The intuitive root of classical logic, an associated decision problem and the middle way

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Abstract

We revisit Boole's reasoning regarding the equation "$x.x = x$" that sowed the seeds of classical logic. We discuss how he considered both "$0.0 = 0$" and "$0.0 \neq 0$" in the "same process of reasoning". This can either be seen as a contradiction, or it can be seen as a situation where Boole could not decide whether "$0.0 = 0$" is universally valid – an elementary "decision problem" in the words of Hilbert and Ackermann. We conclude that Boole's reasoning, that included a choice of ignorance, was founded upon the middle way of the Buddha, later mastered by Nagarjuna. From the modern standpoint, the situation can be likened to Turing's halting problem which resulted from the use of automatic machines and the exclusion of choice machines.

As far as the development of Western logic, or simply logic, is concerned one can trace back the roots to the ideas of Leibniz who wished to calculate reason like the calculations of algebra[1]. However, it was Boole who actually made such thoughts operational by connecting his method of reasoning to the calculations of algebra[2]. As we understand, from pp 31-33 of ref.[2], the basic intuition[1] of Boole was that if "x" and "y" stand for "*white things*" and "*sheep*" respectively, then "xy" stands for "*white sheep*". In particular, he considered that, since "*GOOD, GOOD men, is equivalent to GOOD men*", it can be intuitively expressed by the algebraic equation "$x^2 = x$" i.e. a truth told twice is the same truth, same goes for a lie. Of course, here he also considered that the "good" carries an absolute sense, as he declared the following: "*The case supposed in the demonstration of the equation ($x^2 = x$) is that of absolute identity of meaning.*" Then, recognizing the roots of the equation to be 1 and 0, Boole considered representing truth and falsehood by 1 and 0 respectively. This provided the notion of an algebra of logic which, nevertheless, was an intuitive leap taken by Boole to draw the connection between verbal reasoning and algebraic calculation. Further, due to the fact that either 1 or 0 can satisfy the equation "$x^2 = x$", and not both at the same time, Boole's intuition put his work on a stronger footing from the Aristotelian perspective that a statement can either be true or false, but not both at the same time[8]. That is, if we consider a proposition P to be absolute[2], then there are the following two possibilities.

1. P is TRUE.
2. P is FALSE.

It is due to the intuitive mathematical basis, that Boole provided to verbal reasoning, classical logic continued to get accepted as the only scientific path to truth analysis. And it is due to the lack of such mathematical

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[1] This is intuitive reasoning because there was/is no proof of such analogy that was given by Boole, neither we could find any elsewhere i.e. the root of Boole's algebra of logic is founded on intuition.

[2] By "absolute" we mean "complete expression of some truth".

basis (as far as our knowledge of the scientific literature is concerned) that the middle way of the Buddha[5] (then mastered by Nagarjuna e.g. see refs.[4, 6]), an epitome of Eastern literature, has remained unaccepted by the so called "scientific" minded persons and often have been stamped as "pseudoscience"[7]. To mention, according to the middle way, there are four possibilities.

1. P is TRUE.
2. P is FALSE.
3. P is both TRUE and FALSE (contradiction).
4. P is neither TRUE nor FALSE (undecidable).

What needs to be understood, and what we shall explain further, is that the third and fourth options convey the possibility of the loss of absoluteness of the proposition P in accord with the context of its use or application. To be more elaborate, the proposition that is considered to be a complete expression of some truth, and therefore absolute, can possibly lead to a contradiction i.e. the third option. This contradiction can be resolved by taking into consideration the context of the use or application of P and therefore, P gets modified accordingly and loses its absoluteness due its relation to the context i.e. the fourth option. Essentially, the third and the fourth options are only assertions regarding the contextual truth or validity of any proposition, which however can be only assumed to be complete, and therefore absolute, so that one can work with the first two options. In this sense, the first two options are the distinct options created by the middle that is contained in the third and the fourth options.

While the above explanation of ours, regarding the third and the fourth options, may look utterly ridiculous right at the outset, however, a bit of patient and skeptic observation can lead to an understanding of the associated subtlety. To trigger the necessary skepticism to make such observations let us take the help of another feature of Eastern literature, namely self-inquiry[3], which is indeed the way to realize the essence of the third and the fourth options[4]. That is, instead of believing that "x" can be either 0 or 1, and not both at the same time, for the equation "$x^2 = x$", we inquire (ourselves) *how we reach the conclusion about the two roots of "$x^2 = x$"*. We get the following two answers.

- First answer can be obtained by quoting Boole from p 37 of ref. [2]: *"We know that $0^2 = 0$, and that $1^2 = 1$; and the equation $x^2 = x$, considered as algebraic, has no other roots other than 0 and 1."* That is, using the knowledge of "$0.0 = 0$" and "$1.1 = 1$", we go through a verification process by putting 0 and 1, one at a time, in place of "x" in the equation "$x^2 = x$". This process is *inductive* in nature.
- Secondly, we can reason in a *deductive* fashion as follows. We factorize "$x^2 = x$" as "$x.(x - 1) = 0$" and identify the roots to be 0 and 1 by setting the each of the factors to 0. This is indeed the general process to solve any quadratic equation.

Now, we focus on the second answer to make the following inquiry. *Assuming* by standard belief that the two forms "$x^2 = x$" and "$x.(x - 1) = 0$" are equivalent, after factorizing "$x^2 = x$" as "$x.(x - 1) = 0$", we generally identify the roots from the factors by setting *each equal to* 0*, but one at a time.* So we wonder, what mathematical obstacle restricts us from setting *both the factors to* 0 *at the same time.*

The reason for wondering in such a way can be elaborated as follows.

- **Question:** *Given $a.b = 0$, where are a and b are two numbers, what possible conclusions can we draw regarding the values of a and b?*
- **Answer:** From our basic knowledge of arithmetic, we can draw the following three conclusions.
    1. $a = n \neq 0 = b \quad$ using $\quad n.0 = 0 \ni n \neq 0$.

[3]While Brouwer may call it "inner-inquiry"[9], the other term in the Western literature which comes close to "self-inquiry" is "self-referencing".

[4]This ultimately leads to the realization of the underlying emptiness of any reasoning. Brouwer might have called it "subtilization of logic" with intuition – an "inner inquiry"[9].

2. $a = 0 \neq n = b$ using $0.n = 0 \ni n \neq 0$.
3. $a = 0 = b$ using $0.0 = 0$.

Therefore, from the above observation we do not find any mathematical obstacle that restricts us from setting both the factors of "$x.(x-1) = 0$" (i.e. x and $(x-1)$) to 0 at the same time by the use of "$0.0 = 0$". Hence, we can have three separate conclusions, which obviously defies the standard belief, that we demonstrate in WEBOX.

WEBOX

1. $\underbrace{\text{“}\underbrace{x}_{=0}.\underbrace{(x-1)}_{=n\neq 0}}_{\text{“}0.n=0\text{”}} = 0\text{”} \leftarrow \text{“}x = 0\text{” AND “}(x-1) \neq 0\text{”} \equiv \text{“}x = 0\text{” AND “}x \neq 1\text{”}$
2. $\underbrace{\text{“}\underbrace{x}_{=n\neq 0}.\underbrace{(x-1)}_{=0}}_{\text{“}n.0=0\text{”}} = 0\text{”} \leftarrow \text{“}x \neq 0\text{” AND “}(x-1) = 0\text{”} \equiv \text{“}x \neq 0\text{” AND “}x = 1\text{”}$
3. $\underbrace{\text{“}\underbrace{x}_{=0}.\underbrace{(x-1)}_{=0}}_{\text{“}0.0=0\text{”}} = 0\text{”} \leftarrow \text{“}x = 0\text{” AND “}(x-1) = 0\text{”} \equiv \text{“}x = 0\text{” AND “}x = 1\text{”}$

Now, let us point out that Boole simply ignored the third possibility in WEBOX, by choice[5]. We may call this as a *choice of ignorance*, which we have shown in BBOX below and we consider this as equivalent to the denial of "$0.0 = 0$", or equivalently the consideration of "$0.0 \neq 0$".

BBOX

1. $\underbrace{\text{“}\underbrace{x}_{=0}.\underbrace{(x-1)}_{=n\neq 0}}_{\text{“}0.n=0\text{”}} = 0\text{”} \leftarrow \text{“}x = 0\text{” AND “}(x-1) \neq 0\text{”} \equiv \text{“}x = 0\text{” AND “}x \neq 1\text{”}$
2. $\underbrace{\text{“}\underbrace{x}_{=n\neq 0}.\underbrace{(x-1)}_{=0}}_{\text{“}n.0=0\text{”}} = 0\text{”} \leftarrow \text{“}x \neq 0\text{” AND “}(x-1) = 0\text{”} \equiv \text{“}x \neq 0\text{” AND “}x = 1\text{”}$
3. $\cancel{\underbrace{\text{“}\underbrace{x}_{=0}.\underbrace{(x-1)}_{=0}}_{\text{“}0.0=0\text{”}} = 0\text{”} \leftarrow \text{“}x = 0\text{” AND “}(x-1) = 0\text{”} \equiv \text{“}x = 0\text{” AND “}x = 1\text{”}}$ $\underbrace{\text{(choice of ignorance)}}_{\text{“}0.0 \neq 0\text{”}}$

We may note that Boole, in order to claim 0 to be a root of "$x^2 = x$", had to consider the validity of "$0^2 = 0$", which is just a symbolic abbreviation of "$0.0 = 0$". Therefore, the claim that x can be *either* 0 *or* 1 (not both at the same time) is founded on the use of both "$0.0 = 0$" and "$0.0 \neq 0$" in one and the same process of reasoning. Interpreting verbally, the claim that a proposition P can be *either true or false* (not both at the same time) is founded on the use of both the truth and the falsehood of the proposition "$0.0 = 0$" in one and the same process of reasoning.

One way to see the situation is that Boole's reasoning is founded on a contradiction. The other way to see the situation is that Boole's reasoning is plagued with a problem to decide whether "$0.0 = 0$" or "$0.0 \neq 0$". We may identify this as a *decision problem* by borrowing the words of Hilbert and Ackermann[3]. Therefore, Boole's choice of ignorance, itself, reflects a situation which is not acceptable in classical logic i.e. the contradiction. It can also be interpreted as an inadequacy of Boole's process of reasoning where the universal validity of "$0.0 = 0$" remains undecidable. Hence, assuming the equivalence of the two forms "$x^2 = x$" and "$x.(x-1) = 0$", we can

[5] We can not decide whether Boole's choice of ignorance was made consciously or unconsciously. Possibly belief played a role. Further, the reader may object to our reasoning by noting that the issue arises because we have considered the factorized form "$x.(x-1) = 0$" instead of "$x^2 = x$". However, this objection stands on the assumption that these two forms are not equivalent to each other which, however, goes against the standard belief. Also, then the reader needs to justify on what ground one form can be preferably chosen over the other.

safely conclude that Boole's intuition that sowed the seeds of classical logic, itself, was founded on the middle way as explained below.

1. "$0.0 = 0$" is true while claiming 0 to be a root of "$x^2 = x$".
2. "$0.0 = 0$" if false while making the choice of ignorance.
3. In the whole process of reasoning "$0.0 = 0$" is both true and false.
4. It can not be decided whether "$0.0 = 0$" holds universally or not throughout the whole process of reasoning.

In view of the present discussion, we may further conclude that while classical logic is the basis of mechanical reasoning, the middle way is the foundation of human reasoning[4, 6, 5]. While the former can halt at a contradiction or a decision problem, the latter always runs by finding a way to resolve the contradiction by refining the middle through intuition and making choices to take decisions. Considering the modern literature of logic and computer science, this can be likened to what Turing encountered as the "*halting problem*" while dealing with "*automatic machines*" only, by excluding "*choice machines*". Turing's automatic machines, which operate through mechanical reasoning, can not make choices and take decisions, for which an external operator is needed. Such external operator is certainly a human being who can make choices and take decisions while the automatic machine has halted at a contradiction or a decision problem[10].

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References

[1] M. Davis, *Engines of logic*, W. W. Norton and Company (2001). 1

[2] G. Boole, *An Investigation of the Laws of Thought*, Cambridge: Macmillan and Co. (1854). 1, 2

[3] D. Hilbert, W. Ackermann,*Principles of Mathematical Logic*, Chelsea Publishing Company (1950). 3

[4] J. Garfield, *The Fundamental Wisdom of the Middle Way: Nagarjuna's Mulamadhyamakakarika*, Oxford University Press, USA (1995). 2, 4

[5] F. Stcherbatsky, *Buddhist Logic* (two volumes), Dover Publications (1962). 2, 4

[6] J. Westerhoff, *Nagarjuna's Madhyamaka: A Philosophical Introduction*. Oxford University Press (2009). 2, 4

[7] S. Radhakrishnan, *Indian Philosophy*, Vol. 1, George Allen & Unwin (1948). 2

[8] Aristotle, *Metaphysics*, Stanford Encyclopedia of Philosophy (2000). https://plato.stanford.edu/entries/aristotle-metaphysics/. 1

[9] L. E. J. Brouwer, *Consciousness, philosophy, and mathematics*, Proceedings of the Tenth International Congress of Philosophy (Amsterdam, August 11–18, 1948), North-Holland Publishing Company, Amsterdam1949, pp. 1235–1249. 2

[10] A. Turing, *On Computable Numbers, with an application to the Enstscheidungsproblem*, Proceedings of the London Mathematical Society, Vol. s2-42, Issue 1, pp. 230-265 (1937); *A correction*, Proceedings of the London Mathematical Society, Vol. s2-43, Issue 1, pp. 544-546 (1938). 4