Geodesic Deviations and the MOND Paradigm

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Abstract

We sketch an argument suggesting that the sensitivity of geodesics to initial conditions explains away the Modified Newtonian Dynamics (MOND) paradigm. Accounting for both transversality constraints and the Jacobi equation leads to a non-vanishing correction to Newtonian dynamics, which replicates the effect of the Milgrom parameter. Our work falls in line with the Planck data on the nearly vanishing curvature of the large-scale Universe.

Key words: Modified Newtonian Dynamics, geodesics, sensitive dependence to initial conditions, Jacobi equation, transversality conditions, Dark Matter, Planck data.

A geodesic in Riemannian spacetime having curvilinear coordinates $x^{\mu}, (\mu = 0,1,2,3)$ is represented by the set of parametric equations

$$x^{\mu} = x^{\mu}(s) \tag{1}$$

where s denotes the proper time. The components of the four-velocity are defined as

$$\dot{x}^{\mu}(s) = v^{\mu} = \frac{dx^{\mu}(s)}{ds} \tag{2}$$

and comply with

$$g_{\mu\nu} v^{\mu} v^{\nu} = 1 \tag{3}$$

The action functional

$$S[x^{\mu}] = \int_{\tau_1}^{\tau_2} L(x^{\mu}(s), \dot{x}^{\mu}(s), s)\, ds \tag{4}$$

stays stationary with respect to arbitrary infinitesimal variations of coordinates leaving the boundary points fixed

$$\delta x^{\mu}(s_1) = \delta x^{\mu}(s_2) = 0 \tag{5}$$

The equations of motion derived from (4) and (5) read

$$\frac{\partial L}{\partial x^{\mu}} - \frac{d}{ds}\left(\frac{\partial L}{\partial v^{\mu}}\right) = 0 \tag{6}$$

Constraining (4) to the fixed boundary conditions (5) is not a universal choice. A more general requirement is that the boundary points lie on two given curves, $\zeta = \zeta_1(s)$ and $\zeta = \zeta_2(s)$, respectively [1-2]. A constraint of this type is suitable for the analysis of *geodesic sensitivity to initial conditions*, whereby two adjacent geodesics separate from each other according to the Jacobi equation (A1) introduced in the Appendix. If s_1 and s_2 denote the boundary points of a non-relativistic geodesic, demanding stationarity of the action integral leads to the so-called *transversality conditions* [2]

$$\left[\frac{\partial L}{\partial v^{\mu}} + \frac{\partial \zeta_1}{\partial x^{\mu}}\left(L - v^{\mu}\frac{\partial L}{\partial v^{\mu}}\right)\right|_{s=s_1} = 0 \tag{7}$$

$$\left[\frac{\partial L}{\partial v^{\mu}} + \frac{\partial \zeta_2}{\partial x^{\mu}}\left(L - v^{\mu}\frac{\partial L}{\partial v^{\mu}}\right)\right|_{s=s_2} = 0 \tag{8}$$

under the assumption that $\delta x^{\mu}(s_1)$ and $\delta x^{\mu}(s_2)$ do not depend on each other.

Consider next a toy model describing the planar problem in polar coordinates, where

$$x^1 = r \tag{9}$$

$$x^2 = \theta \tag{10}$$

Let us assume that the gravitational field is created by a centrally located mass M. In the weak field approximation, the expression of the line is given by $(c=1)$ [5]

$$ds^2 = (1+2\varphi)dt^2 - (1-2\varphi)(dr^2 + r^2 d\theta^2) \tag{11}$$

where Newton's potential is

$$\varphi(r) = -\frac{G_N M}{r} \tag{12}$$

If $\zeta(s)$ is the geodesic deviation entering (A1), a reasonable approximation of its partial derivatives is

$$\frac{\partial \zeta}{\partial r} \approx \frac{\partial \zeta}{\partial s}\frac{ds}{dr} = \frac{\partial \zeta}{\partial s}\frac{1}{v_r} \tag{13}$$

$$\frac{\partial\zeta}{\partial\theta}\approx\frac{\partial\zeta}{\partial s}\frac{ds}{d\theta}=\frac{\partial\zeta}{\partial s}\frac{r}{v_\theta}=\frac{\partial\zeta}{\partial s}\frac{1}{\omega} \tag{14}$$

in which ω stands for angular velocity. Current astrophysical data indicate that we live in a Universe which is flat or nearly flat [7]. The solution of (A1) for a nearly flat spacetime $(K(s)\to 0)$ can be presented as

$$\frac{d\zeta(s)}{ds}=c(s)=c_0+\varepsilon(s);\quad 0<\varepsilon(s)<<1;\quad s\in[s_1,s_2] \tag{15}$$

in which $\varepsilon(s)$ quantifies the overall departure from the constant deviation rate c_0.

Let a set of geodesic measurements be taken on a massive cosmic object whose evolution is modeled by (9) to (12). One pair of measurements is made in a *near* distance range $(Z<Z_0)$, the other in a *far* distance range corresponding to galactic scales $(Z>Z_0)$. Assuming again a nearly flat Universe with a slightly positive horizon curvature, we introduce the plausible hypothesis that the local curvature $K_Z(s)$ scales linearly with Z as in

$$K_Z(s) \propto K_{Z_0}(s)(\frac{Z}{Z_0}); \quad Z \geq Z_0 \tag{16}$$

As shown in the Appendix, increasing the local positive curvature *tends to reduce* the geodesic separation $\zeta(s)$. It follows that the overall correction $\varepsilon(s)$ of (15) is expected to scale with Z in a reciprocal manner to (16), that is,

$$\varepsilon_Z(s) \propto \varepsilon_{Z_0}(s)(\frac{Z_0}{Z}) \tag{17}$$

To simplify notation, in what follows we use N for "near" and F for "far" as in

$$Z < Z_0 \Leftrightarrow N$$

$$Z > Z_0 \Leftrightarrow F$$

The explicit contribution of the second term in (7) and (8) is given by

$$L - v_r \frac{\partial L}{\partial v_r} - v_\theta \frac{\partial L}{\partial v_\theta} = -E \tag{18}$$

where the total energy is [4-5]

$$E = T + V = \frac{\mu}{2}(v_r^2 + v_\theta^2) + \varphi(r) \tag{19}$$

Here, μ represents the mass of the cosmic object and

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta} \tag{20}$$

its radial and orbital velocities. Using polar coordinates and subtracting (7) from (8) gives, for each of the F and N ranges,

$$\Delta v_r\big|_{F,N} = \frac{E}{\mu}\Delta[(c_0+\varepsilon(s))\frac{1}{v_r}]\Big|_{F,N} \tag{21}$$

$$\Delta v_\theta\big|_{F,N} = \frac{E}{\mu}\Delta[(c_0+\varepsilon(s))\frac{1}{\omega}]\Big|_{F,N} \tag{22}$$

in which

$$\Delta v_r = v_r(s_2) - v_r(s_1); \quad \Delta v_\theta = v_\theta(s_2) - v_\theta(s_1) \tag{23}$$

$$\Delta[(c_0+\varepsilon(s))\frac{1}{v_r}] = [(c_0+\varepsilon(s))\frac{1}{v_r}]\Big|_{s_2} - [(c_0+\varepsilon(s))\frac{1}{v_r}]\Big|_{s_1} \tag{24}$$

$$\Delta[(c_0+\varepsilon(s))\frac{1}{\omega}] = [(c_0+\varepsilon(s))\frac{1}{\omega}]\Big|_{s_2} - [(c_0+\varepsilon(s))\frac{1}{\omega}]\Big|_{s_1} \tag{25}$$

Subtracting the corresponding F and N terms in (21) and (22) and demanding that they amount to a *relative drop* in radial and orbital velocities, leads to the following conditions

$$\Delta v_r\big|_F - \Delta v_r\big|_N < 0 \tag{26}$$

$$\Delta v_\theta\big|_F - \Delta v_\theta\big|_N < 0 \tag{27}$$

If (26) and (27) are satisfied, the apparent deficit in kinetic energy ΔT between the "far" and "near" measurements offers an unforeseen explanation for flattening of galactic rotation curves. In particular, this deficit may be mapped to the change in gravitational force attributed to the Milgrom acceleration parameter [6], namely

$$\boxed{\Delta T\big|_F - \Delta T\big|_N \Leftrightarrow a_0} \tag{28}$$

APPENDIX: The Jacobi Equation

Let Γ_0 represent a fixed geodesic whose coordinates are function of the distance s measured along it. Denote a nearby geodesic by Γ. Let the geodesic normal to Γ_0 be called Γ_1 and assume that Γ_1 intersects Γ at point

P. Let the distance between Γ_0 and Γ measured along Γ_1 at P be denoted as $\zeta(s)$. It can be shown that $\zeta(s)$ satisfies the *Jacobi equation* [3]

$$\frac{d^2\zeta(s)}{ds^2} = -K(s)\zeta(s) \tag{A1}$$

in which $K(s)$ is the Gaussian curvature at P. The neighboring geodesic Γ is pulled back towards Γ_0 if $K > 0$, or pushed away from Γ_0 if $K < 0$. It follows that the Gaussian curvature represents a *local measure of geodesic instability*. On a spherical surface, $K > 0$ means stability whereas $K < 0$ on non-spherical surfaces describes instability.

References

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