Hamiltonian Instability and the Classical-to-Quantum Transition

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Abstract

The mechanism of *Arnold diffusion* (AD) describes the dynamic instability of nearly-integrable Hamiltonian systems. Here we argue that AD leads to action quantization for classical systems having an infinite number of degrees of freedom. Planck's constant emerges as long-time value of the action differential applied to large ensembles of oscillators in near equilibrium conditions.

Key words: Arnold diffusion, Hamiltonian chaos, Planck constant, action quantization, classical to quantum transition.

The formalism of *action-angle variables* applies to generic Hamiltonian systems undergoing periodic motion, such as harmonic oscillation or Kepler rotation. It consists of replacing the generalized coordinates and momenta using the transformation [1-3]

$$(q,p) \rightarrow (\theta, I) \tag{1}$$

Action-angle variables are canonically conjugate and introduced through the generating function [3]

$$S(q,I) = \int_q p(q,H)\,dq \tag{2}$$

such that

$$I = \frac{1}{2\pi}\int_C p(q,H)\,dq = I(H) \tag{3a}$$

$$\theta = \frac{\partial S(q,I)}{\partial I} \tag{3b}$$

Since I is a cyclic variable, the corresponding drift of (2) per each period of I amounts to [1]

$$\Delta S = 2\pi I \tag{4}$$

Consider a nearly-integrable periodic system with n degrees of freedom defined by the Hamiltonian [4-6]

$$H = H_0(I) + \varepsilon H_1(I, \theta, \varepsilon) \tag{5}$$

where $0 \le \varepsilon \le \varepsilon_0$ is a small perturbation parameter and $H_0(I)$ is the unperturbed Hamiltonian, taken to be fully integrable in the limit $\varepsilon = 0$. The frequency of the unperturbed motion is determined by

$$\omega(I) = \frac{\partial H_0}{\partial I} \tag{6}$$

For $\varepsilon \le \varepsilon_0 << 1$, the equations of motion read

$$\dot{I} = -\frac{\partial H(\theta, I)}{\partial \theta} = 0 \tag{7}$$

$$\dot{\theta} = \frac{\partial H(\theta, I)}{\partial I} = \omega(I) \tag{8}$$

in which $I, \theta \in \mathbf{R}^n$. The solution of (7)-(8) lies on invariant n- tori residing in the phase-space of dimension $\mathbf{R}^{2n}$. For $n \le 2$, all solutions are stable since 2 -tori confine trajectories on a 3-dimensional energy surface. This is no longer the case for $n \ge 3$ where, according to the *Arnold diffusion conjecture*, the action of near-integrable systems changes by $O(1)$ over a sufficiently long time. In particular, assuming that

$$\left|H_0\right| < c_1,\ \left|H_1\right| < c_1 \tag{9}$$

where c_1 is a positive constant, and taking the unperturbed Hamiltonian to represent a quasi-convex function of the action variable, the following condition holds [6]

$$\delta I = \left|I(t) - I(0)\right| < C_1 \varepsilon^{1/2n} \tag{10}$$

over sufficiently long-times satisfying

$$0 \le t \le \exp\left(C_2^{-1} \middle/ \varepsilon^{1/2n}\right) \tag{11}$$

In (10) and (11), C_1, C_2 are also positive constants. By (4), the corresponding drift in action is given by

$$\delta(\Delta S) = 2\pi \delta I \le O(C_1 \varepsilon^{1/2n}) \tag{12}$$

Normalizing (12) to C_1 confirms that the drift in action is of $O(1)$, which naturally replicates the process of *action quantization* for $n >> 1$. It follows that Planck's constant may be mapped to the long-time value of (12) applied to large ensembles of oscillators in near equilibrium conditions. Stated differently, the transition from classical to quantum behavior is expected to occur when

$$0 \leq t \leq \exp\left(C_2^{-1} / \varepsilon^{1/2n} \right);\ n >> 1 \tag{13}$$

leading to

$$\boxed{\delta(\Delta S) = 2\pi \delta I = O(1)} \tag{14}$$

These findings fall in line with the approach taken in [7].

<u>References</u>

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