The hypergeometric $F\left(\frac{1}{2},\ \frac{3}{4},\ \frac{7}{4},\ \frac{3}{4}\right)$

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Abstract

We give some formulas related to $F\left(\frac{1}{2},\ \frac{3}{4},\ \frac{7}{4},\ \frac{3}{4}\right)$

Introduction

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right) \tag{1}$$

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \ ,\ R(z) > 0 \tag{2}$$

where $\Gamma(z)$ is the Gamma function

$$\Gamma(z+1) = z\,\Gamma(z) \tag{3}$$

The Gauss Hypergeometric function is defined by

$$F(a,\ b,\ c,\ x) = \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(c)_n\, n!}\, x^n \ ,\ |x| < 1 \tag{4}$$

where $(a)_n = a\,(a+1)\,(a+2)\dots(a+n-1)$; $(a)_0 = 1$; $(a)_n = \Gamma(a+n)\,/\,\Gamma(a)$, $a \neq 0, -1, -2, \dots$, See Olver et al., [3].

In this note we give some formulas related to $F\left(\frac{1}{2},\ \frac{3}{4},\ \frac{7}{4},\ \frac{3}{4}\right)$.

Some formulas related to $F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right)$

$$\sqrt[4]{27}\,\ln\left(\frac{2}{\sqrt{3}}\right) + \frac{1}{\sqrt[4]{3}}\,F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \int_0^{\sqrt[4]{27}} \ln\left(\sqrt[3]{\frac{1}{x^2} + \sqrt{\frac{1}{x^4} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{x^2} - \sqrt{\frac{1}{x^4} - \frac{1}{27}}}\right) dx \tag{5}$$

$$\frac{2}{\sqrt{\pi}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2 - \sqrt[4]{27}\,\ln\left(\frac{2}{\sqrt{3}}\right) - \frac{1}{\sqrt[4]{3}}\,F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \int_{\sqrt[4]{27}}^{\infty} \ln\left(\frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \cos^{-1}\left(\frac{3\sqrt{3}}{x^2}\right)\right)\right) dx \tag{6}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 3 \sum_{n=0}^{\infty} \frac{2^{-4n} \cdot 3^n}{(4n+3)} \binom{2n}{n} \tag{7}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \frac{3}{4} \int_0^1 x^{-1/4} \left(1 - \frac{3}{4} x\right)^{-1/2} dx \tag{8}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \frac{\pi}{\sqrt{3}} - \frac{3}{2} \sum_{n=0}^{\infty} \frac{2^{-4n} \cdot 3^n}{(2n+1)(4n+3)} \binom{2n}{n} \tag{9}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \sqrt[4]{\frac{3}{4}}\,\sqrt{\pi}\,\frac{\Gamma(3/4)}{\Gamma(5/4)} - \sqrt[4]{\frac{3}{4}}\,F\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}\right) \tag{10}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 6 - 4\,F\left(\frac{1}{4}, 1, \frac{7}{4}, \frac{3}{4}\right) \tag{11}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 2\,{}_4F_3\left(\begin{matrix} 1/4, 3/4, 3/8, 7/8 \\ 7/8, 11/8, 1/2 \end{matrix} \middle| \frac{9}{16}\right) - F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, -\frac{3}{4}\right) \tag{12}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \frac{9}{28}\,{}_4F_3\left(\begin{matrix} 3/4, 5/4, 7/8, 11/8 \\ 11/8, 15/8, 3/2 \end{matrix} \middle| \frac{9}{16}\right) - F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, -\frac{3}{4}\right) \tag{13}$$

${}_4F_3$ is the generalized hypergeometric function.

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 6 \int_0^1 \frac{1 - x^2}{1 + 6x^2 + \sqrt{1 + 48x^4}}\, dx \tag{14}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 3 \int_0^1 \sqrt{\frac{1 - x}{1 + 6x - 3x^2}}\, dx \tag{15}$$

For $\frac{2}{3} < a < 1$ we have

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = a^{3/4} F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3a}{4}\right) + \frac{3}{2} \sum_{n=0}^{\infty} \frac{(1/4)_n (1-a)^{n+1}}{(n+1)!} F\left(\frac{1}{2}, n+1, n+2, -3(1-a)\right) \tag{16}$$

For $0 < a < 1$ we have

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = a^{3/4} F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3a}{4}\right) + \frac{3}{2\sqrt{4-3a}} \sum_{n=0}^{\infty} \frac{(1/4)_n (1-a)^{n+1}}{(n+1)!} F\left(\frac{1}{2}, 1, n+2, \frac{3-3a}{4-3a}\right) \tag{17}$$

For $0 < a < 3/4$ we have

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \left(\frac{4}{3}\right)^{3/4} a^{3/4} F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, a\right) + \sqrt{2}\, \sqrt[4]{3}\, \sqrt{1-a}\, F\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{2}, 1-a\right) - \sqrt[4]{\frac{3}{4}}\, F\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}\right) \tag{18}$$

For $u = \frac{3^{3/4}}{\sqrt{2}}$ we have

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \frac{1}{2} + \frac{\sqrt{2}}{3^{3/4}} \int_u^{\infty} \left(1 + \cos\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}\left(-1 + \frac{27}{2x^4}\right)\right)\right) dx + \frac{5}{4\sqrt{6}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{n+1} \left(\frac{5}{8}\right)^n F\left(\frac{1}{4}, 1, n+2, \frac{5}{9}\right) \tag{19}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \frac{1}{2} + \frac{\sqrt{2}}{3^{3/4}} \int_u^{\infty} \left(1 + \cos\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}\left(-1 + \frac{27}{2x^4}\right)\right)\right) dx + \frac{5}{8} \sum_{n=0}^{\infty} \frac{(1/4)_n}{n!} \left(\frac{75}{128}\right)^n \sum_{k=0}^{n} \frac{(-1)^k 2^{n-k}}{2n+k+1} \binom{n}{k} \left(\frac{5}{6}\right)^k \tag{20}$$

Endnote

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 2\sqrt[4]{\frac{3}{4}} \int_0^{\pi/3} \sqrt{\sin x}\, dx \tag{21}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = 2\sqrt[4]{\frac{3}{4}} \int_{\pi/6}^{\pi/2} \sqrt{\cos x}\, dx \tag{22}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = \frac{\pi}{\sqrt{3}} - 2\sqrt[4]{\frac{3}{4}} \int_0^{\sqrt[4]{3/4}} \sin^{-1}(x^2)\, dx \tag{23}$$

$$F\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{3}{4}\right) = -\frac{\pi}{2\sqrt{3}} + 2\sqrt[4]{\frac{3}{4}} \int_0^{\sqrt[4]{3/4}} \cos^{-1}(x^2)\, dx \tag{24}$$

$$\int_0^{\pi/3} \sqrt{\sin x}\, dx = \sqrt{\frac{2}{\pi}} \left(\Gamma\left(\frac{3}{4}\right)\right)^2 - 2\,\text{EllipticE}\left(\frac{\pi}{12}, 2\right) \tag{25}$$

where

$$\text{EllipticE}\left(\frac{\pi}{12}, 2\right) = \int_0^{\pi/12} \sqrt{1 - 2\sin^2 x}\, dx \tag{26}$$

is the elliptic integral of the second kind.

References

[1] Boros, G., and Moll, V., Irresistible Integrals, Cambridge University Press, 2004.

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