**FINSIM ROCKET EQUATION BURNOUT VELOCITY ACCURACY COMPARED TO FINITE DIFFERENCE AND TR-10 PREDICTION**

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## ABSTRACT

The **rocket equation** in a form that accounts for the force of gravity or rocket weight was included in the latest version of FinSim to determine drag-free burnout velocity and drag-free burnout altitude for single stage rockets launched from the surface of the Earth. This new feature was implemented on the *Fin Geometry for Aeroelastic Analysis* screen, where burnout velocity and burnout altitude are plotted with the red Vb designation on the flutter velocity verses altitude plot when using the *NACA 4197 Flutter Velocity Tool*. This new feature in FinSim makes it possible to immediately compare predicted flutter velocity to rocket equation burnout velocity and burnout altitude for any single stage rocket using FinSim. However, the question by several FinSim users was *"how accurate is the rocket equation for predicting burnout velocity and burnout altitude"*. This simple question was the genesis for this paper which attempts to quantify the accuracy of the rocket equation compared to the finite difference method and TR-10 model rocket altitude prediction method for computing burnout velocity and burnout altitude. One not so surprising result is the accuracy of the rocket equation verses altitude increases as the dimensions, mass and rocket motor performance including thrust and burn time is increased. This analysis helps to quantify the accuracy of the rocket equation for burnout velocity and burnout altitude verses mass fraction using FinSim's new *NACA 4197 Flutter Velocity Tool*.


## Nomenclature

| | | |
|---|---|---|
| $A$ | = | Reference area of the rocket, typically just behind the nose cone |
| $Cd$ | = | Average drag coefficient from liftoff to burnout |
| $\rho$ | = | Average atmospheric air density from liftoff to burnout |
| $\delta V$ | = | Change in velocity from liftoff to burnout |
| $\delta Z$ | = | Change in altitude from liftoff to burnout |
| $g$ | = | Acceleration of gravity from liftoff to burnout |
| $Isp$ | = | Rocket motor specific impulse |
| $W_0$ | = | Rocket weight at liftoff including propellant |
| $W_f$ | = | Final rocket weight at burnout at time, $T_b$ |
| $W_p$ | = | Propellant weight at liftoff |
| $T_b$ | = | Propellant burn time to rocket motor burnout |
| $\Delta T_b$ | = | Time increment for the finite difference method |
| $\beta_0$ | = | Average rocket ballistic coefficient from liftoff to burnout |
| $a_0$ | = | Average rocket acceleration from liftoff to burnout |
| $\zeta$ | = | Propellant mass fraction, $W_p/W_0$ |
| $\dot{m}$ | = | Mass flow rate of propellant, kg/sec |

## FINITE DIFFERENCE METHOD - 1

The basic equation of rocket motion is required for deriving a finite difference solution, which is obtained from Newton's First Law of Motion[1], $\sum \boldsymbol{F} = \boldsymbol{ma}$. Where, $\sum \boldsymbol{F}$ is the summation of all external forces applied to the rocket, **m** is the mass of the rocket and **a** is the acceleration of the rocket. Acceleration is also expressed as $dV/dt$ or the rate of change of velocity with respect to time. The forces acting on a rocket during the thrusting phase of flight are its weight, W, thrust, T, and aerodynamic drag, $D = Cd\,A\,{}^{1}\!/_{2}\,\rho\,V^2$. Where Cd is drag coefficient, $\rho$ is air density, V is velocity and A is the reference area of the rocket, typically the section behind the nose cone. To start, the burn time array can be defined knowing the burn time increment.

$$tb_{n+1} = tb_n + \Delta T_b \tag{1}$$

For vertical flight, Newton's equation of motion for the thrusting phase of flight becomes.

$$m\,\frac{dV}{dt} = T - Cd\,A\,\frac{1}{2}\,\rho\,V^2 - W \tag{2}$$

The acceleration term, $dV/dt$ determines the velocity increment for each time step, $\Delta T_b$ during the flight integration process where $dV = (dV/dt)\ dt$ is the incremental velocity. The finite difference equation for velocity increment for the results presented in this paper becomes the following and is the basic form used by the Mathcad spreadsheet analysis.

$$dV(V, W, F, Cd, \theta, \rho) = \frac{F}{\frac{W}{g}} - \frac{Cd\,A\,\frac{1}{2}\,\rho\,V^2}{\frac{W}{g}} - g\sin\theta \quad and \quad \theta = 90\ deg \tag{3}$$

Time dependent rocket weight knowing the initial weight, number of burn time increments and propellant weight for n = 1 to $Nburn_{max} - 1$ become.

$$W_n = W_0 - \frac{n}{Nburn_{max} - 1}\,W_p \tag{4}$$

Velocity and altitude at each n + 1 time step are determined from the following equation knowing velocity and altitude at each time step, n. Typically, the initial thrust phase boundary conditions are $V_1$ and $Z_1$ at n = 1. The equations of motion are integrated by performing the analysis using time step, $\Delta T_b$. These equations can be integrated using a variety of techniques including the Euler method or ordinary time stepping. The finite difference equation for velocity as a function of time during the thrusting phase becomes.

$$V_{n+1} = V_n + dV(V_n, W_n, F, Cd, \theta, \rho)\,\Delta T_b \tag{5}$$

Finally, the finite difference equation for altitude as a function of time becomes.

$$Z_{n+1} = Z_n + V_n\,\Delta T_b \tag{6}$$

The burnout velocity and burnout altitude for a rocket launched vertically in the atmosphere is determined by using the variable index specified at the $Nburn_{max}$ time step.

## ROCKET EQUATION METHOD INCLUDING FORCE OF GRAVITY - 2

The ideal rocket equation is a simplified derivation for rocket burnout velocity and altitude that does not include the force of gravity or aerodynamic drag. However, the form of the **rocket equation**[2, 6] presented in Equation-7 has been slightly modified to include the force of gravity from launch to burnout. The following relationships for the change of velocity and altitude for an ideal rocket not including aerodynamic drag are described below.

The change in rocket velocity during the thrusting phase of flight becomes.

$$\delta V = g\, I_{sp}\, ln\left(\frac{W_0}{W_f}\right) - g\, T_b \tag{7}$$

The change in rocket altitude during the thrusting phase of flight becomes.

$$\delta Z = \frac{\delta V}{2}\, T_b \tag{8}$$

## TR-10 VELOCITY AND ALTITUDE METHOD - 3

The equations presented in this section are from the well-known report, TR-10 a model rocket altitude and velocity prediction analysis[3] based on the integral form of the equation of motion in the vertical direction. The equations for burnout velocity and burnout altitude include the effects of drag and average weight which are equal and opposite to the thrust force or, $\sum F = ma$. A complete derivation and description of these equations may be found in the report, *Model Rocket Altitude Prediction Charts including Aerodynamic drag*. TR-10's equations for burnout velocity and burnout altitude are derived on page 38 and repeated below as Equation-12 and Equation-13 respectively. Equation-9 to Equation-13 is necessary to properly derive burnout velocity and burnout altitude for comparison to the previous two methods described as the **Rocket Equation** and **Finite Difference** methods.

The average rocket weight from liftoff to burnout becomes.

$$W_{avg} = \frac{W_0 + W_f}{2} \tag{9}$$

The average rocket ballistic coefficient becomes.

$$\beta_0 = \frac{W_{avg}}{Cd\, A\, \frac{1}{2}\rho} \tag{10}$$

The average rocket acceleration from liftoff to burnout becomes.

$$a_0 = \frac{F}{W_{avg}} - 1 \tag{11}$$

The rocket velocity at burnout or change in velocity becomes.

$$\delta V = \sqrt{\beta_0\, a_0}\, \tanh\left(g\, \sqrt{\frac{a_0}{\beta_0}}\, T_b\right) \tag{12}$$

The change in rocket altitude during the thrusting phase of flight becomes.

$$\delta Z = \frac{\beta_0}{g} \ln\left(\cosh\left(g \sqrt{\frac{a_0}{\beta_0}}\, T_b\right)\right) \tag{13}$$

**ROCKET EQUATION ACCURACY**

To access the accuracy of the **rocket equation** as a stand-alone design tool within FinSim it is necessary to compare this method which includes the effects of gravity but not aerodynamic drag to other methods that not only include gravity but also include aerodynamic drag[4, 5]. The two supplemental methods to perform this comparison with the rocket equation are the **finite difference** solution to the equation of motion described by Equation-2 and the **TR-10** solution based on an integral solution procedure also described by Equation-2. To simplify the analysis some assumptions were made. First, the aerodynamic drag coefficient, Cd was assumed to be a constant value over a range of Mach number based on the average Cd expected from liftoff to burnout. Then, to compute drag, the average atmospheric air density is determined based on the assumption for an isothermal atmosphere where air temperature is constant from liftoff to burnout and drag force is determined using the equation for aerodynamic drag, $D = Cd\, A\, {}^1\!/_2\, \rho\, V^2$. To provide insight into how rocket length, diameter and mass effect burnout velocity and altitude accuracy it was decided that plotting burnout velocity and altitude verses propellant mass fraction for two different size rockets could determine over what range of mass fraction the rocket equation is most accurate. Where, mass fraction is defined as the ratio of total propellant weight to initial rocket weight.

Two different size rocket designs were utilized to determine accuracy of the rocket equation. The first model is based on a NACA report design that was 55 inches long, 5 inches diameter, weighed 50 pounds and powered by a rocket motor having an average thrust of 500 pounds. This first model corresponds roughly to a large high-power rocket. The second design was upscaled to 275 inches long, 25 inches diameter, weighed 6,250 pounds and powered by a rocket motor having a thrust of 62,500 pounds. The second model corresponds roughly to a professional sounding rocket intended to probe the upper atmosphere. The average Cd for each model was determined using HyperCFD[8] to generate the Cd verses Mach number expected during the flight as illustrated in Figure-1 for the small rocket and the large rocket. The average drag coefficient over the range of velocity expected from liftoff to burnout for the small model was determined to be Cd = 0.295 and a burnout velocity of Mach 4. Then, the average drag coefficient over the range of velocity expected from liftoff to burnout for the large rocket was determined to be Cd = 0.229 and a burnout velocity of Mach 6. Finally, the average air density from liftoff to burnout was computed assuming an isothermal atmosphere from launch altitude to the maximum altitude predicted by the finite difference equation. For the specified design parameters, a Mathcad spreadsheet analysis compared the relative difference between the **rocket**

**equation** and **TR-10** methods relative to results generated by the **finite difference** method for predicting burnout velocity and burnout altitude verses propellant mass fraction for a high-power class rocket (small rocket) and a professional sounding rocket (large rocket). The relative accuracy for these analyses is plotted in Figure-2, Figure-3, Figure-4, and Figure-5 where burnout velocity and burnout altitude accuracy have been normalized by the finite difference method as a function of rocket propellant mass fraction, Wp/Wo.

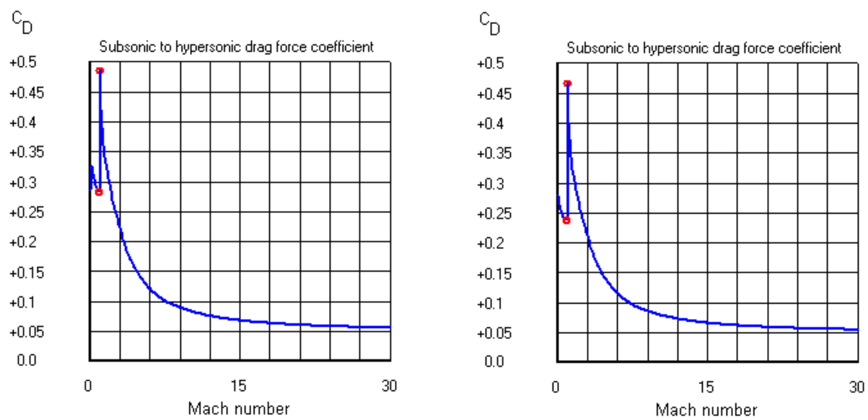


Figure-1, Model-1 and Model-2 Cd verses Mn determined using **HyperCFD** then averaged over the expected Mach range

SMALL ROCKET BURNOUT VELOCITY AND ALTITUDE COMPARISON

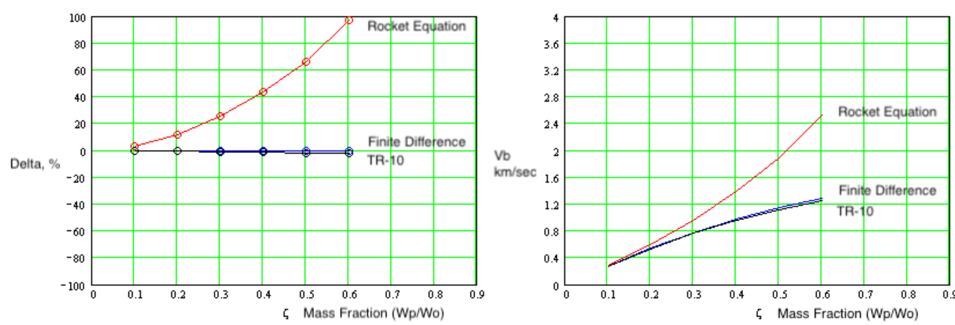


Figure-2, Small rocket (Model-1) **burnout velocity (km/sec)** accuracy in percent (%) for the rocket equation and TR-10 methods normalized by the finite difference method verses mass fraction, $\zeta$

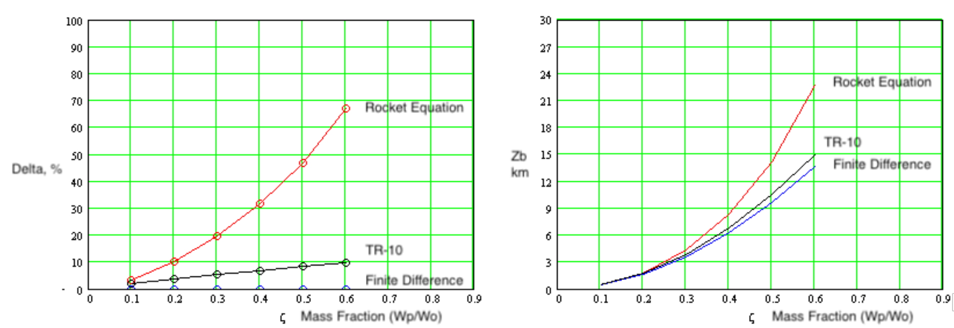


Figure-3, Small rocket (Model-1) **burnout altitude (km)** accuracy in percent (%) for the rocket equation and TR-10 methods normalized by the finite difference method verses mass fraction, $\zeta$

LARGE ROCKET BURNOUT VELOCITY AND ALTITUDE COMPARISON

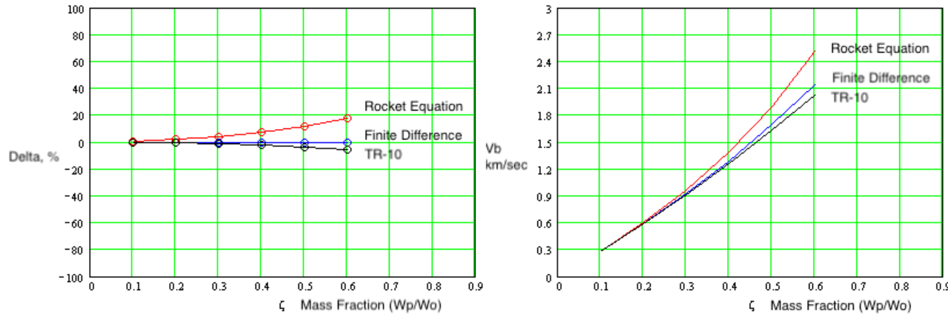


Figure-4, Large rocket (Model-2) **burnout velocity (km/sec)** accuracy in percent (%) for the rocket equation and TR-10 methods normalized by the finite difference method verses mass fraction, $\zeta$

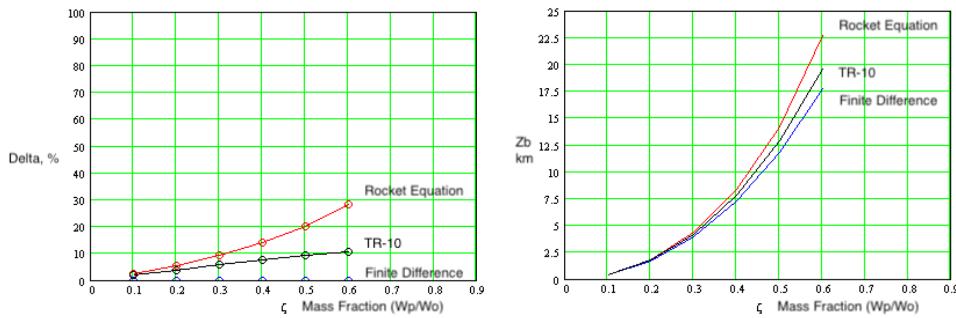


Figure-5, Large rocket (Model-2) **burnout altitude (km)** accuracy in percent (%) for the rocket equation and TR-10 methods normalized by the finite difference method verses mass fraction, $\zeta$

Normalizing velocity and altitude generated by the rocket equation and TR-10 by the velocity and altitude predicted by the finite difference method allows the user to quickly determine each method's relative accuracy over a range of propellant mass fraction. Results plotted in Figure-3 to Figure-5 illustrate that as propellant mass fraction, $\zeta$ is decreased the rocket equation provides results increasingly closer to the finite difference method because drag effects are less significant for slower rockets that achieve lower velocity and altitude. Conversely, as performance is increased, aerodynamic drag becomes more significant for rockets that have the capacity for achieving greater burnout velocity and burnout altitude. Finally, for similar mass fraction, $\zeta$ the rocket equation is more accurate for computing burnout velocity and burnout altitude for large rockets than small rockets confirmed by the ballistic coefficient, $\beta_0$ or ratio of mass to frontal area. Equation-14 and Equation-15 determine the relative difference between the rocket equation and the TR-10 methods for computing burnout velocity and burnout altitude compared to the finite difference method.

$$\Delta \mathrm{V}_{rocket\ equation} = \frac{V_{rocket\ equation} - V_{finite\ difference}}{V_{finite\ difference}} * 100 \tag{14}$$

$$\Delta \mathrm{V}_{TR-10} = \frac{V_{TR-10} - V_{finite\ difference}}{V_{finite\ difference}} * 100 \tag{15}$$

Computed raw data for the difference in percent between rocket equation burnout velocity and TR-10 burnout velocity is tabulated in Table-1 and Table-2.

| $\frac{W_{propellant}}{W_0}$ | $\Delta V_{rocket\ equation}$ % | $\Delta V_{TR-10}$ % | $\Delta Z_{rocket\ equation}$ % | $\Delta Z_{TR-10}$ % | Mn FDE | $\beta_0$ $kg/m^2$ |
|---|---|---|---|---|---|---|
| 0.6 | 97.414 | -2.194 | 66.971 | 9.544 | 3.75 | 4,242 |
| 0.5 | 65.819 | -1.928 | 46.571 | 8.242 | 3.35 | 4,544 |
| 0.4 | 43.375 | -1.449 | 31.628 | 6.810 | 2.84 | 4,848 |
| 0.3 | 25.457 | -0.914 | 19.673 | 5.313 | 2.25 | 5,150 |
| 0.2 | 11.776 | -0.374 | 10.135 | 3.686 | 1.57 | 5,453 |
| 0.1 | 2.930 | 0.0 | 3.440 | 1.966 | 0.80 | 5,756 |

Table-1, Rocket equation and TR-10 burnout velocity and altitude verses mass fraction for a **small rocket** relative to results generated by the finite difference method. **Yellow cells indicate engineering accuracy**.

| $\frac{W_{propellant}}{W_0}$ | $\Delta V_{rocket\ equation}$ % | $\Delta V_{TR-10}$ % | $\Delta Z_{rocket\ equation}$ % | $\Delta Z_{TR-10}$ % | Mn FDE | $\beta_0$ $kg/m^2$ |
|---|---|---|---|---|---|---|
| 0.6 | 17.765 | -5.283 | 28.103 | 10.656 | 6.29 | 27,372 |
| 0.5 | 11.425 | -3.475 | 19.937 | 9.221 | 4.99 | 29,327 |
| 0.4 | 7.198 | -2.090 | 18.834 | 7.478 | 3.80 | 31,283 |
| 0.3 | 4.117 | -1.083 | 9.052 | 5.648 | 2.71 | 33,328 |
| 0.2 | 1.874 | -5.110 | 5.223 | 3.815 | 1.73 | 35,193 |
| 0.1 | 0.717 | 0.0 | 2.433 | 1.946 | 0.82 | 37,148 |

Table-2, Rocket equation and TR-10 burnout velocity and altitude verses mass fraction for a **large rocket** relative to results generated by the finite difference method. **Yellow cells indicate engineering accuracy**.

## CONCLUSION AND SUMMARY

These results illustrate that as propellant mass fraction, $\zeta$ is decreased the **rocket equation** provides results increasingly closer to the finite difference method because drag effects are less significant for smaller rockets that achieve lower velocity and altitude. Conversely, as rocket motor performance is increased, aerodynamic drag becomes more significant for all rockets that have the capacity for achieving greater burnout velocity and burnout altitude. Finally, for similar mass fraction, $\zeta$ the **rocket equation** is more accurate for computing burnout velocity and burnout altitude for large rockets compared to small rockets based on the ballistic coefficient, $\beta_0$ or ratio of rocket mass to frontal area. These observations are illustrated in Table-1 and Table-2 where the yellow cells signify results that are within engineering accuracy. Therefore, for rockets having a mass fraction, $\zeta$ less than 0.15 like model rockets and even some high power rockets the **rocket equation** provides burnout velocity and burnout altitude within engineering accuracy i.e., less than 10 percent. Finally, as the large rocket results illustrate the **rocket equation** provides results within engineering accuracy for mass fraction less than 0.40 where rocket propellant accounts for less than 40% of the entire rocket mass. Final note about the rocket equation's use in the new version of FinSim. This investigation indicates the **rocket equation** provides results for burnout velocity and burnout altitude within engineering accuracy for model rockets having mass fraction, $\zeta$ less than 15% and sounding rockets with mass fraction less than 40%. However, the following question may arise. *"If the finite difference and the TR-10 methods are more accurate than the rocket equation why doesn't FinSim use either of these two methods* *instead".* The answer to this question is the rocket equation corrected for gravity provides reasonable results without the FinSim user required to supply drag coefficient and air density data. During this investigation, Cd was predicted over the intended Mach number range using HyperCFD a standalone supersonic and hypersonic CFD computer program. On the other hand, FinSim is a flutter analysis computer program intended for rapid flutter velocity and aerodynamic loading predictions. Therefore, this analysis confirmed the rational for not burdening FinSim with unnecessary input data requirements.

## DERIVATION OF THE ROCKET EQUATION

This derivation of the modified form of the rocket equation includes the force of gravity. The rocket equation in the following form was derived using the **principal of impulse and momentum**[6] between time $t + \Delta t$ for systems that lose mass as a function of time. Based on this theory the change in rocket velocity during the thrusting phase of flight becomes.

$$\delta V = g\, I_{sp}\, ln\left(\frac{W_0}{W_f}\right) - g\, T_b \qquad (16)$$

The following partial derivation of the rocket equation is based on the principal of impulse and momentum and is the form of the equation used in this analysis.

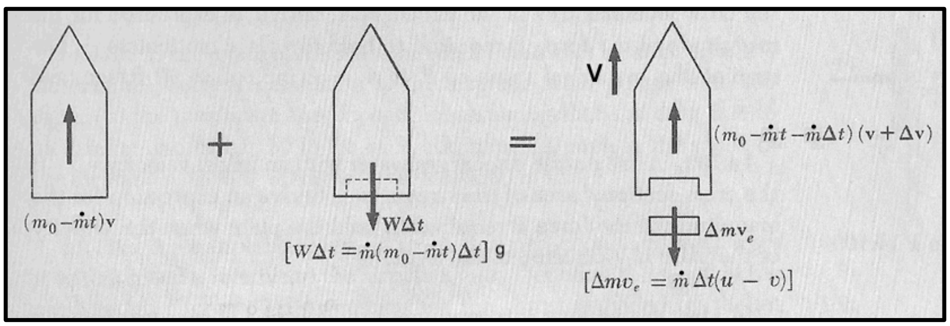


Figure-6, Impulse and momentum terms required for Equation-17

Equation describing the principal of impulse and momentum from reference 6.

$$\sum W\Delta t = m\,\Delta v - (\Delta m)\,u \qquad (17)$$

Impulse and momentum terms for Equation-17 from the rocket described in Figure-6

$$(m_0 - \dot{m}\,t)\,v - g(m_0 - \dot{m}\,t)\,\Delta t = (m_0 - \dot{m}\,t - \dot{m}\,\Delta t)\,(v + \Delta v) - \dot{m}\,\Delta t(u - v) \qquad (18)$$

$$-g(m_0 - \dot{m}\,t)\,\Delta t = (m_0 - \dot{m}\,t - \dot{m}\,\Delta t)\,(v + \Delta v) - (m_0 - \dot{m}\,t)\,v - \dot{m}\,\Delta t(u - v) \qquad (19)$$

Dividing both sides by $\Delta t$ and letting $\Delta t$ approach zero, we obtain the following equation.

$$-g(m_0 - \dot{m}\,t) = (m_0 - \dot{m}\,t)\,\frac{dv}{dt} - \dot{m}\,u \qquad (20)$$

Separating variables and generating the integral from t = 0, v = 0 to t = t and v = v.

$$\frac{dv}{dt} = \frac{\dot{m}u}{m_0 - \dot{m}\,t} - g \qquad (21)$$

$$v = \dot{m}u \int_0^t \frac{1}{m_0 - \dot{m}\,t}\,dt - gt \qquad (22)$$

After integrating from 0 to t the rocket equation becomes.

$$v = u \ln\left(\frac{m_0}{m_0 - \dot{m}\, t}\right) - gt \tag{23}$$

Average speed of propellant expelled at the base of the rocket from reference 1.

$$u = g\ Isp \tag{24}$$

The rocket equation in more complex form that includes the force of gravity becomes.

$$v = g\ Isp \ln\left(\frac{m_0}{m_0 - \dot{m}\, t}\right) - gt \tag{25}$$

The final form for the rocket equation that includes the force of gravity becomes.

$$\delta V = g\ Isp \ln\left(\frac{W_0}{W_f}\right) - g\, T_b \tag{26}$$

**SINGLE STAGE ROCKET PEAK TRAJECTORY NEGLECTING DRAG**

The following section although not technically part of the discussion to determine burnout velocity and burnout altitude is provided to complete the discussion for a means to estimate the complete flight profile of a rocket using the drag-free rocket equation from liftoff to peak altitude. Where, a rocket's peak altitude, Zmax, presented as Equation-28 is the maximum altitude[7] a rocket reaches when launched in a vertical trajectory from the Earth's surface in the presence of gravity. The results presented in Figure-7 plot Zmax for the small and large rocket verses propellant mass fraction. The plots for small rocket and large rocket Zmax verses propellant mass fraction is identical and are plotted as a function of propellant mass fraction. Because the equation for maximum altitude[7] is a function of mass ratio, MR the following equation defines MR in terms of propellant mass fraction, $\zeta$ using the following equation.

$$MR = 1 - \zeta \tag{27}$$

Finally, the following equation determines rocket peak altitude as a function of mass ratio.

$$Z_{max} = \frac{1}{2} Isp^2 g\ \ln\left(\frac{1}{MR}\right) \left(\ln\left(\frac{1}{MR}\right) - \frac{W_0}{F} + \frac{W_0 MR}{F}\right) \tag{28}$$

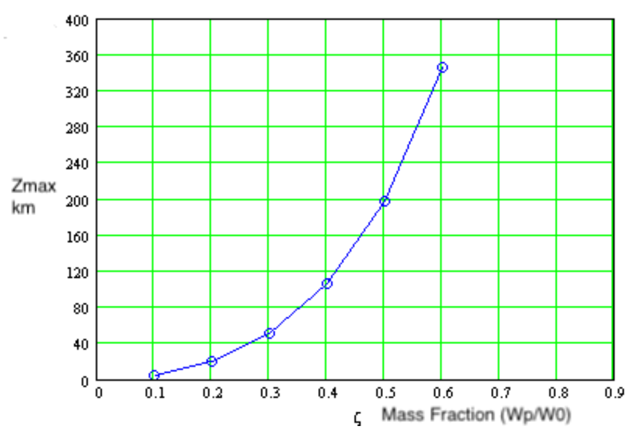

Figure-7, Peak altitude verses propellant mass fraction

## VALIDATING THE PEAK TRAJECTORY EQUATION

Drag free peak altitude reached by a rocket under the influence of gravity described in Equation-28 will not be derived here but can be validated using the equations for uniformly accelerated vertical motion[6] described by Equation-29 and Equation-30 below.

$$V = V_0 - g\,t \tag{29}$$

$$y = V_0 t - \frac{1}{2} g\,t^2 \tag{30}$$

Then, applying the boundary condition for Equation-29 that $V = 0 \;@\; t = t_{coast}$ .

$$t_{coast} = \frac{\delta V}{g} \tag{31}$$

Finally, applying the boundary condition for Equation-30 that $y = y_{coast} \;@\; t = t_{coast}$ .

$$y_{coast} = \frac{1}{2}\frac{\delta V^2}{g} \tag{32}$$

$$Z_{max} = \delta Z + y_{coast} \tag{33}$$

As expected, results using Equation-33 agrees exactly with the drag free rocket equation for peak altitude presented in Equation-28 where $\delta V$ is the burnout velocity and $\delta Z$ is the burnout altitude previously described in Equation-7 and Equation-8. Finally, the error analysis conducted previously for burnout velocity and burnout altitude does not apply to the peak altitude predicted by the drag free rocket equation presented in Equation-28 and Equation-33 because drag induced effects after burnout as the rocket coasts to its peak trajectory, $y_{coast}$ is considerable.

| $Z_{max}$ $km$ | $\zeta$ $\frac{W_p}{W_0}$ | **MR** $\frac{Final\ weight}{Takeoff\ weight} = \frac{W_0 - W_p}{W_0}$ |
|---|---|---|
| 346.248 | 0.6 | 0.4 |
| 196.729 | 0.5 | 0.5 |
| 106.137 | 0.4 | 0.6 |
| 51.419 | 0.3 | 0.7 |
| 20.004 | 0.2 | 0.8 |
| 4.434 | 0.1 | 0.9 |

Table-3, Peak altitude verses propellant mass fraction, $\zeta$ and mass ratio, MR

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