How the twins each age less than the other

Tom Fuchs
tomfuchs@gmail.com

This paper shows that the twin paradox hasn't heretofore been solved. It's shown that the asymmetry in a twin paradox experiment resulting in a difference in aging is that the twins move different distances relative to each other, due to length contraction. It's shown that the twins each age less than the other, yet unparadoxically after accounting for relativity of simultaneity. An equation is derived to calculate the constant speed at which a spaceship must travel so that its occupants age a given time during a trip. Code is given to numerically integrate a twin paradox experiment involving acceleration.

1 Simplifying the puzzle

Experiment #1: Sue travels from Earth to the Alpha Centauri star system and back, while Bob stays on Earth. She launches from Earth, accelerates and decelerates at 1 g to arrive at Alpha Centauri at relative rest, and immediately returns to Earth the same way.

Earth's acceleration of gravity is negligible for the experiments herein. See the equations of special relativity at The Relativistic Rocket, for a rocket having a constant proper acceleration $a > 0$. The equations that predict the elapsed times for each quarter of the trip are

$$t = \sqrt{(d/c)^2 + 2d/a} \tag{1}$$

$$T = \frac{c}{a}\text{acosh}(ad/c^2 + 1) \tag{2}$$

The distance d for each quarter of the trip = 2.17 ly. The speed of light is c. Use $c = 1$ ly/yr and $a = 1.03$ ly/yr$^2 \approx g$. Multiply the results of (1) for Bob and (2) for Sue by 4 (quarters) to see that Bob ages $t \approx 11.9$ years while Sue ages $T \approx 7.2$ years.

Each twin sees the other as moving. The puzzle of the twin paradox is that each should then paradoxically age less than the other, as predicted by the gamma factor equation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{t}{T} \tag{3}$$

Since the paradox is presented using only two inertial frames (one for each twin), a twin paradox experiment needn't involve acceleration or a turnaround, eliminating those as solutions in whole or in part. Moreover, the clock postulate tells us that:

> [An] accelerating clock will count out its time in such a way that at any one moment, its timing has slowed by a factor (γ) that depends only on its current speed [v]; its acceleration has no effect at all.

This means that the paradox for an experiment involving acceleration is solvable by considering only momentarily co-moving inertial frames, using the acceleration just to determine the speed and gamma factor at any given moment.

Other texts also fail to solve the paradox by showing asymmetry between the spacetime paths of the twins. Since a twin paradox experiment can be limited to two inertial frames, it's clear that the twins do each age less than the other, in concordance with the gamma factor (3). An actual solution must show how a paradox is nonetheless ruled out.

2 A clue: missing lines of simultaneity

Missing lines of simultaneity give a clue to solving the twin paradox:

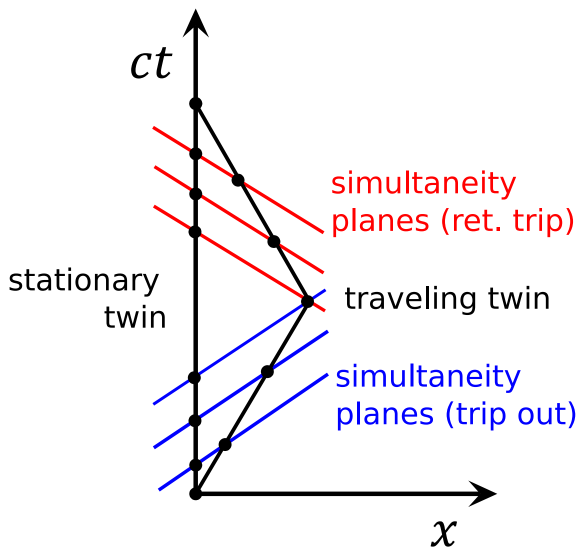

Figure 1: Spacetime diagram of a twin paradox where the traveling twin is actually two travelers moving at constant speed, one outgoing from the starting point and another returning to it, passing by each other where the turnaround point would be. At this moment the clock reading for the first traveler is transferred to the second one. Their trip times are summed at the end of their journey. By Acdx, CC BY-SA 3.0, via Wikimedia Commons.

In Fig. 1, notice that the middle part of the stationary twin's world line (on the vertical axis) has no possible line of simultaneity that connects to the traveling twins' world lines, whereas any event on the traveling twins' world lines can connect to the stationary twin's world line. These "missing" lines of simultaneity allow for the possibility that the twins in any twin paradox experiment each age less than the other, but without paradox, because some events that occur during the experiment in the stationary twin's frame—including ticks of that twin's clocks—don't occur during the experiment in the traveling twin's frame. Which is to say that the traveling twin's experiment is a subset of the stationary twin's experiment.

3 Speeding to Andromeda, or elsewhere

See the problem “Speeding to Andromeda” in the chapter “Speeding” at Exploring Black Holes:

> At approximately what constant speed v_{Sun} with respect to our Sun must a spaceship travel so that its occupants age only 1 year during a trip from Earth to the Andromeda galaxy?

The method therein to get the answer ($1 - v_{\text{Sun}} \approx 1.25 \times 10^{-13}$) takes several steps and requires that the speed is close to the speed of light. The following equation works in every case:

$$v = \frac{d/T}{\sqrt{1 + [d/(Tc)]^2}} \tag{4}$$

When $d = 2$ million ly, $T = 1$ yr, and $c = 1$ ly/yr, (4) returns the same answer as above.

Here is the derivation of (4). From basic physics,

$$t = \frac{d}{v} \tag{5}$$

From The Relativistic Rocket,

$$v = \frac{at}{\sqrt{1 + (at/c)^2}} \tag{6}$$

$$\gamma = \sqrt{1 + (at/c)^2} \tag{7}$$

Refer to the gamma factor (3). Substituting and rearranging:

$$v = \frac{at}{\gamma} \tag{8}$$

$$t = T\gamma = \frac{d}{v} \tag{9}$$

$$at = v\gamma = \frac{d}{T} \tag{10}$$

Substituting the two terms at in (6) with the d/T from (10) gives (4). This completes the derivation of (4).

The denominator in (4) is the gamma factor γ, as shown by rearranging (10):

$$v = \frac{d}{T\gamma} \tag{11}$$

Eq. (11) shows that the speed v needed to get to a destination, while aging the time T, is the speed that length contracts the distance d to that which is covered in the time T at that speed.

Experiment #2: Joy ages 10 years while traveling from the star Vega to Earth at a constant speed v relative to our Sun. Dan stays on Earth.

Rearranging (11):

$$T = \frac{d}{v\gamma} \tag{12}$$

When the Earth-Vega distance d in Dan's frame =	25.0 ly
and the given aging T for Joy =	10 yr
then, as calculated by (4), the speed v needed by the twins toward each other ≈	0.928c
and, as calculated by (5), Dan's aging, his time t taken to reach Joy ≈	26.9 yr
and, as calculated by the gamma factor (3), the gamma factor γ ≈	2.69
and the Earth-Vega distance d/γ in Joy's frame ≈	9.28 ly
and, as calculated by (12), Joy's aging, her time T taken to reach Earth =	10.0 yr

which matches the given aging T for Joy that was input into (4), as expected.

4 Length contraction explains the difference in aging

The asymmetry in a twin paradox experiment resulting in a difference in aging is that the twins move less distance relative to each other in the traveling twin's frame, due to length contraction, so that the experiment completes faster for that twin. This is shown when their relative speed is constant by plugging (5) and (12) into the gamma factor (3) to get

$$\frac{t}{T} = \frac{d/v}{d/(v\gamma)} = \gamma \tag{13}$$

For example, in experiment #2 the twins move 25.0 light years relative to each other in Dan's frame, but just ~9.28 light years in Joy's frame. Moving less distance at the same speed (~0.928c) takes less time, so she ages less (10 years) than he does (~26.9 years). Although they both travel relative to each other, Dan is the stationary twin because he's at rest relative to Earth and Vega, the starting and ending points between which Joy moves.

5 How a paradox is ruled out

The twins each age less than the other, yet unparadoxically after accounting for relativity of simultaneity.

To see this for experiment #2, overlay the barn-pole paradox experiment. In this version the barn doors stay open. Dan, representing the runner, holds the trailing end of the pole that has a proper length d, the Earth-Vega distance in his frame. Joy stays at the far door of the barn. The experiment starts when she passes Vega and ends when she passes Earth. Let the proper distance between the barn doors be $D \stackrel{\text{def}}{=} d/\gamma$, the Earth-Vega distance in her / the barn's frame, so that in her frame he's at the near door of the barn when the experiment starts. In his frame, she's distance d away from him when the experiment starts, and the barn is length contracted to $d_{\mathrm{b}} = D/\gamma$. During Dan's experiment (the experiment in his frame), he ages $t = d/v$ and she ages $T = d/(v\gamma) = D/v$. During Joy's experiment (the experiment in her frame), she ages $T = D/v$ and he ages $t_{\mathrm{b}} = D/(v\gamma) = d_{\mathrm{b}}/v$, his aging while traversing the barn. They each age the same percentage less than the other, as shown by $t/T = T/t_{\mathrm{b}} = \gamma$. The exact part of Dan's aging that's needed to rule out a paradox, $t - t_{\mathrm{b}} = (d - d_{\mathrm{b}})/v =$ his aging while covering the distance to the barn, doesn't occur during Joy's experiment.

Experiment #3: Meg ages 10 years while traveling from Earth to the star Vega at a constant speed v relative to our Sun. Han stays on Earth.

For this experiment we can run the barn-pole paradox version of experiment #2 in reverse, to reason that the twins still each age the same percentage less than the other. The experiment starts when Meg passes Earth and ends when she passes Vega. The exact part of Han's aging that's needed to rule out a paradox, $t - t_{\mathrm{b}} =$ his aging while covering the distance *from* the barn, doesn't occur during Meg's experiment.

Section 4 said that the experiment completes faster for the traveling twin. This comes with a caveat: we ignore that the traveling twin's experiment is a subset of the stationary twin's experiment. For example, for experiment #2 we pretend that the experiment starts in Joy's frame at $t = T = 0$, as it does in Dan's frame. In reality in her frame, when the experiment starts at $T = 0$ (when she passes Vega), he's already covered the distance "to the barn" by then, so that $t > 0$. When we account for relativity of simultaneity, we find that each twin's experiment (the experiment in their own frame) completes faster for the other twin.

6 Numerical integration for an experiment involving acceleration

The clock postulate tells us that a twin paradox experiment can be divided into segments or steps wherein the twins have a constant speed relative to each other, and the agings in the steps summed to get agings for the whole experiment. For example, we divide experiment #2 into three equal steps. For each step:

$d = 25.0$ ly/3	8.33	ly
$T = 10$ yr/3	3.33	yr
v	0.928	c
γ	2.69	
$D \stackrel{\text{def}}{=} d/\gamma$	3.09	ly
$d_{\mathrm{b}} = D/\gamma$	1.15	ly
$t = d/v$	8.98	yr
$t_{\mathrm{b}} = d_{\mathrm{b}}/v$	1.24	yr

These variables are further explained in section 5.

During each step, Dan ages $t \approx 9.0$ years in his frame, ages $t_b \approx 1.2$ years in Joy's frame, and she ages $T \approx 3.3$ years in either frame. The $t_b < t$ because in her frame, when the experiment starts at $T = 0$ (when she passes Vega), it's already underway for him. In her frame at $T = 0$, using the values above for each step, Dan has already aged $3\,(t - t_b) \approx 23.2$ years (verifiable using the values in section 3 and the Lorentz transformation for time).

Revisit experiment #1. The twins' agings can be calculated by numerically integrating one quarter of Sue's round trip, calculating their agings for each segment or step that's at constant speed, and then multiplying the sum of those results by 4 (quarters). Here is code to show this. Click the Run button to get the output:

```
Predicted by the Relativistic Rocket equations:
	Bob ages 11.9 yr
	Sue ages 7.2 yr

Predicted by this code:
	During Bob's experiment:
		Bob ages 11.9 yr
		Sue ages 7.2 yr
		Sue moves 8.7 ly in Bob's frame
		Bob moves 4.6 ly in Sue's frame

	During Sue's experiment:
		Bob ages 4.9 yr
		Sue ages 7.2 yr
		Sue moves 2.7 ly in Bob's frame
		Bob moves 4.6 ly in Sue's frame
```

Sue's experiment (the experiment in her frame) is a subset of Bob's experiment (the experiment in his frame), even though the experiment starts and ends when they're together on Earth. That's how they can each age less than the other without paradox. In each step, t_b (his aging in her frame) $< t$ (his aging in his frame), whereas her aging is T in either frame. With all of her aging accounted for in his frame, her step must be a subset of his step, and none of his aging $> t_b$ can occur during any part of her whole experiment. Bob still ages more than Sue does; her experiment just doesn't share all of the events that occur during his experiment, including most of the ticks of his clocks. Lines of simultaneity connect every event on her world line to his world line, but not vice versa.

Appendix A – Code for the program

Below is the Go language code for the numerical integration program that's referenced in section 6, in case the link to the code is broken. You can run the code at the Go Playground after fixing the formatting.

```
package main

import (
	"fmt"
	"math"
```

```
)

const (
	// Sue's acceleration in ly/yr^2, = ~1 g
	a = 1.03

	// Half of the distance in ly between Earth and Alpha Centauri as
measured in Bob's frame
	d = 2.17

	// The number of steps per year in Bob's frame, in the numerical
integration below
	// More steps gives greater accuracy
	stepCountPerYear int = 400

	// The speed of light = 1 ly/yr
	c = 1

	// End of user input

	// Bob's aging in years during each step
	tStep = 1 / float64(stepCountPerYear)
)

func main() {
	// For further explanation of these variables, see section 5
	// During Bob's experiment:
	t := 0.0 // Bob's aging
	dCheck := 0.0 // The distance Sue moves in Bob's frame
	// During either twin's experiment:
	T := 0.0 // Sue's aging
	D := 0.0 // The distance Bob moves in Sue's frame
	// During Sue's experiment:
	tb := 0.0 // Bob's aging
	db := 0.0 // The distance Sue moves in Bob's frame

	for {
		t += tStep

		// Get their velocity v relative to each other, which could
be measured instead
		at := a * t
		// These equations are from the Relativistic Rocket site
		gamma := math.Sqrt(1 + math.Pow(at / c, 2))
		v := at / gamma

		// Get Sue's aging during this step
		TStep := tStep / gamma

		T += TStep

		// Get the distance Bob moves in Sue's frame during this
step
		DStep := v * TStep
```

```
			D += DStep

			tb += TStep / gamma

			db += DStep / gamma

			dCheck += v * tStep

			if dCheck >= d {
				// The distance d has been reached
				break
			}
		}

		// These equations are from the Relativistic Rocket site
		// Bob's aging
		tExpected := math.Sqrt(math.Pow(d / c, 2) + 2 * d / a)
		// Sue's aging
		TExpected := (c / a) * math.Acosh(a * d / math.Pow(c, 2) + 1)

		// Make a round trip
		dCheck *= 4
		t *= 4
		T *= 4
		D *= 4
		tb *= 4
		db *= 4
		tExpected *= 4
		TExpected *= 4

		fmt.Printf("Predicted by the Relativistic Rocket equations:\n")
		fmt.Printf("\tBob ages %0.1f yr\n", tExpected)
		fmt.Printf("\tSue ages %0.1f yr\n", TExpected)

		fmt.Printf("\nPredicted by this code:\n")
		fmt.Printf("\tDuring Bob's experiment:\n")
		fmt.Printf("\t\tBob ages %0.1f yr\n", t)
		fmt.Printf("\t\tSue ages %0.1f yr\n", T)
		fmt.Printf("\t\tSue moves %0.1f ly in Bob's frame\n", dCheck)
		fmt.Printf("\t\tBob moves %0.1f ly in Sue's frame\n", D)

		fmt.Printf("\n\tDuring Sue's experiment:\n")
		fmt.Printf("\t\tBob ages %0.1f yr\n", tb)
		fmt.Printf("\t\tSue ages %0.1f yr\n", T)
		fmt.Printf("\t\tSue moves %0.1f ly in Bob's frame\n", db)
		fmt.Printf("\t\tBob moves %0.1f ly in Sue's frame\n", D)

		// fmt.Printf("\nThere were %0.0f steps in the numerical
	integration\n", t / tStep)
	}
```