Navier-Stokes equations for protoplanetary disks and planetary rings

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Abstract

The axisymmetric solution of the cylindrical Navier-Stokes equations are simple, so that it is possible to search the equilibrium solution, and the particular exact solutions near the equilibrium for the disk dynamics of a fluid.

The Navier-Stokes equation for an incompressible fluid in cylindrical coordinates are:

$$
\begin{aligned}
&\rho\left(\partial_t u_r + u_r\partial_r u_r + \frac{u_\theta \partial_\theta u_r}{r} + u_z\partial_z u_r - \frac{u_\theta^2}{r}\right) = \rho f_r - \partial_r p + \eta\left(2\partial^2_{rr}u_r + \frac{\partial^2_{r\theta}u_\theta}{r} + \partial^2_{rz}u_z + \frac{\partial^2_{\theta\theta}u_r}{r^2} + \partial^2_{zz}u_r + \frac{2\partial_r u_r}{r} - \frac{3\partial_\theta u_\theta}{r^2} - \frac{2u_r}{r^2}\right) + \\
&\qquad +\partial_r\eta\left(2\partial_r u_r\right) + \partial_\theta\eta\left(\partial_r u_\theta + \frac{\partial_\theta u_r}{r} - \frac{u_\theta}{r}\right) + \partial_z\eta\left(\partial_r u_z + \partial_z u_r\right) \\
&\rho\left(\partial_t u_\theta + u_r\partial_r u_\theta + \frac{u_\theta \partial_\theta u_\theta}{r} + u_z\partial_z u_\theta + \frac{u_r u_\theta}{r}\right) = \rho f_\theta - \partial_\theta p + \eta\left(\partial^2_{rr}u_\theta + \frac{\partial^2_{r\theta}u_r}{r} + \frac{2\partial^2_{\theta\theta}u_\theta}{r^2} + \frac{\partial^2_{\theta z}u_z}{r} + \partial^2_{zz}u_\theta + \frac{\partial_r u_\theta}{r} + \frac{3\partial_\theta u_r}{r^2} - \frac{u_\theta}{r^2}\right) + \\
&\qquad +\partial_r\eta\left(\partial_r u_\theta + \frac{\partial_\theta u_\theta}{r} - \frac{u_\theta}{r}\right) + \partial_\theta\eta\left(\frac{2\partial_\theta u_\theta}{r} + \frac{2u_r}{r}\right) + \partial_z\eta\left(\frac{\partial_\theta u_z}{r} + \partial_z u_\theta\right) \\
&\rho\left(\partial_t u_z + u_r\partial_r u_z + \frac{u_\theta \partial_\theta u_z}{r} + u_z\partial_z u_z\right) = \rho f_z - \partial_z p + \eta\left(\partial^2_{rr}u_z + \partial^2_{rz}u_r + \frac{\partial^2_{\theta\theta}u_z}{r^2} + \frac{\partial^2_{\theta z}}{r} + 2\partial^2_{zz}u_z + \frac{\partial_r u_z}{r} + \frac{\partial_z u_r}{r}\right) + \\
&\qquad +\partial_r\eta\left(\partial_r u_z + \partial_z u_r\right) + \partial_\theta\eta\left(\frac{\partial_\theta u_z}{r} + \partial_z u_\theta\right) + \partial_z\eta 2\partial_z u_z
\end{aligned}
$$

a stationary fluid satisfy:

$$
\begin{aligned}
&0 = \partial_t\rho + \frac{1}{r}\partial_r(\rho r u_r) \\
&0 = \partial_r(\rho r u_r) \\
&\rho \propto (r u_r)^{-1}
\end{aligned}
$$

if $u_r = 0$ then $\rho(r)$ is an arbitrary function.

The axisymmetric stationary solution for a disk are[1]:

$$
\begin{aligned}
&\rho \propto (r u_r)^{-1} \\
&\rho\left(\partial_t u_r + u_r\partial_r u_r - \frac{u_\theta^2}{r}\right) = -\partial_r p - \frac{GM\rho}{r^2} + 2\eta\left(\partial^2_{rr}u_r + \frac{1}{r}\partial_r u_r - \frac{u_r}{r^2}\right) + 2\partial_r\eta\partial_r u_r \\
&\rho\left(\partial_t u_\theta + u_r\partial_r u_\theta + \frac{u_\theta u_r}{r}\right) = \eta\left(\partial^2_{rr}u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{u_\theta}{r^2}\right) + \partial_r\eta\left(\partial_r u_\theta - \frac{u_\theta}{r}\right) \\
&u_r = u_z = 0
\end{aligned}
$$

The stationary solution for disk (and planetary rings) are[2]:

$$
\begin{aligned}
&-\frac{\rho u_\theta^2}{r} = -\partial_r p - \frac{GM\rho}{r^2} \\
&0 = \left(\partial^2_{rr}u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{u_\theta}{r^2}\right)\eta + \left(\partial_r u_\theta - \frac{u_\theta}{r}\right)\partial_r\eta \\
&u_r = u_z = 0
\end{aligned}
$$

if $\partial_r p \neq 0$ then the probability distribution of the velocity[3] induces a flow of particles into the disk, which we have forbidden with the statement $u_r = 0$, so that:

$$
\boxed{
\begin{aligned}
&\eta = A r^{-\frac{1}{2}} \\
&u_\theta = \sqrt{\frac{GM}{r}} = \gamma r^{-\frac{1}{2}} \\
&u_r = u_z = 0
\end{aligned}
}
$$

if η has a power law, then it is possible the Keplerian velocity for u_θ, if $\partial_r\eta = 0$ then the Keplerian velocity is obtainable only with null viscosity $\eta = 0$, because of for viscosity different from zero $\eta \neq 0$ is possible only $u_\theta = Ar^{-1} + \omega r$ that are

[1] These equations are compressible equations so that are different from the Landau-Lifshitz cylindrical Navier-Stokes equations

[2] the stationary solution has null radial velocity, because of there are not fluid sources. This stationary solution is true for compressible and incompressible fluid

[3] this is the weak point of the proof

the rigid body (constant angular velocity ω) or a constant viscous force on the annulus[4]

The Navier-Stokes equations have only two stationary solution with constant dinamics viscosity η with central forces: the harmonic, and inverse cube, attractive forces. It is possible to obtain a stationary solution with a gravitational force using a variable viscosity: this is not true for long times, because of the viscosity depend on microscopic collision of fluid particles in the kinetic theory of the gas; so that a statistical equilibrium is obtained in long times through the conversion of the microscopic collision in kinetic energy of the disk (the disc expands until it reaches the cancellation of collisions, and a zero viscosity, where the Navier-Stokes equation are satisfied for each central force)

It is possible to obtain a particular solution of the Navier-Stokes equation, that could be measured in particular protoplanetary disk.

A protoplanetary disk, away from the formation, maintains a statistical balance between the internal energy absorbed by the disk from the continuous impact of objects on not perfectly circular orbits, and the viscosity on the orbits that transfers the energy in the expansion of the disk; a probable form of the solution should be (the collisions correct elliptical orbits in circular orbits):

$$\begin{cases} \eta = A\epsilon(t)r^{\beta} \\ u_\theta = \gamma r^{-\frac{1}{2}} \\ u_r = \epsilon(t)r^{\alpha} \end{cases}$$

so that the Navier-Stokes equations are:

$$\begin{cases} \rho\left(\partial_t u_r + u_r\partial_r u_r - \frac{u_\theta^2}{r}\right) = -\partial_r p - \frac{GM\rho}{r^2} + 2\eta\left(\partial^2_{rr}u_r + \frac{1}{r}\partial_r u_r - \frac{u_r}{r^2}\right) + 2\partial_r\eta\partial_r u_r \\ \rho\left(\partial_t u_\theta + u_r\partial_r u_\theta + \frac{u_r u_\theta}{r}\right) = \eta\left(\partial^2_{rr}u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{u_\theta}{r^2}\right) + \partial_r\eta\left(\partial_r u_\theta - \frac{u_\theta}{r}\right) \end{cases}$$

$$\begin{cases} \rho\left[\partial_t(\epsilon r^\alpha) + (\epsilon r^\alpha)\partial_r(\epsilon r^\alpha)\right] = 2A\epsilon r^\beta\left[\partial^2_{rr}(\epsilon r^\alpha) + \frac{1}{r}\partial_r(\epsilon r^\alpha) - \frac{1}{r^2}(\epsilon r^\alpha)\right] + 2\partial_r(A\epsilon r^\beta)\partial_r(\epsilon r^\alpha) \\ \rho\left[(\epsilon r^\alpha)\partial_r(\gamma r^{-\frac{1}{2}}) + \frac{1}{r}(\epsilon r^\alpha)(\gamma r^{-\frac{1}{2}})\right] = A\epsilon r^\beta\left[\partial^2_{rr}(\gamma r^{-\frac{1}{2}}) + \frac{1}{r}\partial_r(\gamma r^{-\frac{1}{2}}) - \frac{1}{r^2}(\gamma r^{-\frac{1}{2}})\right] + \partial_r(A\epsilon r^\beta)\left[\partial_r(\gamma r^{-\frac{1}{2}}) - \frac{1}{r}(\gamma r^{-\frac{1}{2}})\right] \end{cases}$$

$$\begin{cases} \rho\left(r^\alpha\partial_t\epsilon + \epsilon^2\alpha r^{2\alpha-1}\right) = \epsilon^2 2Ar^{\alpha+\beta-2}\left[\alpha(\alpha-1) + \alpha - 1\right] + \epsilon^2 2A\alpha\beta r^{\alpha+\beta-2} \\ \rho\left(-\epsilon\frac{1}{2}\gamma r^{\alpha-\frac{3}{2}} + \epsilon\gamma r^{\alpha-\frac{3}{2}}\right) = \epsilon A\gamma r^{\beta-\frac{5}{2}}\left(\frac{3}{4} - \frac{1}{2} - 1\right) + \epsilon A\beta\gamma r^{\beta-\frac{5}{2}}\left(-\frac{1}{2} - 1\right) \end{cases}$$

$$\begin{cases} \partial_t\epsilon = \left[-\alpha r^{\alpha-1} + \frac{2A}{\rho}r^{\beta-2}\left(\alpha^2 + \alpha\beta - 1\right)\right]\epsilon^2 \\ \rho r^{\alpha-1} = -\frac{3A}{2}r^{\beta-2} - 3A\beta r^{\beta-2} \end{cases}$$

the ϵ function is a time function, so that the radial distance must have null exponents:

$$\begin{array}{l} \alpha = 1 \\ \beta = 2 \\ A = -\frac{2}{15}\rho \end{array}$$

so that it is possible to obtain the particular solution:

$$\begin{array}{l} \partial_t\epsilon = -\frac{23}{15}\epsilon^2 \\ \epsilon(t) = \frac{15}{23}\frac{1}{t-t_0} \end{array}$$

then it is possible to write the particular protoplanetary disk solution:

$$\boxed{\begin{array}{l} \eta = -\frac{2}{23}\frac{\rho r^2}{t-t_0} \\ u_\theta = \gamma r^{-\frac{1}{2}} \\ u_r = \frac{15}{23}\frac{r}{t-t_0} \end{array}}$$

the viscosity is negative, because of contiguous layer of protoplanetary disk exchange particles of different energy, with an energy transfer from the inner layers to the extern layer (like a heat flux): an annulus have an increase in kinetic energy due to the lower annulus.

This solution is particular: the incompressible fluid has $u_r \propto r^{-1}$ because of the $0 = \partial_r(ru_r)$, so that this solution is for compressible fluid

Navier-Stokes equations in cylindrical coordinates

The transformation between cartesian and cylindrical coordinates are:

$$\begin{cases} \boldsymbol{r} = \boldsymbol{x}\cos\theta + \boldsymbol{y}\sin\theta \\ \boldsymbol{\theta} = -\boldsymbol{x}\sin\theta + \boldsymbol{y}\cos\theta \\ \boldsymbol{z} = \boldsymbol{z} \end{cases} \quad \begin{cases} \boldsymbol{x} = \cos\theta\,\boldsymbol{r} - \sin\theta\,\boldsymbol{\theta} \\ \boldsymbol{y} = \sin\theta\,\boldsymbol{r} + \cos\theta\,\boldsymbol{\theta} \\ \boldsymbol{z} = \boldsymbol{z} \end{cases}$$

[4] $0 = \partial_r(r\partial_r u_\theta) = r\partial_{rr}u_\theta + \partial_r u_\theta \implies \alpha^2 = 0,\ \partial_r(2\pi r\partial_r u_\theta) = 0$

the versor trasformation are:

$$\begin{cases} \partial_\theta \widehat{\boldsymbol{r}} = -\sin\theta\, \widehat{\boldsymbol{x}} + \cos\theta\, \widehat{\boldsymbol{y}} = \widehat{\boldsymbol{\theta}} \\ \partial_\theta \widehat{\boldsymbol{\theta}} = -\cos\theta\, \widehat{\boldsymbol{x}} - \sin\theta\, \widehat{\boldsymbol{y}} = -\widehat{\boldsymbol{r}} \end{cases}$$

the cylindrical del operator used in the Navier-Stokes equations are:

$$\begin{aligned}
&\boldsymbol{\nabla} f = \widehat{\boldsymbol{x}}\partial_x + \widehat{\boldsymbol{y}}\partial_y + \widehat{\boldsymbol{z}}\partial_z = (\widehat{\boldsymbol{r}}\cos\theta - \widehat{\boldsymbol{\theta}}\sin\theta)(\cos\theta\partial_r - \tfrac{1}{r}\sin\theta\partial_\theta) + (\widehat{\boldsymbol{r}}\sin\theta + \widehat{\boldsymbol{\theta}}\cos\theta)(\cos\theta\partial_r - \tfrac{1}{r}\sin\theta\partial_\theta) + \widehat{\boldsymbol{z}}\partial_z = \\
&= \widehat{\boldsymbol{r}}\partial_r + \tfrac{\widehat{\boldsymbol{\theta}}}{r}\partial_\theta + \widehat{\boldsymbol{z}}\partial_z \\
&\boldsymbol{\nabla}\cdot\boldsymbol{u} = \left(\widehat{\boldsymbol{r}}\partial_r + \tfrac{\widehat{\boldsymbol{\theta}}}{r}\partial_\theta + \widehat{\boldsymbol{z}}c\partial_z\right)\left(\widehat{\boldsymbol{r}}u_r + \widehat{\boldsymbol{\theta}}u_\theta + \widehat{\boldsymbol{z}}u_z\right) = \tfrac{1}{r}\partial_r(ru_r) + \tfrac{1}{r}\partial_\theta u_\theta + \partial_z u_z \\
&(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \left(u_r\partial_r + \tfrac{u_\theta\partial_\theta}{r} + u_z\partial_z\right)\left(\widehat{\boldsymbol{r}}u_r + \widehat{\boldsymbol{\theta}}u_\theta + \widehat{\boldsymbol{z}}u_z\right) = \widehat{\boldsymbol{r}}\left(u_r\partial_r u_r + \tfrac{u_\theta\partial_\theta u_r}{r} + u_z\partial_z u_r\right) + \widehat{\boldsymbol{\theta}}\tfrac{u_r u_\theta}{r} + \\
&+\widehat{\boldsymbol{\theta}}\left(u_r\partial_r u_\theta + \tfrac{u_\theta\partial_\theta u_\theta}{r} + u_z\partial_z u_\theta\right) - \widehat{\boldsymbol{r}}\tfrac{u_\theta^2}{r} + \widehat{\boldsymbol{z}}\left(u_r\partial_r u_z + \tfrac{u_\theta\partial_\theta u_z}{r} + u_z\partial_z u_z\right) \\
&\boldsymbol{\nabla}\boldsymbol{u} = (\widehat{\boldsymbol{r}}\partial_r + \tfrac{\widehat{\boldsymbol{\theta}}}{r}\partial_\theta + \widehat{\boldsymbol{z}}\partial_z)(\widehat{\boldsymbol{r}}u_r + \widehat{\boldsymbol{\theta}}u_\theta + \widehat{\boldsymbol{z}}u_z) = \widehat{\boldsymbol{r}\boldsymbol{r}}\partial_r u_r + \widehat{\boldsymbol{r}\boldsymbol{\theta}}\partial_r u_\theta + \widehat{\boldsymbol{r}\boldsymbol{z}}\partial_r u_z + \widehat{\boldsymbol{\theta}\boldsymbol{r}}\left(\tfrac{1}{r}\partial_\theta u_r - \tfrac{1}{r}u_\theta\right) + \widehat{\boldsymbol{\theta}\boldsymbol{\theta}}\left(\tfrac{1}{r}\partial_\theta u_\theta + \tfrac{1}{r}u_r\right) + \\
&+\widehat{\boldsymbol{\theta}\boldsymbol{z}}\tfrac{1}{r}\partial_\theta u_z + \widehat{\boldsymbol{z}\boldsymbol{r}}\partial_z u_r + \widehat{\boldsymbol{z}\boldsymbol{\theta}}\partial_z u_\theta + \widehat{\boldsymbol{z}\boldsymbol{z}}\partial_z u_z \\
&(\boldsymbol{\nabla}\boldsymbol{u})^T = \widehat{\boldsymbol{r}\boldsymbol{r}}\partial_r u_r + \widehat{\boldsymbol{\theta}\boldsymbol{r}}\partial_r u_\theta + \widehat{\boldsymbol{z}\boldsymbol{r}}\partial_r u_z + \widehat{\boldsymbol{r}\boldsymbol{\theta}}\left(\tfrac{1}{r}\partial_\theta u_r - \tfrac{1}{r}u_\theta\right) + \widehat{\boldsymbol{\theta}\boldsymbol{\theta}}\left(\tfrac{1}{r}\partial_\theta u_\theta + \tfrac{1}{r}u_r\right) + \widehat{\boldsymbol{z}\boldsymbol{\theta}}\tfrac{1}{r}\partial_\theta u_z + \widehat{\boldsymbol{r}\boldsymbol{z}}\partial_z u_r + \widehat{\boldsymbol{\theta}\boldsymbol{z}}\partial_z u_\theta + \widehat{\boldsymbol{z}\boldsymbol{z}}\partial_z u_z \\
&\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T = 2\widehat{\boldsymbol{r}\boldsymbol{r}}\partial_r u_r + \widehat{\boldsymbol{r}\boldsymbol{\theta}}\left(\partial_r u_\theta + \tfrac{1}{r}\partial_\theta u_r - \tfrac{1}{r}u_\theta\right) + \widehat{\boldsymbol{r}\boldsymbol{z}}\left(\partial_r u_z + \partial_z u_r\right) + \widehat{\boldsymbol{\theta}\boldsymbol{r}}\left(\partial_r u_\theta + \tfrac{1}{r}\partial_\theta u_r - \tfrac{1}{r}u_\theta\right) + 2\widehat{\boldsymbol{\theta}\boldsymbol{\theta}}\left(\tfrac{1}{r}\partial_\theta u_\theta + \tfrac{1}{r}u_r\right) + \\
&\qquad + \widehat{\boldsymbol{\theta}\boldsymbol{z}}\left(\partial_z u_\theta + \tfrac{1}{r}\partial_\theta u_z\right) + \widehat{\boldsymbol{z}\boldsymbol{r}}\left(\partial_r u_z + \partial_z u_r\right) + \widehat{\boldsymbol{z}\boldsymbol{\theta}}\left(\partial_z u_\theta + \tfrac{1}{r}\partial_\theta u_z\right) + 2\widehat{\boldsymbol{z}\boldsymbol{z}}\partial_z u_z
\end{aligned}$$

the transformation for the Navier-Stokes equation are:

$$\begin{aligned}
&\rho\left[\partial_t\boldsymbol{u} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = \rho\boldsymbol{f} - (\widehat{\boldsymbol{r}}\partial_r + \tfrac{\widehat{\boldsymbol{\theta}}}{r}\partial_\theta + \widehat{\boldsymbol{z}}\partial_z)p + (\widehat{\boldsymbol{r}}\partial_r + \tfrac{\widehat{\boldsymbol{\theta}}}{r}\partial_\theta + \widehat{\boldsymbol{z}}\partial_z)\cdot\{\eta[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T]\} = \\
&= \widehat{\boldsymbol{r}}\rho f_r + \widehat{\boldsymbol{\theta}}\rho f_\theta + \widehat{\boldsymbol{z}}\rho f_z - \widehat{\boldsymbol{r}}\partial_r p - \tfrac{\widehat{\boldsymbol{\theta}}}{r}\partial_\theta p + \widehat{\boldsymbol{z}}\partial_z p + \eta\left\{\widehat{\boldsymbol{r}}2\partial^2_{rr}u_r + \widehat{\boldsymbol{\theta}}\left(\partial^2_{rr}u_\theta + \tfrac{\partial^2_{r\theta}u_r}{r} - \tfrac{\partial_\theta u_r}{r^2} - \tfrac{\partial_r u_\theta}{r} + \tfrac{u_\theta}{r^2}\right) + \widehat{\boldsymbol{z}}\left(\partial^2_{rr}u_z + \partial^2_{rz}u_r\right) + \right.\\
&+\widehat{\boldsymbol{r}}\tfrac{2\partial_r u_r}{r} + \widehat{\boldsymbol{\theta}}\left(\tfrac{\partial_r u_\theta}{r} + \tfrac{\partial_\theta u_r}{r^2} - \tfrac{u_\theta}{r^2}\right) + \widehat{\boldsymbol{z}}\left(\tfrac{\partial_r u_z}{r} + \tfrac{\partial_z u_r}{r}\right) + \widehat{\boldsymbol{\theta}}\left(\tfrac{\partial_r u_\theta}{r} + \tfrac{\partial_\theta u_r}{r^2} - \tfrac{u_\theta}{r^2}\right) - \widehat{\boldsymbol{r}}\left(\tfrac{2\partial_\theta u_\theta}{r^2} + \tfrac{2u_r}{r^2}\right) + \widehat{\boldsymbol{r}}\left(\tfrac{\partial^2_{r\theta}u_\theta}{r} + \tfrac{\partial^2_{\theta\theta}u_r}{r^2} - \tfrac{\partial_\theta u_\theta}{r^2}\right) + \\
&+\widehat{\boldsymbol{\theta}}\left(\tfrac{2\partial^2_{\theta\theta}u_\theta}{r^2} + \tfrac{2\partial_\theta u_r}{r^2}\right) + \widehat{\boldsymbol{z}}\left(\tfrac{\partial^2_{\theta\theta}u_z}{r^2} + \tfrac{\partial^2_{z\theta}u_\theta}{r}\right) + \widehat{\boldsymbol{r}}\left(\partial^2_{zr}u_z + \partial^2_{zz}u_r\right) + \widehat{\boldsymbol{\theta}}\left(\tfrac{\partial^2_{z\theta}u_z}{r} + \partial^2_{zz}u_\theta\right) + \left.\widehat{\boldsymbol{z}}2\partial^2_{zz}u_z\right\} + \\
&+\partial_r\eta\left[\widehat{\boldsymbol{r}}2\partial_r u_r + \widehat{\boldsymbol{\theta}}\left(\partial_r u_\theta + \tfrac{1}{r}\partial_\theta u_\theta - \tfrac{1}{r}u_\theta\right) + \widehat{\boldsymbol{z}}\left(\partial_r u_z + \partial_z u_r\right)\right] + \\
&+\partial_\theta\eta\left[\widehat{\boldsymbol{r}}\left(\partial_r u_\theta + \tfrac{1}{r}\partial_\theta u_r - \tfrac{u_\theta}{r}\right) + \widehat{\boldsymbol{\theta}}\left(\tfrac{2\partial_\theta u_\theta}{r} + \tfrac{2u_r}{r}\right) + \widehat{\boldsymbol{z}}\left(\tfrac{\partial_\theta u_z}{r} + \partial_z u_\theta\right)\right] + \\
&+\partial_z\eta\left[\widehat{\boldsymbol{r}}\left(\partial_r u_z + \partial_z u_r\right) + \widehat{\boldsymbol{\theta}}\left(\tfrac{\partial_\theta u_z}{r} + \partial_z u_\theta\right) + \widehat{\boldsymbol{z}}2\partial_z u_z\right]
\end{aligned}$$

The Navier-Stokes equations in cylindrical coordinate (with variable dynamics viscosity) are:

$$\begin{aligned}
&\rho\left(\partial_t u_r + u_r\partial_r u_r + \tfrac{u_\theta\partial_\theta u_r}{r} + u_z\partial_z u_r - \tfrac{u_\theta^2}{r}\right) = \rho f_r - \partial_r p + \eta\left(2\partial^2_{rr}u_r + \tfrac{\partial^2_{r\theta}u_\theta}{r} + \partial^2_{rz}u_z + \tfrac{\partial^2_{\theta\theta}u_r}{r^2} + \partial^2_{zz}u_r + \tfrac{2\partial_r u_r}{r} - \tfrac{3\partial_\theta u_\theta}{r^2} - \tfrac{2u_r}{r^2}\right) + \\
&\qquad + \partial_r\eta\left(2\partial_r u_r\right) + \partial_\theta\eta\left(\partial_r u_\theta + \tfrac{\partial_\theta u_r}{r} - \tfrac{u_\theta}{r}\right) + \partial_z\eta\left(\partial_r u_z + \partial_z u_r\right) \\
&\rho\left(\partial_t u_\theta + u_r\partial_r u_\theta + \tfrac{u_\theta\partial_\theta u_\theta}{r} + u_z\partial_z u_\theta + \tfrac{u_r u_\theta}{r}\right) = \rho f_\theta - \partial_\theta p + \eta\left(\partial^2_{rr}u_\theta + \tfrac{\partial^2_{r\theta}u_r}{r} + \tfrac{2\partial^2_{\theta\theta}u_\theta}{r^2} + \tfrac{\partial^2_{\theta z}u_z}{r} + \partial^2_{zz}u_\theta + \tfrac{\partial_r u_\theta}{r} + \tfrac{3\partial_\theta u_r}{r^2} - \tfrac{u_\theta}{r^2}\right) + \\
&\qquad + \partial_r\eta\left(\partial_r u_\theta + \tfrac{\partial_\theta u_\theta}{r} - \tfrac{u_\theta}{r}\right) + \partial_\theta\eta\left(\tfrac{2\partial_\theta u_\theta}{r} + \tfrac{2u_r}{r}\right) + \partial_z\eta\left(\tfrac{\partial_\theta u_z}{r} + \partial_z u_\theta\right) \\
&\rho\left(\partial_t u_z + u_r\partial_r u_z + \tfrac{u_\theta\partial_\theta u_z}{r} + u_z\partial_z u_z\right) = \rho f_z - \partial_z p + \eta\left(\partial^2_{rr}u_z + \partial^2_{rz}u_r + \tfrac{\partial^2_{\theta\theta}u_z}{r^2} + \tfrac{\partial^2_{\theta z}}{r} + 2\partial^2_{zz}u_z + \tfrac{\partial_r u_z}{r} + \tfrac{\partial_z u_r}{r}\right) + \\
&\qquad + \partial_r\eta\left(\partial_r u_z + \partial_z u_r\right) + \partial_\theta\eta\left(\tfrac{\partial_\theta u_z}{r} + \partial_z u_\theta\right) + \partial_z\eta 2\partial_z u_z
\end{aligned}$$

The cylindrical Navier-Stokes equation for an axisymmetric disk are simple:

$$\begin{aligned}
&\rho\left(\partial_t u_r + u_r\partial_r u_r - \tfrac{u_\theta^2}{r}\right) = \rho f_r - \partial_r p + 2\eta\left(\partial^2_{rr}u_r + \tfrac{\partial_r u_r}{r} - \tfrac{u_r}{r^2}\right) + 2\partial_r\eta\left(\partial_r u_r\right) \\
&\rho\left(\partial_t u_\theta + u_r\partial_r u_\theta + \tfrac{u_r u_\theta}{r}\right) = \rho f_\theta - \partial_\theta p + \eta\left(\partial^2_{rr}u_\theta + \tfrac{\partial_r u_\theta}{r} - \tfrac{u_\theta}{r^2}\right) + \partial_r\eta\left(\partial_r u_\theta - \tfrac{u_\theta}{r}\right)
\end{aligned}$$

the stationary solution for $u_\theta = \gamma r^{-\frac{1}{2}}$ is:

$$\begin{aligned}
&0 = -\tfrac{\gamma}{2}r^{-\frac{3}{2}}\partial_r\eta - \gamma r^{-\frac{3}{2}}\partial_r\eta + \tfrac{3\gamma}{4}r^{-\frac{5}{2}}\eta - \tfrac{\gamma}{2}r^{-\frac{5}{2}}\eta - \gamma r^{-\frac{5}{2}}\eta = -\tfrac{3}{2}r\partial_r\eta - \tfrac{3}{4}\eta \\
&\int dr\partial_r \ln\eta = -\int dr\tfrac{1}{2r} = -\tfrac{1}{2}\ln r \\
&\eta = Ar^{-\frac{1}{2}}
\end{aligned}$$

then the viscosity power law is equal to the tangential velocity power law, in the stationary case.