

# Theoretical Treatment on the Hydrostatic Stability of Ships (Part 1:) Stable Conditions for the Upright State

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## Summary

In this paper, a theoretical treatment on the hydrostatic stability of ships is presented.

As the simplest hull form, a columnar ship with rectangular cross-section, which is made of homogeneous squared timber with arbitrary breadth and arbitrary material, is chosen. In the 1st. paper, the conditions under which the ship is stable in the upright state with horizontal deck are analyzed by means of ship's hydrostatics.

By doing so, the dependence of the stable conditions on the breadth and material of the ship will be clarified.

*Keywords : Hydrostatic Stability, Columnar Ship, Rectangular Cross-Section,  
Arbitrary Breadth and Material, Stable Conditions, Upright State*

## 1. Introduction

One of the authors lectures on the hydrostatic stability of ships to 2<sup>nd</sup>. year students of the naval architectural engineering course<sup>(1),(2)</sup> in the faculty of engineering at the university where the 1<sup>st</sup>. author<sup>(3)</sup> works. In the 1<sup>st</sup>. semester, the basics of the hydrostatic of floating bodies, such as buoyant force and center of buoyancy<sup>(4)-(10)</sup> are taught in the course “Hydrostatics of Floating Bodies” as a compulsory subject. In the 2<sup>nd</sup>. semester, the theory of derivation of metacentric radius<sup>(11)-(14)</sup> which is the main theme of the lecture on “Theory of Ship Stability” is explained, and then some simple examples are given to deepen the understanding of the theory. Probably, universities and colleges of technology in naval architecture, marine engineering and nautical mechanics all over the country also teach the above-mentioned flow of lectures, although the subject titles may differ.

As a typical example, many textbooks on naval architecture<sup>(15)</sup> and nautical mechanics<sup>(16),(17)</sup> describe that a columnar ship with a square cross-section, which is made of timber with half the specific weight of water, cannot float stably when one side of the square is horizontal, but the ship is stable when it is inclined laterally and the diagonal of the square is parallel to the water line. This consequence is explained by the positional relationship between the metacenter and the center of gravity.

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Taking above typical example one step further, how wide of breadth will can a columnar ship of rectangular cross-section, made of timber with half the specific weight of water, float stably with its long side horizontal ? Or, what specific weight of material (*i.e.* lighter or heavier than timber) will can a square cross-sectional columnar ship float stably with one side horizontal ? By setting such examples, one of the authors has been lecturing on this problem for several years in the “Theory of Ship' Stability” subject at the 1<sup>st</sup>. author’s university.

As a result, the degree of understanding of the hydrostatic stability of ships has improved significantly compared to before the lecture, so we think that there will should be provided to teachers and students who will teach and learn this field in the future, and we will give some examples. One of the authors gave an explanation<sup>(18)</sup> of the effective examples and published it in the Journal “NAVIGATION” of Japan Institute of Navigation in 2021.

We summarize the above examples consistently and introduce them in this paper.

## 2. Stable Conditions for a Columnar Ship of Rectangular Cross-Section with Arbitrary Material $\alpha$ and Arbitrary Breadth $\beta$

Fig. 1 shows a columnar ship of length  $L$  with a rectangular cross-section of depth  $h$  and breadth  $\beta h$ , which is a squared timber of specific weight  $\gamma_t$  made of homogeneous material. Let’s consider determining the conditions under which the columnar ship can float stably with its long side  $\beta h$  parallel to the water line (*i.e.* upright state) in water of specific weight  $\gamma_w$ . The left side of Fig. 1 shows the upright state, and the right side shows the forces and moment acting on the cross-section inclined by heel angle  $\theta$ .

As a setting variable, The ratio of the specific weight of the columnar ship,  $\gamma_t$  (where  $t$  is the initial letter of timber), to the specific weight of water,  $\gamma_w$  (where  $w$  is the initial letter of water), is defined as  $\alpha$  (hereafter referred to as **material**), and the ratio of the breadth,  $\beta h$ , to the depth,  $h$ , of the cross-section (*i.e.* aspect ratio) is defined as  $\beta$  (hereafter referred to as **breadth**), as follows :

$$\left. \begin{aligned} \alpha &\equiv \frac{\gamma_t}{\gamma_w} \quad (\text{where, } 0 < \alpha \leq 1) \\ \beta &\equiv \frac{\text{breadth}}{\text{depth}} = \frac{\beta h}{h} \quad (\text{where, } \beta > 0) \end{aligned} \right\} \dots\dots\dots(1)$$

Here, when  $\gamma_w$  is fresh water,  $\alpha$  represents the specific gravity of the columnar ship.

First, let us consider the determination of the draft. The weight  $W$  and the buoyant force  $F_B$  (to be described separately from the center of buoyancy  $B$ ) of this columnar ship can be obtained as follows, respectively :

$$\left. \begin{aligned} W &= \gamma_t V_t = \gamma_t \cdot \beta h \cdot h \cdot L \\ F_B &= \gamma_w V_w = \gamma_w \cdot \beta h \cdot d \cdot L \end{aligned} \right\} \dots\dots\dots(2)$$

Theoretical Treatment on the Hydrostatic Stability of Ships  
 (Part 1:) Stable Conditions for the Upright State

Here, the weight  $W$  of the former is obtained as the product of the specific weight  $\gamma_t$  and the total volume  $V_t$  of the columnar ship, while the buoyancy  $F_B$  of the latter is obtained as the product of the specific weight  $\gamma_w$  of water and the displacement volume  $V_w$  of underwater portion, according to Archimedes' principle.

The floating body is stable in the following conditions where the weight  $W$  and buoyancy  $F_B$  are in equilibrium.

$$W = F_B \dots\dots\dots(3)$$

Substituting  $W$  and  $F_B$  in Eq. (2) into both sides of the above, we obtain as :

$$\gamma_t \cdot \beta h \cdot h \cdot L = \gamma_w \cdot \beta h \cdot d \cdot L \dots\dots\dots(4)$$

By solving the above equation, the undetermined draft  $d$  can be determined as  $\alpha$  times the depth  $h$ , as follows :

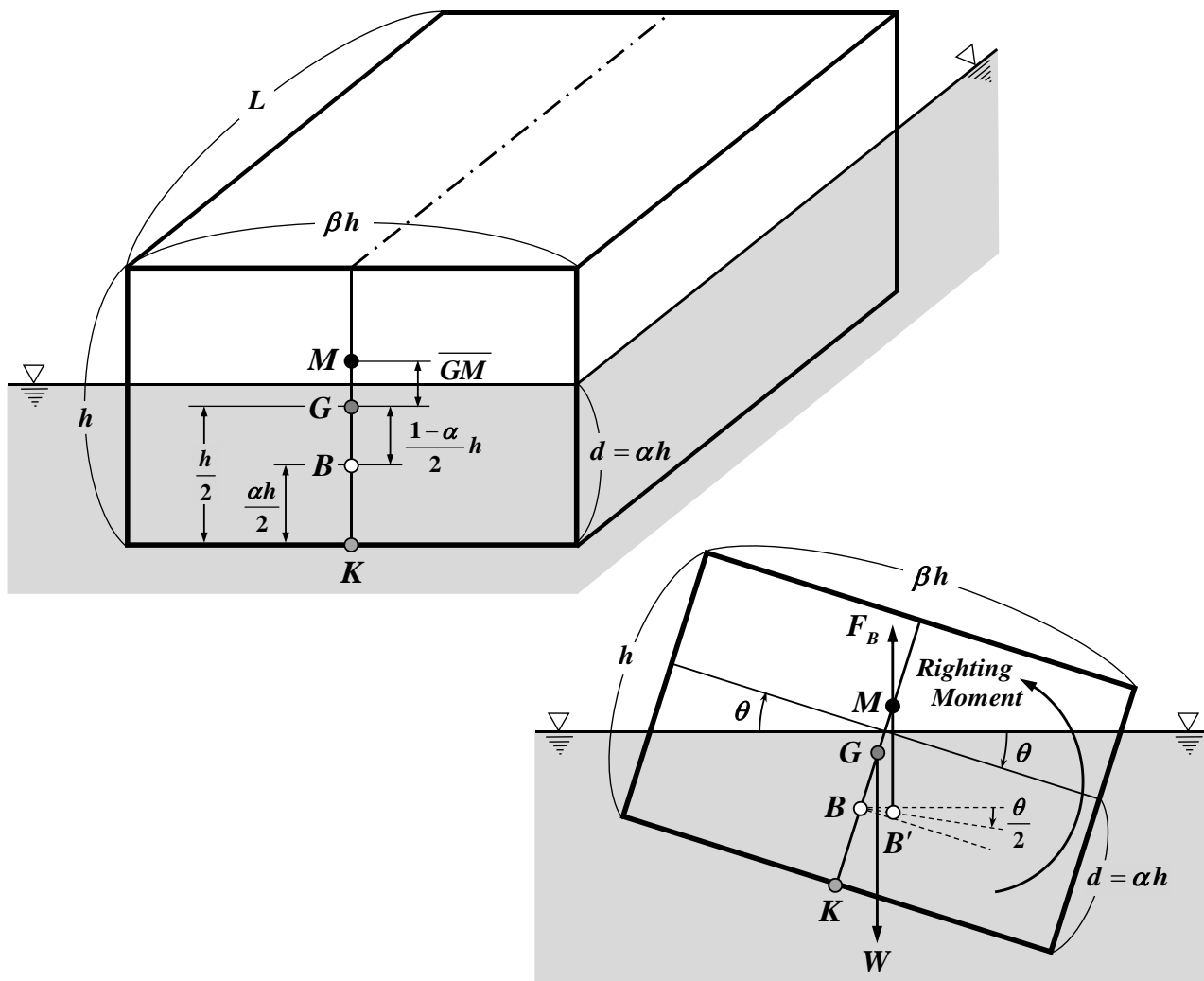


Fig.1 Upright (left) and laterally inclined (right) states of a columnar ship with rectangular cross-section.

$$d = \frac{\gamma_t}{\gamma_w} h = \alpha h \dots\dots\dots(5)$$

Next, let's consider determining the location of the metacenter  $M$  (center of inclination).

The metacentric radius  $\overline{BM}$  (distance between the center of buoyancy  $B$  and the metacenter  $M$ ) can be calculated by using basic formula<sup>(11)-(14)</sup> of naval architecture as follows :

$$\overline{BM} = \frac{I_{CL}}{V_w} \dots\dots\dots(6)$$

Here, the numerator,  $I_{CL}$ , is the quadratic moment about the center line of water plane (the single-dotted chain line in the left side of Fig. 1, where the subscript  $CL$  is the abbreviation of Center Line), and the denominator,  $V_w$ , is the underwater volume of a ship.

In this case, the numerator and denominator of Eq. (6) can be calculated as follows :

$$\left. \begin{aligned} I_{CL} &= \frac{1}{12} (\beta h)^3 L \\ V_w &= \beta h \cdot d \cdot L = \beta h \cdot \alpha h \cdot L = \alpha \beta h^2 L \end{aligned} \right\} \dots\dots\dots(7)$$

In the above equation, the former,  $I_{CL}$ , refers to the fact that the water plane is a rectangle of length  $L$  and breadth  $\beta h$ , as shown in Fig. 1 (left), and the denominator,  $V_w$ , refers to the fact that the draft is  $d = \alpha h$ , as determined by Eq. (5).

By using the result of Eq. (7) into Eq. (6),  $\overline{BM}$  can be calculated independently of the ship's length  $L$  as follows :

$$\overline{BM} = \frac{\frac{1}{12} (\beta h)^3 L}{\alpha \beta h^2 L} = \frac{\beta^2}{12\alpha} h \dots\dots\dots(8)$$

Furthermore, let's find  $\overline{BG}$  (distance between the center of buoyancy  $B$  and the center of gravity  $G$ ).

As shown in Fig. 1 (left), *the Center of Gravity*,  $G$  is located at the centroid of the rectangular cross-section and *the Center of Buoyancy*,  $B$  is located at the centroid of the rectangle below the water surface<sup>(4),(7),(8)</sup> and 1st. half of (9),(10). And the point on the centerline of the bottom of the ship is designated as  $K$  (abbreviation of Keel). Then, the distances from  $K$  to  $G$  and  $B$  are determined respectively as follows :

$$\left. \begin{aligned} \overline{KG} &= \frac{h}{2} \\ \overline{KB} &= \frac{d}{2} = \frac{\alpha h}{2} \end{aligned} \right\} \dots\dots\dots(9)$$

Therefore, the distance  $\overline{BG}$  between  $B$  and  $G$  can be obtained as follows :

$$\begin{aligned} \overline{BG} &= \overline{KG} - \overline{KB} \\ &= \frac{h}{2} - \frac{\alpha h}{2} = \frac{1-\alpha}{2} h \dots\dots\dots(10) \end{aligned}$$

Theoretical Treatment on the Hydrostatic Stability of Ships  
 (Part 1 :) Stable Conditions for the Upright State

From the above preparatory calculations, the metacentric height  $\overline{GM}$  (the distance between the center of gravity  $G$  and the metacenter  $M$ ) can be determined by subtracting Eq. (10) from Eq. (8), as follows :

$$\begin{aligned} \overline{GM} &= \overline{BM} - \overline{BG} \\ &= \frac{\beta^2}{12\alpha} h - \frac{1-\alpha}{2} h = \frac{\beta^2 - 6\alpha + 6\alpha^2}{12\alpha} h \quad \dots\dots\dots(11) \end{aligned}$$

In order to float stably in the upright state as shown in Fig. 1 (left), where the long side of the columnar ship is parallel to the water line, it is required that the stability force (mechanically, the righting moment) acts to return the ship from the inclined state to the upright state, as shown in Fig. 1 (right). For this purpose, the metacenter  $M$  must be located above the center of gravity  $G$ . In other words, the metacentric height should be positive value, and the stable condition can be described as follows :

$$\overline{GM} > 0 \quad \dots\dots\dots(12)$$

If we use the result obtained in Eq. (11) for the left-hand side of the above inequality n, it can be written as follows :

$$\overline{GM} = \frac{\beta^2 - 6\alpha + 6\alpha^2}{12\alpha} h > 0 \quad \dots\dots\dots(13)$$

Both  $h$  in the above equation and  $\alpha$  in the denominator are positive value, mathematically, it is only necessary that the numerator is positive. As a result, the stable condition of this example shown in Fig. 1 is obtained as follows :

$$\beta^2 - 6\alpha + 6\alpha^2 > 0 \quad \dots\dots\dots(14)$$

**2.1 Stable conditions of a columnar ship for breadth  $\beta$  with fixed material  $\alpha$**

First, in Section 2. 1, we will fix the material  $\alpha$  of the columnar ship and consider what breadth  $\beta$  will make it float stably with its long side is horizontal, as shown in Fig. 1 (left).

By solving the stable condition in Eq. (14) for  $\beta$ , we obtain as :

$$\begin{aligned} \beta^2 &> 6\alpha - 6\alpha^2 = 6\alpha(1-\alpha) \\ &= \frac{3}{2} - 6\left(\alpha - \frac{1}{2}\right)^2 \equiv \Omega \quad \dots\dots\dots(15) \end{aligned}$$

If the right-hand side of the above equation is written as  $\Omega$ , it can be seen that it is stable when the following Eq. (16) is satisfied :

$$\beta > \sqrt{\Omega} = \sqrt{6\alpha(1-\alpha)} \quad \dots\dots\dots(16)$$

As a result, it shows that  $\sqrt{\Omega}$  is the limiting value of breadth for stable floating.

For example, in the case of a timber with  $\alpha = \frac{1}{2}$ , the stable  $\beta$  is calculated as follows. Thus, it indicates that the timber is stable if the breadth is at least about 1.3 times wider than the depth.

$$\beta]_{\alpha=\frac{1}{2}} > \sqrt{6 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{\sqrt{6}}{2} \doteq 1.225 \dots\dots\dots(17)$$

As a result, a square with  $\beta = 1$  cannot float stably with one side horizontal. And it encompasses what is written in many textbooks as typical examples <sup>(16),(17)</sup> and problems <sup>(15)</sup>.

Let us examine the dependence of the stable breadth limit  $\sqrt{\Omega}$  on the material  $\alpha$ .

The relationship between  $\alpha$  and  $\sqrt{\Omega}$  is shown in Fig. 2 from the result of completing the square of the right-hand side  $\Omega$  in Eq. (15). From this figure, we can see that the shape is convex upward and has a maximum value of  $\frac{\sqrt{6}}{2}$  at  $\alpha = \frac{1}{2}$ . Here, the value  $\Omega$  is positive between  $0 < \alpha \leq 1$  in the setting range of material  $\alpha$  and becomes zero at  $\alpha = 0, 1$  of both ends.

Therefore, Eq. (17) for  $\alpha = \frac{1}{2}$  above is the most stringent condition of breadth. When the material  $\alpha$  is lighter or heavier than above, the limiting value  $\sqrt{\Omega}$  of  $\beta$  will be smaller, and it is stable even if the breadth is narrower than 1.3 times the depth.

As shown in the following example of calculation, the limiting value  $\sqrt{\Omega}$  of  $\beta$  becomes smaller than 1.225 as it moves away from the center of  $\alpha = \frac{1}{2}$  to both sides (the light and heavy sides). In particular, when  $\alpha$  is  $\frac{1}{5}$  or  $\frac{4}{5}$ , the limit value of  $\beta$  is 0.98, and it can be seen that the timber is stable even if the breadth is narrower than the square ( $\beta = 1$ ).

$$\left. \begin{aligned} \alpha = \frac{1}{3}, \frac{2}{3} &\rightarrow \beta > \sqrt{6 \cdot \frac{1}{3} \cdot \frac{2}{3}} = \frac{2\sqrt{3}}{3} \doteq 1.155 \\ \alpha = \frac{1}{4}, \frac{3}{4} &\rightarrow \beta > \sqrt{6 \cdot \frac{1}{4} \cdot \frac{3}{4}} = \frac{3\sqrt{2}}{4} \doteq 1.061 \\ \alpha = \frac{1}{5}, \frac{4}{5} &\rightarrow \beta > \sqrt{6 \cdot \frac{1}{5} \cdot \frac{4}{5}} = \frac{2\sqrt{6}}{5} \doteq 0.980 \end{aligned} \right\} \dots\dots\dots(18)$$

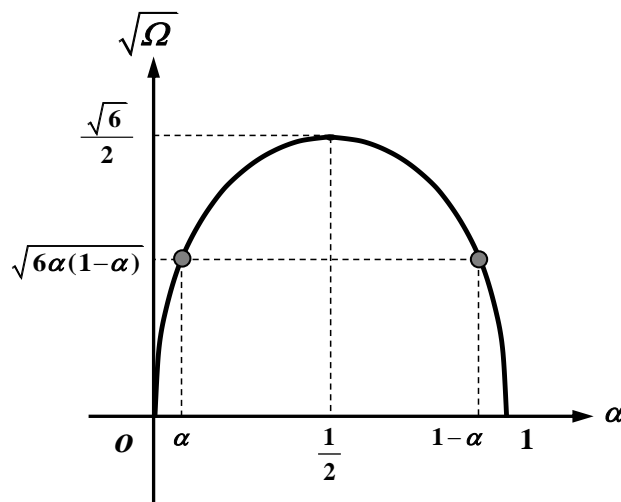


Fig. 2 Relationship between limiting value  $\sqrt{\Omega}$  of stable breadth and material  $\alpha$ .

It is also physically interesting to note, as we can see from the factors in Eq. (16) and the results in Eq. (18), that the limiting value  $\sqrt{\Omega}$  of stable  $\beta$  is the same for materials  $\alpha$  and  $1-\alpha$ , as shown by the symbols of  $\bullet$  in Fig. 2.

Theoretical Treatment on the Hydrostatic Stability of Ships  
 (Part 1 :) Stable Conditions for the Upright State

**2.2 Stable conditions of a columnar ship for material  $\alpha$  with fixed breadth  $\beta$**

Next, in section 2.2, we will fix the breadth  $\beta$  of the columnar ship and consider what kind of material  $\alpha$  will make it float stably with its long side parallel to the water line, as shown in Fig. 1 (left). Let's consider about this.

In order to solve the stable condition in Eq. (14) for  $\alpha$ , we put  $\Gamma$  on the left-hand side and complete the square as follows :

$$\begin{aligned} \Gamma &= 6\alpha^2 - 6\alpha + \beta^2 \\ &= 6\left(\alpha - \frac{1}{2}\right)^2 + \frac{2\beta^2 - 3}{2} \dots\dots\dots(19) \end{aligned}$$

Then, the stable condition in Eq. (14) can be written as :

$$\Gamma > 0 \dots\dots\dots(20)$$

Since the situation of the above quadratic equation  $\Gamma$  with respect to  $\alpha$  differs depending on whether the constant term  $2\beta^2 - 3$  is positive or negative value, the following cases i ) and ii ) are examined separately.

**i) the case of  $2\beta^2 > 3$  ( i.e.  $\beta > \frac{\sqrt{6}}{2}$  ) for wide breadth**

In this case,  $\Gamma$  in Eq. (19) is a downwardly convex shape and it is always positive in the range painted in gray, as shown in Fig. 3. Therefore, since the stable condition of Eq. (20) is satisfied regardless of  $\alpha$ , the floating body is always stable in the upright state.

This case ( i ) coincides with the stable condition of  $\beta$  for  $\alpha = \frac{1}{2}$  in Eq. (17) of Section 2. 1.

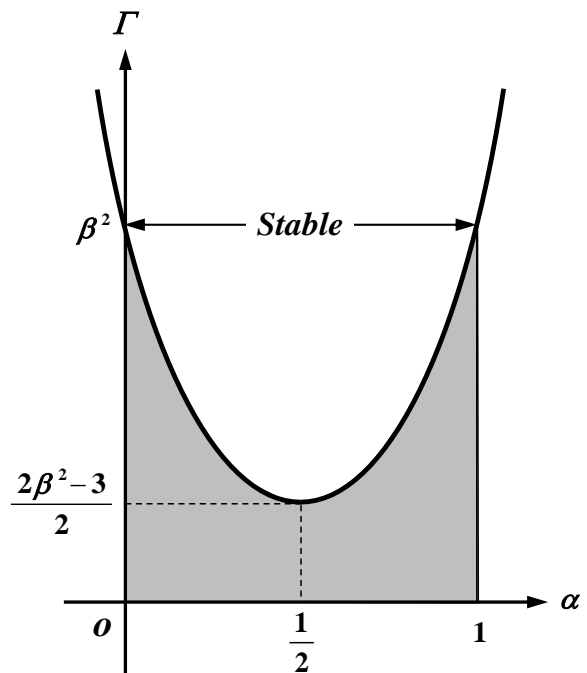


Fig.3 the case of  $2\beta^2 > 3$  for wide breadth.

ii) the case of  $2\beta^2 < 3$  (i.e.  $\beta < \frac{\sqrt{6}}{2}$ ) for narrow breadth

In this case, there are two solutions for  $\Gamma = 0$  in Eq. (19), as follows :

$$\left. \begin{aligned} \alpha &= \frac{1}{2} \pm \frac{\sqrt{3(3-2\beta^2)}}{6} \equiv \frac{1}{2} \pm \kappa \\ \left( \text{where, } \kappa &\equiv \frac{\sqrt{3(3-2\beta^2)}}{6} \right) \end{aligned} \right\} \dots\dots\dots(21)$$

These are the points of intersection with the  $\alpha$ -axis, as indicated by the mark of  $\circ$  in Fig. 4. Since the quadratic equation  $\Gamma$  is a downwardly convex shape, the range painted in gray, which satisfies the stable condition  $\Gamma > 0$  in Eq. (20), can be written as follows :

$$\left. \begin{aligned} 0 < \alpha < \frac{1}{2} - \kappa & \quad (\text{Light Material}) \\ \frac{1}{2} + \kappa < \alpha \leq 1 & \quad (\text{Heavy Material}) \end{aligned} \right\} \dots\dots\dots(22)$$

From the above, it can be seen that the light and heavy materials on both sides around  $\alpha = \frac{1}{2}$  are stable.

Furthermore, the unstable region  $2\kappa$ , where  $\Gamma < 0$ , is obtained centering on  $\alpha = \frac{1}{2}$  as follows :

$$2\kappa = \frac{\sqrt{3(3-2\beta^2)}}{3} \dots\dots\dots(23)$$

As a result,  $2\kappa = 0$  at  $\beta = \frac{\sqrt{6}}{2}$  and

$2\kappa = 1$  at  $\beta = 0$ . Thus, it can be seen that the unstable region  $2\kappa$  expands as the breadth  $\beta$  becomes narrower.

Below, for 3 specific examples of  $\beta$ , the value of  $\alpha$  which satisfies  $\Gamma = 0$  is calculated by using  $\kappa$  in Eq.(21) as follows :

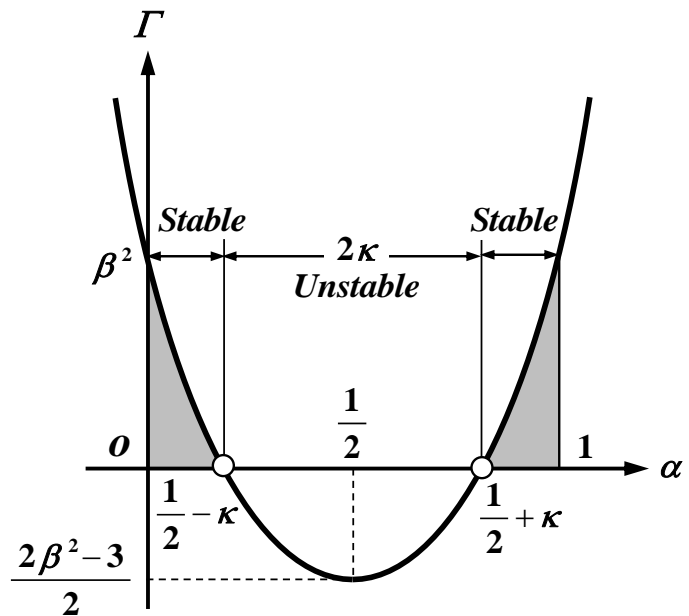


Fig.4 the case of  $2\beta^2 < 3$  for narrow breadth.



Theoretical Treatment on the Hydrostatic Stability of Ships  
 (Part 1:) Stable Conditions for the Upright State

$$\left. \begin{aligned}
 \beta = 1 \rightarrow \kappa &= \frac{\sqrt{3(3-2 \cdot 1)}}{6} = \frac{\sqrt{3}}{6} \doteq 0.289 \\
 \therefore \alpha &= 0.5 \pm \kappa = 0.211, 0.789 \\
 \\
 \beta = \frac{1}{\sqrt{2}} \rightarrow \kappa &= \frac{\sqrt{3\left(3-2 \cdot \frac{1}{2}\right)}}{6} = \frac{\sqrt{6}}{6} \doteq 0.408 \\
 \doteq 0.707 \quad \therefore \alpha &= 0.5 \pm \kappa = 0.092, 0.908 \\
 \\
 \beta = \frac{1}{2} \rightarrow \kappa &= \frac{\sqrt{3\left(3-2 \cdot \frac{1}{4}\right)}}{6} = \frac{\sqrt{30}}{12} \doteq 0.456 \\
 \therefore \alpha &= 0.5 \pm \kappa = 0.044, 0.956
 \end{aligned} \right\} \dots\dots\dots(24)$$

From the above results, it can be seen that as the breadth  $\beta$  becomes narrower, the stable regions outside the two  $\alpha$  in Eq. (21) which satisfy  $\Gamma = 0$  decrease.

**2.3  $\alpha, \beta, \overline{GM}$  in the rectangular cross-section of Fig. 1**

The material  $\alpha$  and breadth  $\beta$  of the rectangular cross-section in Fig. 1 are as follows :

$$\left. \begin{aligned}
 \alpha &= 0.58 \\
 \beta &= 1.62
 \end{aligned} \right\} \dots\dots\dots(25)$$

Then  $\overline{GM}$  is calculated by using Eq. (13) as follows :

$$\overline{GM} = 0.167 h > 0 \quad \dots\dots\dots(26)$$

In fact,  $B, G,$  and  $M$  in Fig. 1 (Left) shows the positional relationship drawn correctly, and as shown in Fig. 1 (Right), the floating body is stable state due to the righting moment which brings it back to the upright state from the laterally inclined state.

Further, in Fig. 1 (Right), the center of buoyancy  $B$  in the upright state moves to the direction of half angle<sup>(11)-(14)</sup>  $\frac{\theta}{2}$  when it is inclined laterally by  $\theta$ . Then, the position of the center of buoyancy  $B'$  after the inclination can be determined as the intersection of the above-mentioned half-angle directional line and the vertical line lowered from the metacenter  $M$ . Therefore, the position of  $B'$  shown in Fig. 1 (Right) is also the correct position under the setting variables of Eq. (25).

### 3. Stable Conditions for a Columnar Ship of Rectangular Cross-Section with specified Material $\alpha$ and Breadth $\beta$

In Chapter 2, we have set up a problem in which both the material  $\alpha$  and the breadth  $\beta$  take arbitrary values, and have shown how to solve it and determine the stable conditions.

When lecturing to students, it would be easier for them to understand if we specify a representative value for either  $\alpha$  or  $\beta$ . Chapter 3 is described from such a perspective.

#### 3.1 Stable condition for breadth $\beta$ of a columnar ship with material $\alpha = \frac{1}{2}$ (timber)

First, let's try to solve the example problem in Section 2.1 by using timber with  $\alpha = \frac{1}{2}$  as the material from the beginning.

In this case, the stable condition of Eq. (15) becomes a very simple inequality, since the right-hand side is  $\Omega = \frac{3}{2}$ , as follows :

$$\beta^2 - \frac{3}{2} > 0 \dots\dots\dots(27)$$

The positive value of  $\beta$  satisfying the above equation can be obtained by mental calculation as follows. Then it coincides with the result of Eq. (17) in Section 2.1.

$$\beta > \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2} \doteq 1.225 \dots\dots\dots(28)$$

This makes it easy to conclude that a rectangular columnar ship made of timber will float stably with upright state if its breadth is at least 1.3 times wider than its depth.

#### 3.1.a In the case of breadth $\beta = \frac{\sqrt{6}}{2}, \sqrt{3}$ for a columnar ship with material $\alpha = \frac{1}{2}$ (timber)

Here, for the stable condition related to the breadth  $\beta$  shown in Eq. (28) in Section 3.1, we will take two specific examples,  $\beta = \frac{\sqrt{6}}{2}$ , which is its limit value implying the neutral state, and  $\beta = \sqrt{3}$ , which satisfies its condition. And we will show the two status as follows :

By setting  $\alpha = \frac{1}{2}$  in Eq. (13) in Chapter 2,  $\overline{GM}$  in this case can be obtained as :

$$\overline{GM} \Big]_{\alpha=\frac{1}{2}} = \frac{2\beta^2 - 3}{12} h \dots\dots\dots(29)$$

Then,  $\overline{GM}$  for the above two cases can be calculated, respectively, as follows :

Theoretical Treatment on the Hydrostatic Stability of Ships  
(Part 1 :) Stable Conditions for the Upright State

$$\left. \begin{aligned}
 \beta &= \frac{\sqrt{6}}{2} (\doteq 1.225) \\
 \rightarrow \overline{GM} &= \frac{2 \cdot \frac{3}{2} - 3}{12} h = 0 \quad (\text{Neutral}) \\
 \\
 \alpha &= \sqrt{3} (\doteq 1.732) \\
 \rightarrow \overline{GM} &= \frac{2 \cdot 3 - 3}{12} h = \frac{h}{4} \quad (\text{Stable})
 \end{aligned} \right\} \dots\dots\dots(30)$$

The shapes of the rectangular cross-sections and the positional relationship between  $B$ ,  $G$ , and  $M$  for the above two states are shown in Fig. 5. The left figure shows the neutral state where  $M$  and  $G$  coincide. And if the breadth is even a little wider than left it, the timber can float stably with upright state as shown in the right figure. Here, in the right state of  $\beta = \sqrt{3}$ ,  $G$  is located exactly midway between  $B$  and  $M$ .

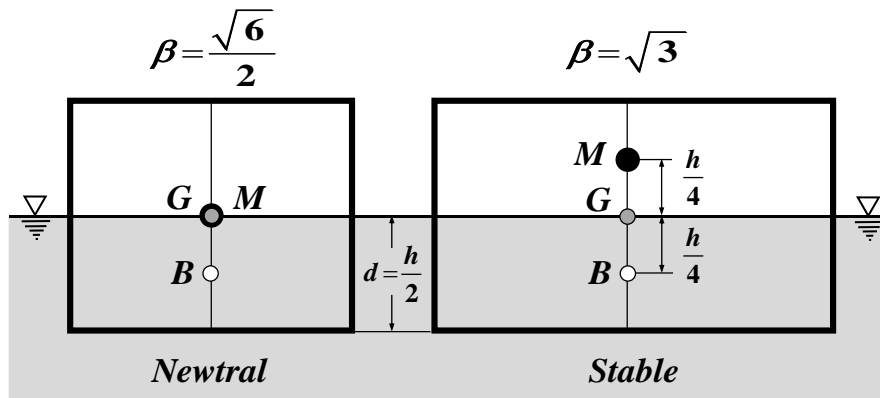


Fig. 5 In the case of material  $\alpha = \frac{1}{2}$ , breadth  $\beta = \frac{\sqrt{6}}{2}, \sqrt{3}$ .

**3.2 Stable condition for material  $\alpha$  of a columnar ship with breadth  $\beta = 1$  (square)**

Next, let's solve the example in Section 2.2 by setting a square cross-section with breadth  $\beta = 1$ .

In this case, the stable conditions in Eqs. (19) and (20) become quadratic inequality about  $\alpha$ , as follows :

$$\begin{aligned}
 \Gamma &= 6\alpha^2 - 6\alpha + 1 \\
 &= 6\left(\alpha - \frac{1}{2}\right)^2 - \frac{1}{2} > 0 \quad \dots\dots\dots(31)
 \end{aligned}$$

$\alpha$  satisfying  $\Gamma = 0$  can be easily solved by the above equation for the latter completing the square, as follows :

$$\alpha = \frac{1}{2} \pm \frac{\sqrt{3}}{6} \doteq 0.211, 0.789 \dots\dots\dots(32)$$

This coincides with the result of the 1<sup>st</sup>. case of Eq. (24) for the narrower breadth of Section 2.2 (ii).

Since  $\Gamma$  is a quadratic equation with downward convexity, as shown in Fig. 4, the range of  $\alpha$  which satisfies the stable condition  $\Gamma > 0$  in Eq. (31) is can be obtained as follows<sup>(17)</sup> :

$$\left. \begin{array}{l} 0 < \alpha < 0.211 \text{ (Light Material : cork and Styrofoam etc.)} \\ 0.789 < \alpha \leq 1 \text{ (Heavy Materials : rubber and leather etc.)} \end{array} \right\} \dots\dots\dots(33)$$

Here, in the above states, the draft of floating body for each  $\alpha$  is  $d = \alpha h$  , as shown in Eq. (5).

On the other hand, the range of unstable  $\alpha$  is as follows :

$$0.211 < \alpha < 0.789 \text{ (Woods : Japanese cypress and larch etc.)} \dots\dots\dots(34)$$

The results show that a columnar ship of square cross-section floats stably with one side parallel to the water line for light materials such as cork and Styrofoam, and for heavy materials such as rubber and leather, as shown in Eq. (33). On the contrary, for woods such as Japanese cypress and larch, as shown in Eq. (34), the timber cannot float when one side is horizontal.

**3.2.a In the case of material  $\alpha = \frac{1}{6}, \frac{5}{6}$  for a columnar ship with breadth  $\beta = 1$  (square)**

Here, let us specifically take up light  $\frac{1}{6}$  and heavy  $\frac{5}{6}$  as the stable material  $\alpha$  shown in Eq. (33) in Section 3.2, and show their stats.

The  $\overline{GM}$  in this case can be calculated by setting  $\beta = 1$  in Eq. (13) as follows :

$$\overline{GM} \Big]_{\beta=1} = \frac{1-6\alpha+6\alpha^2}{12\alpha} h = \frac{1-6\alpha(1-\alpha)}{12\alpha} h \dots\dots\dots(35)$$

Therefore, using the above equation,  $\overline{GM}$  for each light and heavy case can be obtained as follows:

$$\left. \begin{array}{l} \alpha = \frac{1}{6} (\doteq 0.167) \\ \rightarrow \overline{GM} = \frac{1-6 \cdot \frac{1}{6} \cdot \frac{5}{6}}{12 \cdot \frac{1}{6}} h = \frac{\frac{h}{6}}{2} = \frac{h}{12} \\ \\ \alpha = \frac{5}{6} (\doteq 0.833) \\ \rightarrow \overline{GM} = \frac{1-6 \cdot \frac{5}{6} \cdot \frac{1}{6}}{12 \cdot \frac{5}{6}} h = \frac{\frac{h}{6}}{10} = \frac{h}{60} \end{array} \right\} \dots\dots\dots(36)$$

Theoretical Treatment on the Hydrostatic Stability of Ships  
(Part 1:) Stable Conditions for the Upright State

The floating states of these light and heavy materials are shown in Fig. 6, including the positional relationships of  $B$ ,  $G$ , and  $M$ .

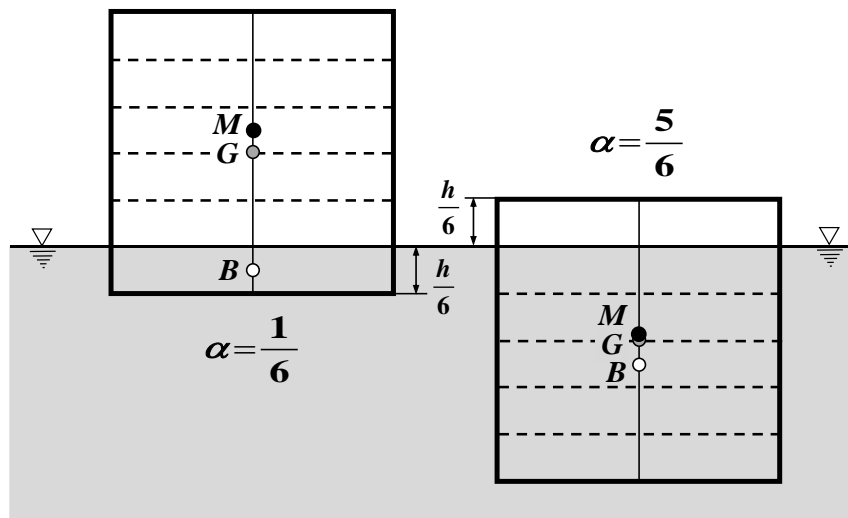


Fig. 6 In the case of breadth  $\beta=1$ , material  $\alpha = \frac{1}{6}, \frac{5}{6}$ .

#### 4. Afterword

In this paper, we have presented some examples which are effective in understanding the hydrostatic stability of ships from the 1<sup>st</sup>. author's empirical point of view. It would be the authors' great pleasure if this paper could be help to teachers and students who will teach and learn this field in the future.

#### Acknowledgments

In closing this paper, we would like to say the following words of thanks from the 1<sup>st</sup>. author<sup>(3)</sup>. I would like to communicate my deepest gratitude to my late teacher, *Pr. Masato KURIHARA*<sup>(19),(20),(21)</sup>, who cordially taught me the theory of "Hydrostatics of Ships" with detailed figures and formulas on the blackboard when I was a first-year undergraduate student and learned my first specialized subject of naval architecture in *the College of Naval Architecture of Nagasaki*. Therefore, I am following the appearance of my teacher at that time from more than 40 years ago as an exemplary example, when the author gives lectures to students on Hydrostatics of Floating Bodies<sup>(7),(8)</sup> and Theory of Ship Stability<sup>(12),(13),(22),(23)</sup>.

Finally, I would like to express my heartfelt gratitude to *Dr. Yoshihiro KOBAYASHI*, former professor at Sojo University and current president of Como-Techno Ltd., who always gave warm encouragement to the author's research and recommended that this study should be published in English. I am greatly inspired by the vigorous academic spirit with which he writes about the results of his research in books<sup>(24),(25)</sup>.

## Appendix

### A-1. Lecture videos on the hydrostatic stability of ships

One of the authors teaches the theory of ship's hydrostatic stability, which is developed in this paper, to 2nd. year students of a naval architectural engineering course<sup>(1),(2)</sup> in a lecture entitled "Theory of Ship Stability" at the 1<sup>st</sup>. author's<sup>(3)</sup> university. We have uploaded the three recorded videos of the lecture to YouTube as on-demand materials because of the recent trend of remote lectures, and introduce them in this Appendix A-1.

The 1<sup>st</sup>. video<sup>(22)</sup> is a theory for determining the breadth condition for a columnar ship with a rectangular cross-section, whose specific weight is half that of water  $\alpha = \frac{1}{2}$ , to float stably in an upright position, which is explained in Section 3.1.

The 2<sup>nd</sup>. video<sup>(26)</sup> shows that the above theory was confirmed experimentally in a small water tank for high school students' inquiry learning online.

The 3<sup>rd</sup>. video<sup>(23)</sup> explains that a theory for determining the conditions of specific weight (*i.e.* lightness or heaviness of the material) for a columnar ship with square cross-section  $\beta=1$  to float stably in an upright position, which is described in Section 3.2.

Three lecture videos are explained in Japanese, but if you are interested, please watch them.

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‡ Bold text in the list means that there is a HyperLink.

Theoretical Treatment on the Hydrostatic Stability of Ships  
(Part 1:) Stable Conditions for the Upright State

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