On some conjectures concerning perfect powers

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Abstract. The starting point of our paper is Kashihara's open problem #30, concerning the sequence A001292 of the OEIS, asking how many terms are perfect squares. We confirm his last conjecture up to the 100128-th term and provide a general theorem which rules out 4/9 of the candidates. Moreover, we formulate a new, provocative, conjecture involving the OEIS sequence A352991 (which includes all the terms of A001292, except the first one). Our risky conjecture states that all the perfect powers belonging to the sequence A352991 are perfect squares and they cannot be written as higher order perfect powers. This challenging conjecture has been checked for any integer smaller than 10111121314151617181920212223456789 and no counterexample has been found so far.

Keywords: Open Problem, Perfect power, Perfect square, Conjecture, Integer sequence.

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1 Introduction

In late 2010, the author of this paper found a recreative open problem by Kenichiro Kashihara (see [1], open problem #30, p. 25) concerning the sequence A001292 of the On-Line Encyclopedia of Integer Sequences (OEIS) [2]. Kashihara's problem #30 consists of two independent parts and the author solved the first one quite easily at the time (the complete solution can be found in [6], Section 3.3, pp. 12–15), since it asks to find the probability $0 < \mathrm{p}(c) < 1$ that the trailing digit of the generic term of the sequence A001292 is $c \in \{0, 1, 2, \ldots, 9\}$ and the formula provided in [6] shows that $\mathrm{p}(c) = \frac{11-c}{55}$ for any $c \neq 0$, whereas $\mathrm{p}(0) = 0.0\overline{18}$ (e.g., if $c = 7$, then $\mathrm{p}(7) = \frac{4}{55} = 0.0\overline{72}$).

In the present paper, we will focus ourselves on the second part of the above mentioned Kashihara's problem #30, asking how many elements of the sequence A001292 are perfect powers, since Kashihara conjectured that there are none.

Now, bearing in mind that a perfect power of an integer $d > 0$ is a natural number $k \geq 2$ such that $a^k = d$, where also a is a positive integer, we could point out that A001292 (1) = 1 can be considered as a solution and argue how this disproves the conjecture, but (from here on) we will disregard this special case and assume that we are looking for a nontrivial counterexample to Kashihara's conjecture.

Lastly, Section 3 is devoted to introduce a (quite improbable) conjecture concerning perfect powers belonging to the OEIS sequence A352991 [4, 5].

2 The $\{2, 3, 4, 5, 6, 7, 8\}(\text{mod } 9)$ exclusion criterion

In order to be clear on the invoked OEIS sequences, let us introduce a few useful definitions.

Definition 1. We define the m-th term of the sequence A007908 as A007908$(m) :=$ $123\_\ldots\_(m-1)\_m$, where $m \in \mathbb{Z}^+$.

Definition 2. We define the sequence A001292 of the OEIS as the concatenations (sorted in ascending order) of every cyclic permutation of the elements of the sequence A007908 (e.g., given $m = 3$, A001292(A007908(3)) $= 123, 231, 312$).

Definition 3. We define the sequence A352991 of the OEIS as the concatenation of all the distinct permutations of the first strictly positive m integers, sorted into ascending order (e.g., 12345671089 is a term of the sequence, while 12345670189 does not belong to A352991, even if all the digits of the string $123\ldots910$ appear once and only once, since "10" is missed).

After having checked the first 100128 terms of the sequence A001292 (see Appendix), exploring any exponent at or above 2, we have not found any perfect power, so that Kashihara's conjecture has been verified up to 10^{1235} (i.e., the 100129-th term of A001292 is the smallest cyclic permutation of A007908(448) and is greater than 10^{1235} by construction).

Moreover, we can prove the following Theorem 1, concerning the sequence A352991 which includes every term of A001292.

Theorem 1. For any $m > 1$, A352991(n) cannot be a perfect power of an integer if A352991(n) is a permutation of A007908(m) and $m : m \equiv \{2, 3, 5, 6\}(\text{mod } 9)$.

Proof. By definition, A007908(m) [3] cannot be a perfect power if $123\_\ldots\_(m-1)\_m$ is divisible by 3 and it is not divisible by 3^2. Thus, from the well-known divisibility by 3 and 9 criteria, $m :$ $\left(3 \mid \sum_{j=1}^{m} j\right) \wedge \left(3^2 \nmid \sum_{j=1}^{m} j\right)$ is a sufficient, but not necessary, condition for letting us disregard any permutation of $123\_\ldots\_(m-1)\_m$ (i.e., given m, if a generic permutation of A007908(m) is divisible by 3 and is not congruent to $0(\text{mod } 9)$, then all the permutations of A007908(m) are divisible by 3 once and only once, since the commutativity property holds for addition).

It follows that, for any $n \in \mathbb{Z}^+$, A352991(n) cannot be a perfect power if it is a permutation of the string $123\_\ldots\_(m-1)\_m$, where m is such that A134804(m) is divisible by 3. Therefore, the residue modulo 9 of every perfect power belonging to A352991 cannot be 2 or 3 or 5 or 6, and this concludes the proof of Theorem 1. ☐

Corollary 1. Kashihara's conjecture is true for the concatenation of any cyclic permutation of A007908(m), where $m : (m \equiv \{2, 3, 5, 6\}(\text{mod } 9) \vee m < 448)$.

Proof. We observe that A001292 is a subsequence of A352991 [2, 4]. By invoking Theorem 1, we can state that every perfect power candidate has to be the concatenation of a (cyclic) permutation of A007908(m), where m is such that $m \equiv \{0, 1, 4, 7, 8\}(\text{mod } 9)$. On the other hand, all the remaining terms up to $99\_100\_101\_\ldots\_445\_446\_447\_1\_2\_3\_\ldots\_96\_97\_98$ have been directly checked (see Appendix for details) and no perfect power has been found.

Therefore, Corollary 1 confirms Kashihara's conjecture for any term of A001292 such that m is congruent to $\{2, 3, 5, 6\}(\text{mod } 9)$ or $m \leq 447$. ☐

Corollary 2. $\nexists n :$ A353025$(n) \equiv \{2, 3, 4, 5, 6, 7, 8\}(\text{mod } 9)$, and any term of A001292 cannot be a perfect power if its digital root is not equal to 0 or 1.

Proof. Trivially, $10 \equiv 1 (\bmod 9)$ and also $(1+0) \equiv 1 (\bmod 9)$, so that any positive integer is congruent modulo 9 to its digital root.

Now, we observe that any terms of A001292 belongs to A353025.

Since from Theorem 1 it follows that every term of the sequence A353025 [5] is a special permutation of A007908(m) which is characterized by $m \equiv \{0, 1, 4, 7, 8\} (\bmod 9)$, in order to prove Corollary 2, it is sufficient to note that

$$\sum_{j=1}^{m} j \equiv \begin{cases} 0 (\bmod 9) & \text{if } m : m \equiv 0 (\bmod 9) \\ 1 (\bmod 9) & \text{if } m : m \equiv 1 (\bmod 9) \\ 1 (\bmod 9) & \text{if } m : m \equiv 4 (\bmod 9) \\ 1 (\bmod 9) & \text{if } m : m \equiv 7 (\bmod 9) \\ 0 (\bmod 9) & \text{if } m : m \equiv 8 (\bmod 9) \end{cases}. \tag{1}$$

□

Remark 1. A well-known property of integers is that every perfect power which is congruent modulo 5 to 0 is also necessarily congruent to $\{0, 25, 75\} (\bmod 100)$, while if a perfect power is congruent modulo 10 to 6, then its second last digit is odd.

Thus, we are free to combine these additional constraints with Corollary 2 in order to reduce the number of perfect power candidates among the terms of A352991.

3 Perfect cubes in A353025

In the first half of April 2022, playing with Kashihara's conjecture, a more risky (very likely false but really hard to disprove by brute force) conjecture arose, it is as follows.

Conjecture 1. Let $n \in \mathbb{N} - \{0, 1\}$ be given. We (provocatively) conjecture that if n is such that A352991(n) is a perfect power of an integer, then $\nexists k \in \mathbb{N} - \{0, 1, 2\} : \text{A352991}(n) = c^k$, $c \in \mathbb{N}$.

On April 16 2022, a direct search was performed by the author on the first 10^7 terms of the sequence and no counterexample has been found (42 perfect squares only).

A few days later, Aldo Roberto Pessolano, performed a deeper search running the Mathematica codes published in Appendix, without finding any counterexample and thus confirming Conjecture 1 (at least) up to the smallest permutation of A007908(22) (i.e., for any term of A352991 which is greater than 1 and smaller than $10111112131415161718192021222345678 9$).

Remark 2. If confirmed, Conjecture 1 would imply that all the perfect powers (greater than 1) in A352991 are perfect squares and nothing more (no cube, no square of square, and so forth). Nevertheless, under the (arbitrary, but perfectly reasonable) assumption of a standard probability distribution of the cubes in A352991 (i.e., we are assuming that $\frac{\left|\left\{n \in \mathbb{N} : \text{A352991}(n) \le m \ \wedge \ \left(\text{A352991}(n)\right)^{\frac{1}{3}} \in \mathbb{N}\right\}\right|}{|\{n \in \mathbb{N} : \text{A352991}(n) \le m\}|} \cong \frac{|\{n \in \mathbb{N} : \text{A000578}(n) \le m\}|}{m}$ holds for any sufficiently large $m \in \mathbb{N}$), we would guess the existence of infinitely many counterexamples to Conjecture 1, even if the smallest one is expected to occur in the interval $[10^{58}, 10^{65}]$. On the other hand, the same argument would corroborate Kashihara's conjecture, since the number of perfect powers belonging to A001292 cannot probabilistically exceed $2 \cdot \sum_{k=2}^{+\infty} \left(\sum_{j=308}^{+\infty} \left(\frac{(4 \cdot j+3) \cdot \left(10^{\left(\frac{4 \cdot j+3}{k}\right)} - 10^{\left(\frac{4 \cdot j+2}{k}\right)}\right)}{9 \cdot 10^{(4 \cdot j+2)}} \right) \right) \approx \frac{2}{9} \cdot \sum_{j=308}^{+\infty} \left(\frac{(4 \cdot j+3) \cdot \left(10^{\left(\frac{4 \cdot j+3}{2}\right)} - 10^{\left(\frac{4 \cdot j+2}{2}\right)}\right)}{10^{(4 \cdot j+2)}} \right) \approx 0$.

Additional open problems. *Does the sequence A353025 have infinitely many perfect squares, infinitely many perfect cubes, infinitely many perfect squares of squares, and so forth? Which is the smallest nontrivial perfect cube (if any) belonging to A353025* (we point out that all the terms greater than one and below $1.01 \cdot 10^{40}$ have been checked without finding any cube)*?*

4 Conclusion

Kashihara's open problem #30 has not been completely solved yet. Even if the first part, concerning the probability that the trailing digit of A001292(n) is $c = 1, 2, \ldots, 9$, was solved by the author a dozen of years ago [6], the second part and the related conjecture still needs a proof or a nontrivial counterexample (the smallest candidate has 1236 digits).

Moreover, in the present paper, we have introduced a wider speculation that allow us to ask to ourselves how to find a term of the OEIS sequence A353025 (disregarding A353025(1)) which is not a perfect square; a challenging open problem, considering that there is not any perfect cube among the terms of A352991 in the interval $(1, 10^{40}]$.

5 Appendix

Aldo Roberto Pessolano helped the author of the present paper by verifying Kashihara's conjecture and Conjecture 1 for a very large number of terms. All the provided Mathematica codes run on the M1 processor of his Apple MacBook Air (2020).

Kashihara's conjecture has been currently tested up to the 100128-th term of A001292 and we confirm that it holds for every element of the set $\{\text{A001292}(2), \text{A001292}(3), \ldots, \text{A001292}(100128)\}$. The search reached the term $99\_100\_\ldots\_446\_447\_1\_2\_\ldots\_97\_98 \approx 9.91 \cdot 10^{1232}$ in 28823 seconds (about 8 hours of calculations) and the code is as follows:

```
c = True;
p = Table[Prime[q], {q, 1, 565}];
Do[rn = Range[k];
	n = ToExpression[StringJoin[ToString[#]&/@rn]];
	If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
		Do[r = RotateLeft[rn, i - 1];
			nk = ToExpression[StringJoin[ToString[#]&/@r]];
			If[IntegerQ[nk^(1/#)],
				Print[nk, " = ", nk^(1/#), "^", #]; c = False; Break[]
			]&/@p,
		{i, 1, k}]
	];
	If[c, Print["1..", k, " checked."], Break[]],
```

```
{k, 2, 447}]
```

About our investigation on the perfect powers in A352991, Pessolano has recently completed the direct check of every term of A352991 which falls in the interval (1, 9876543221201918171615141312111110] (see the code below). As expected, the test has not returned any perfect power above two.

```
z = False;
h = 3;
p = Table[Prime[q], {q, 2, 10}];
q[x_, k_, d_, m_] :=(y = x^k;
        If[DigitCount[y] == d,c = True;
                Do[If[Not[StringContainsQ[ToString[x], ToString[i]]],c = False;
                        Break[],c = True],{i, 10, m}],c = False];
                Return[c])
        Do[r = Range[k];
                n = ToExpression[StringJoin[ToString[#]&/@r]];
                If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],d = DigitCount[n];
                        (s = IntegerPart[(10^(IntegerLength[n] - 1))^(1/#)];
                                f = IntegerPart[(10^(IntegerLength[n]))^(1/#)];
                                Do[If[q[x, #, d, k], Print[x, "^", #, " = ", y];
                                        z = True; Break[]],{x, s, f}])&/@p;g = 2^h;
        While[g < n,If[q[#, h, d, k],
                Print[x, "^", h, " = ", y];
                z = True;
                Break[]]&/@{2, 3, 5, 6, 7};
        h++;g = 2^h]];
If[z, Break[],
        Print["1..", k, " checked."]],
{k, 2, 21}]
```

On the other hand, the following code returns the complete list of the smallest 42 perfect squares belonging to A352991.

```
z = 1;
Do[r = Range[k];
        n = ToExpression[StringJoin[ToString[#]&/@r]];
        If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
                d = DigitCount[n];
                s = IntegerPart[Sqrt[10^(IntegerLength[n] - 1)]];
                f = IntegerPart[Sqrt[10^(IntegerLength[n])]];
                Do[y = x^2;
                        If[DigitCount[y] == d,
                                c = True;
                                Do[
                                        If[Not[StringContainsQ[ToString[y], ToString[i]]],
                                                c = False
                                        ],
                                {i, 10, k}];
                                If[c, Print[z, " ", y]; z++]
                        ],
```

```
            {x, s, f}]
      ],
{k, 2, 10}]
```

These 42 perfect squares correspond to all the perfect powers in $(1,\ 10^{16}]$ belonging to A352991, while the next perfect square is 10135681742311129 (we observe that 100676123^2 is a permutation of $123\_\ldots\_16$, as suggested by the statement of Theorem 1).

1	13527684
2	34857216
3	65318724
4	73256481
5	81432576
6	139854276
7	152843769
8	157326849
9	215384976
10	245893761
11	254817369
12	326597184
13	361874529
14	375468129
15	382945761
16	385297641
17	412739856
18	523814769
19	529874361
20	537219684
21	549386721
22	587432169
23	589324176
24	597362481
25	615387249
26	627953481
27	653927184
28	672935481
29	697435281
30	714653289
31	735982641
32	743816529
33	842973156
34	847159236
35	923187456
36	14102987536
37	24891057361
38	27911048356
39	28710591364
40	57926381041
41	59710832164
42	75910168324

In the end, our tests have finally confirmed that all the perfect powers which are smaller than 10^{34} and that belong to the OEIS sequence A352991 are perfect squares (only).

At the present time, Conjecture 1 has been tested for every integer smaller than $10111121314151617181920212223456789$, and no counterexample has been found yet.

Acknowledgments

We sincerely thank Aldo Roberto Pessolano for having helped us very much with the search of perfect powers belonging to A001292 and A352991, letting us confirm Kashihara's conjecture up to 10^{1235} and Conjecture 1 up to $1.01 \cdot 10^{34}$.

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