = (\partial_2 X^\mu)\ (\partial_2 X_\mu) = 1 \tag{34}$$

Eqs-(34) are indeed compatible with the vanishing of the stress energy tensor T_{ab} of the Polyakov string in Minkowski backgrounds resulting from reparametrization invariance

$$T_{12} = T_{21} = (\partial_1 X^\mu)\ (\partial_2 X_\mu) = 0$$
$$T_{11} = T_{22} = \frac{1}{2}\left(\ (\partial_1 X^\mu)\ (\partial_1 X_\mu) + (\partial_2 X^\mu)\ (\partial_2 X_\mu)\ \right) = 0 \tag{35}$$

The general solution of the wave equation (33) is a sum of right-movers and left-movers

$$X^\mu = X^\mu_R(\sigma^1 - \sigma^2) + X^\mu_L(\sigma^1 + \sigma^2) \tag{36}$$

To find the explicit form of X^μ_R, X^μ_L one should require X^μ to be real-valued and satisfying the quadratic constraints (35), and obeying suitable boundary conditions for closed and open strings [2]. The problem is solved by performing a Fourier mode expansion as shown in [2].

Another trivial solution is to set directly the tension to a constant $T = 2T_o \Rightarrow \lambda(\sigma) = 1$ so that eqs-(31,32) are trivially obeyed. The measure $2\{\varphi^1, \varphi^2\} = 2T_o e^{\phi}$ is then provided entirely in terms of the dilaton, and the string equations of motion reduce to the standard ones (33). In the most general case it is desirable to find non-trivial solutions to eqs-(17,30,31,32) leading to a non-constant measure $\Phi = 2\{\varphi^1, \varphi^2\}$, a non-constant tension $T(\sigma)$ and a non-constant dilaton field $\phi(\sigma)$ [4]. Because this is a very difficult task due to the coupled and non-linear nature of the partial differential equations eqs-(17,30,31,32), it is beyond the scope of this work to find non-trivial solutions.

It is important to emphasize that one cannot set a priori T to a constant in eqs-(1,6). What has been done is, firstly, to perform a variation of the action (1) with respect to all the fields, and only afterwards, search for solutions to the equations of motion. And only then, one has found constant solutions for $T = 2T_o$ in a very special case. It is meaningless to try to vary a constant $T = 2T_o$ from the beginning. The underlying reason why the measure (the tension) is dynamical is because the term (6) in the action (1) is quadratic in the field strength associated with the composite gauge field : $\epsilon^{ab}F_{ab}\epsilon_{cd}F^{cd}$, with $A_a = \epsilon_{ij}\varphi^i\partial_a\varphi^j$. This term is what generates dynamics to the scalars φ^i (to the measure), and which in turn, generates dynamics to the tension field. This is another key difference between this work and [1].

[4] In many of our equations we used the short-hand notation $\sigma = \sigma^a = \sigma^1, \sigma^2$ for convenience

The zero and infinite tensions are two interesting cases to explore. An interesting double-scaling limit is when $T \rightarrow 0, \phi \rightarrow \infty$, or $T \rightarrow \infty, \phi \rightarrow -\infty$, such that the measure remains finite and non-zero. If the tension scalar $T(\sigma^1, \sigma^2) = 0$ vanishes at certain *locations* on the world sheet, the measure also vanishes (it is degenerate) at those locations if ϕ does not diverge. Also, having a divergent determinant $|det(\gamma_{ab})| = e^{2\phi} \rightarrow \infty$ at certain locations is a signal of *putative* singularities on the world sheet.

There are many projects worth mentioning, like to generalize this construction to p-branes and to explore the role of a dynamical tension with the running of the world-volume cosmological "constant". The p-brane extension of the action (1) requires to replace Poisson brackets for Nambu-Poisson brackets $\{\varphi^1, \varphi^2, \varphi^3, \ldots, \varphi^{p+1}\}$ involving $p+1$ scalars $\varphi^i, i = 1, 2, \ldots, p+1$. We have shown that the current formulation of string theory based on the Polyakov action is just a very special case of a more complex action (1) involving a dynamical tension field and which is part of the modified measure formalism. Guendelman [1] already found many cosmological applications. The most general and non-trivial solutions to eqs-(17,30,31,32) remain to be found in addition to pursuing the quantization program.

3 Maximal acceleration and Rindler Worldsheets

The Rindler wedge in $2D$ Minkowski space is comprised of an infinite family of hyperbolas associated with the world lines of uniformly accelerated particles and parametrized by ξ with $-\infty \leq \xi \leq +\infty$ representing the spatial coordinate, and $-\infty \leq \eta \leq +\infty$ representing the temporal one. The asymptotes (of the hyperbolas) associated with outgoing and incoming null lines are described $\xi = -\infty; \eta = \pm\infty$, respectively. The latter hyperbolic trajectories in $2D$ Minkowski spacetime that are described by the Rindler coordinates are given by

$$t \ = \ \frac{e^{a\xi}}{a} \ sinh(a\eta), \quad x \ = \ \frac{e^{a\xi}}{a} \ cosh(a\eta), \quad 0 < a < \infty \tag{37}$$

$$(ds)^2 \ = \ (dt)^2 \ - \ (dx)^2 \ = \ e^{2a\xi} \ \left(\ (d\eta)^2 \ - \ (d\xi)^2 \ \right) \tag{38}$$

Because the Rindler metric is conformally flat it obeys $\sqrt{|\gamma|}\gamma^{ab} = \eta^{ab}$, and one can explicitly verify that the hyperbolic trajectories (37) are solutions to the D'Alambert equation in two dimensions

$$\frac{\partial^2 t}{\partial \eta^2} \ - \ \frac{\partial^2 t}{\partial \xi^2} \ = \ 0, \quad \frac{\partial^2 x}{\partial \eta^2} \ - \ \frac{\partial^2 x}{\partial \xi^2} \ = \ 0 \tag{39}$$

One can also verify by simple inspection that the expressions (37) *obey* the quadratic constraints (35) of the Polyakov string in $2D$ Minkowski spacetime. Namely, one can interpret the above solutions as the coordinates $t(\eta, \xi), x(\eta, \xi)$

of the two-dim world sheet of a string embedded in a two-dim Minkowski spacetime, where η, ξ are the corresponding temporal and spatial world sheet parameters.

Physically, the hyperbolic trajectories (37) parametrizing the right side of the Rindler wedge describe an infinitely large open string stretched along the positive axis and comprised of a continuum of point-masses, with each one of them experiencing a continuum of different proper accelerations, and whose values $g = g(\xi)$ depend on their ξ-locations along the string. The left side of the Rindler wedge is just the parity and time reversals of (37) . The proper acceleration associated with each point mass at ξ is $g(\xi) = ae^{-a\xi}$ and its associated hyperbola obeys $x^2 - t^2 = \frac{1}{g^2(\xi)}$. a is a fixed acceleration parameter and must not be confused with the proper acceleration $g(\xi)$.

This open string scenario is space-filling : the open string fills-in the Rindler wedge and for this reason one may label it as a "Rindler" string which sweeps a "Rindler" worldsheet. This scenario must *not* be confused with the one involving accelerating worldsheets of [7] (open strings observed in inertial and non-inertial frames of reference) nor with a static open string whose worldsheet is just a rectangular strip in $2D$ Minkowski spacetime.

Let us choose next the hyperbolic trajectory corresponding to a given point mass located at a given fixed value of $\xi = \xi_o =$ constant, so that $d\xi = 0$, and from the Rindler interval (38) one finds $g(\xi_o) = ae^{-a\xi_o} \Rightarrow e^{-a\xi_o} ds = d\eta$. Thus, the proper force squared acting on the point mass located at the location ξ_o is given by

$$\mathcal{F}^2 \ = \ \frac{dp^\mu}{ds} \frac{dp_\mu}{ds} \ = \ \frac{dp^\mu}{d\eta} \frac{dp_\mu}{d\eta} \left(\frac{d\eta}{ds}\right)^2 \ = \ - \ (mae^{a\xi_o})^2 \ (e^{-a\xi_o})^2 \ = \ - \ m^2 a^2 \quad (40)$$

The phase-space interval $(d\omega)^2$ corresponding to the cotangent of the $2D$ Rindler spacetime is [6]

$$(d\omega)^2 \ = \ dx^\mu dx_\mu \ + \ \frac{dp^\mu dp_\mu}{\mathbf{b}^2} \ = \ (ds)^2 \left(1 \ + \ \frac{\frac{dp^\mu}{ds} \frac{dp_\mu}{ds}}{\mathbf{b}^2} \right) \ =$$

$$(ds)^2 \left(1 \ + \ \frac{\mathcal{F}^2}{\mathcal{F}^2_{max}} \right), \quad (ds)^2 \equiv dx^\mu dx_\mu \quad (41)$$

where the Born constant $\mathbf{b}$ is postulated to be the maximal proper force $\mathbf{b} = \mathcal{F}_{max}$ and needs to be introduced in (41) to match units. Inserting the values of (40) into the phase space interval (41) it becomes

$$d\omega^2 \ = \ (ds)^2 \ (1 \ - \ \frac{m^2 a^2}{\mathcal{F}^2_{max}}) \ = \ (ds)^2 \ (1 \ - \ \frac{m^2 g^2 e^{2a\xi_o}}{\mathcal{F}^2_{max}}) \quad (42)$$

The bound on the maximal proper force requires $ma \leq \mathcal{F}_{max}$. The phase space interval is null $d\omega^2 = 0$, in particular when

$$m \neq 0, \quad a \neq \infty, \quad ma = \mathcal{F}_{max}, \quad (ds)^2 \neq 0 \quad (43)$$

Note that in the double-scaling limit $m \to 0$, $a \to \infty$ and such that the product $ma = \mathcal{F}_{max} = \mathbf{b}$ equals the Born maximal proper force, one also has a null phase space interval $d\omega^2 = 0$. And this case $differs$ from the massless particle case $m = 0, (ds)^2 = 0$ leading directly to $d\omega^2 = 0$.

From the definition of the proper acceleration $g = ae^{-a\xi_o}$ one infers that an infinite value of $g = \infty$ is consistent with $a = finite$ if, and only if, $\xi_o = -\infty$. When $g = \infty$, the Rindler path corresponds to the null line trajectory $ds^2 = 0$ and the hyperbolas degenerate into the asymptotes which coincide with the null paths of the massless particles (photons). Whereas, an infinite value of $g = \infty$ is consistent with $a = \infty$ if, and only if, $\xi_o \leq 0$. If $\xi_o > 0$ then $g \to 0$ as $a \to \infty$ due to the negative exponential argument. .

If there is a Born *bound* in the proper force given by $\mathbf{b}$, then a massive particle must also have a bound to the maximal value of its acceleration parameter $a = a_{max}$, and so would be the value of its proper acceleration $g = ae^{-a\xi_o}$, except in the limiting special case when $\xi_o = -\infty$. But this latter limiting value is associated with the null path trajectory (hyperbolas degenerate into the asymptotes) corresponding to a massless particle, thus contradicting the assumption that the particle was massive. Therefore, in order to reconcile this discrepancy one must have that $\xi_o \neq -\infty$, and such that the values of ξ are confined to the region $-\infty < \xi \leq \infty$, implying that there must be a *lower* value to the throat size (a minimal length) of the Rindler hyperbolas given by

$$\rho_{min} = \frac{e^{a_{max}\xi_o}}{a_{max}} = \frac{1}{g_{max}} > 0, \quad -\infty < \xi_o \leq 0, \quad a_{max} \neq \infty, \quad c = 1 \quad (44)$$

and consequently there must be a *stretched* Rindler horizon.

The whole discussion involving eqs-(40-44) relates to the study of strings moving in target backgrounds associated with the cotangent bundle of spacetime (phase space). A curved-phase space requires the use of Finsler geometry. We only discussed the simple case of flat phase space and introduced the Born parameter $\mathbf{b}$ representing a maximal proper force, which in turn, leads to a maximal proper acceleration, a minimal length (minimal throat size), and consequently to a stretched Rindler horizon. Note that this finding occurs both in open and closed strings. The key point of eqs-(42,43) was to show that a null interval in phase space $(d\omega)^2 = 0$, does *not* necessarily mean that one has a null interval in spacetime $(ds)^2 = 0$. For this reason we found in eqs-(43,44) that the acceleration cannot be infinite.

We believe that the above interpretation of the stretched Rindler horizon in terms of a maximal proper force and strings propagating in phase spaces might be very relevant to the picture of Susskind (and collaborators) of free strings falling towards a black hole horizon, and to the black hole/string transition which occurs when the black hole entropy matches the entropy of a gas of hot open strings living in the stretched horizon. Since a black hole is the ultimate "accelerator", the transverse size of the strings grow very fast as they fall while experiencing a long string phase and diffusing all over the horizon at a distance of the order of the string length l_s, and leading to the notion of a stretched horizon (a membrane). The authors in [7] discussed this work of Susskind et al and remarked that there are some issues to be resolved, since their BMS_3 algebra is only generated when the string completely hits the horizon and which corresponds to the strict infinite acceleration limit.

So far we have discussed the open string. One can envision an accelerated closed string as an expanding loop (circle) in the $x-y$ plane whose radius increases in size with a uniform acceleration. In this case each one of the continuum of point masses comprising the closed string has the *same* proper acceleration and which is chosen to be given by $g(\xi) = ae^{-a\xi}$. The initial loop radius is $\frac{1}{g(\xi)}$ such that the world-sheet corresponding to this radially expanding motion of the loop in a $3D$ Minkowski spacetime background has the shape of a *hyperboloid*, and whose throat size is $\frac{1}{g(\xi)} = \frac{e^{a\xi}}{a}$. As one *varies* the values of the ξ parameter, the throat size (which coincides with the initial radius size), *varies* as well.

These expanding accelerated radial motions of the loops can be visualized by a $family$ of *hyperboloids* (of varying throat sizes) which can be described in terms of the Rindler temporal and spatial parameters η, ξ, and the additional angular coordinate θ of the circles of radii $r(\eta, \xi)$, as follows

$$x = \frac{e^{a\xi}}{a} \, cosh(a\eta) \, cos(\theta), \quad y = \frac{e^{a\xi}}{a} \, cosh(a\eta) \, sin(\theta),$$
$$t = \frac{e^{a\xi}}{a} \, sinh(a\eta), \quad g(\xi) \equiv a \, e^{-a\xi} \tag{45}$$

such that $x^2+y^2-t^2 = r^2-t^2 = \frac{1}{g^2(\xi)}$ is consistent with the algebraic equation of a one-parameter family of hyperboloids in three dimensions parametrized by ξ. The radius of the loop increases in size according to $r = r(\eta, \xi) = \frac{1}{g(\xi)} cosh(a\eta)$.

To sum up, the (x, y, t) target space coordinates displayed in eq-(45) describe a one-parameter family of hyperboloids (parametrized by ξ) associated with the worldsheets swept by a one-parameter family of closed strings (parametrized by ξ). θ, η are the worldsheet spatial and temporal coordinates of the closed string and ξ is the parameter which labels each member of the family of closed strings. The throat size of each hyperboloid coincides with the initial radius size of each closed string. As their radius begin to grow the worldsheets swept by each closed string begin to wrap their corresponding hyperboloids. This scenario must *not* be confused with the one of accelerating worldsheets in [7] (of closed strings observed in inertial and non-inertial frames of reference) nor with a static closed string whose worldsheet is just the lateral surface of a cylinder-like surface in $3D$ Minkowski spacetime.

We saw earlier for the Rindler hyperbolic $2D$ trajectories that $g = \infty$ correspond to null line trajectories : the hyperbolas degenerate into the asymptotes which coincide with the null paths of massless particles (photons). Hence the string analog is a *tensionless* closed string whose *all* points experience an infinite proper acceleration, while sweeping a $null \ surface$ (a light-cone).

Hence, the above family member of hyperboloids degenerates into a light-cone (zero throat size hyperbolas) when $g(\xi) = \infty \Rightarrow \xi = -\infty$. Physically this latter picture belongs to the case where a closed *tensionless* string (circle, loop) has shrunk to a *point* located at the origin of the $3D$ spacetime, and upon experiencing an infinite radial acceleration along all directions in the $x-y$ plane it will begin to expand radially outwards and sweep the world-sheet given by the forward (future) light-cone. The past light-cone is described by a closed string (loop) of infinite radius size at $\eta = -\infty$ that shrinks to zero size at $\eta = 0$.

This idea can be generalized to expanding spherical membranes, and p-branes as well, leading to higher dimensional hyperboloids of topology $S^p \times R$. The braneworlds model is based on the idea that our $4D$ universe is a 3-brane whose $3 + 1$-dim world volume is embedded in higher dimensions. Hence, an accelerated expanding 3-dim spherical bubble can model the accelerated expansion of our Universe and which is described by de-Sitter space whose topology is $S^3 \times R$. Namely, the 4-dim world volume swept by the spherical 3-brane (whose radius increases exponentially) describes a 4-dim hyperboloid embedded in 5-dim. Hence the accelerated expanding bubbles can model the accelerated exponential expansion of our Universe which is described by de Sitter space.

The authors [7] continued to study their *tensionless* limit of string theory that has recently been formulated in terms of world-sheet Rindler physics. They considered closed strings moving in background Rindler spacetimes and showed that strings probing the near-horizon region of a generic non-extremal black hole become tensionless thereby linking a spacetime Carroll limit to a world-sheet Carroll limit. Then, considering strings in d-dimensional Rindler spacetime they found a Rindler structure induced on the world-sheet. Among other findings, they showed that the Bondi-Metzner-Sachs (BMS) or the Conformal Carroll algebra emerges from the closed string Virasoro algebra as the horizon is hit. Crucial in their findings was to start with the **ILST** action for *tensionless* strings [8]

$$S_{ISLT} \ = \ \int \ d^2\sigma \ V^a \ V^b \ (\partial_a X^\mu) \ (\partial_b X^\nu) \ g_{\mu\nu} \tag{46}$$

where V^a are vector densities. In view of our results in this work involving a *dynamical* tension field and our description of accelerated open and closed strings within the framework of Rindler worldsheets it is warranted to explore further the connections to the work of [7]. About the Carrollian worldsheets ($c = 0$) and null tensions in [7], a direct reading of eq-(3) reveals that a null tension coresponds to having worldsheet locations of zero area-measure $\{\varphi^1, \varphi^2\} = 0$, or to the case when the determinant of the auxiliary wordsheet metric γ_{ab} is infinite, which in turn, implies that the inverse worldsheet metric γ^{ab} becomes *degenerate*. The latter case has been studied by the authors in [7] which required the use of the **ILST** action (46). In this work we begin with a different action (1) and emphasized the key role of the area-preserving diffs symmetry that leaves the Poisson brackets $\{\varphi^1, \varphi^2\}$ and the action (1) invariant. The contraction operation described by the authors in [7] involves a scaling of the Tension and a scaling of the wordsheet *temporal* coordinate only which is *not* area-preserving.

Acknowledgements

We thank M. Bowers for very kind assistance and to the referee for numerous and insightful comments and suggestions to improve this work.

References

[1] E. I. Guendelman, "Implications of the Spectrum of Dynamically Generated String Tension Theories", International Journal of Modern Physics D, https://doi.org/10.1142/S0218271821420281.

E.I. Guendelman, "Light Like Segment Compactification and Braneworlds with Dynamical String Tension" Eur. Phys. J. C **81** (2021) 10, 886.

E.I. Guendelman, "Strings and branes with a modified measure", Class. Quant. Grav. **17** (2000) 3673-3680.

E.I. Guendelman, "Cosmology and Warped Space Times in Dynamical String Tension Theories", arXv : 2104.08875.

E.I. Guendelman, "Escaping the Hagedorn Temperature in Cosmology and Warped Spaces with Dynamical String Tension", arXiv : 2105.02279.

[2] K.Becker, M. Becker and J. Schwarz *String Theory and M-Theory (A Modern Introduction)* (Cambridge University Press, 2007).

[3] J.H. Yoon, "Algebraically Special Class of Spacetimes and $1 + 1$ dimensional Field Theories" arXiv : hep-th/9211129.

[4] C. Castro, "Strings and Membranes from Einstein Gravity, Matrix Models and W_∞ Gauge Theories as paths to Quantum Gravity", International Journal of Modern Physics A Vol. 23, No. 24, pp. 3901-3945 (2008).

[5] A. Stominger, "$w_{1+\infty}$ and the Celestial sphere" arXiv : 2105.14346.

[6] Carlos Castro, "Is Dark Matter and Black-Hole Cosmology an Effect of Born's Reciprocal Relativity Theory ?" Canadian Journal of Physics, Published on the web 29 May 2018, https://doi.org/10.1139/cjp-2018-0097

[7] A. Bagchi, A. Banerjee, S. Chakrabotty and R. Chatterjee, "A Rindler Road to Carrollian Worldsheets" arXiv : 2110.01172.

A. Bagchi, A. Banerjee and S. Chakrabortty, "Rindler Physics on the String Worldsheet", Phys. Rev. Lett. **126** (2021) 031601 [2009.01408].

[8] J. Isberg, U. Lindstrom, B. Sundborg and G. Theodoridis, "Classical and quantized tensionless strings", Nucl. Phys. **B 411** (1994) 122.