Representing Fractions in General Number Bases

Timothy W. Jones

June 20, 2022

Abstract

We show all all reduced $c/d \in (0,1)$ can be represented using any base $b \in \mathbb{N} \setminus \{1\}$.

Introduction

Hardy's classic *An Introduction to the Theory of Numbers* doesn't prove that all fractions $c/d \in (0,1)$ can be represented in any base b. His Chapter 9 *The Representation of Numbers by Decimals* does show this is the case for base $b = 10$ [1] and bases that are given by the product of unique primes to the first power. Here we fill in the details for the general proof: all $c/d \in (0,1)$ can be represented in base $b \in \mathbb{N} \setminus \{1\}$.

General Bases

The central patterns are discernible from simple by hand divisions given by $1/d$, $d \in \{2, \ldots, 17\}$. Such divisions convert the fractions $1/d$ to decimals base 10. The decimals generated are dependent on the intersection of D, the prime factors of d and B, the prime factors of b.

If all prime factors of D are also in B then the decimal representation terminates or is said to be finite: $D \subset B$ implies finite. This is the case with $\{2, 4, 8, 10, 16\}$. If $D \cap B = \emptyset$, then the decimal does not terminate. It is said to be pure repeating. For example $(3, 10) = 1$, the two numbers are relatively prime, and $1/3 = .\overline{3}$. This is the case with $\{3, 5, 7, 11, 13, 17\}$: $D \cap B = \emptyset$

implies pure repeating. If $D \cap B \neq \emptyset$ and $D \not\subset B$, that is D and B share some prime factors but not all, then the decimal representation is termed mixed. It consists of a non-repeating part and a repeating part. For example 10 and 6 share the prime factor 2 but not 3 and the representation is given by $1/6 = .1\overline{6}$. This is the case with $\{6, 12, 14, 15\}$: $D \cap B \neq \emptyset$ and $D \not\subset B$ implies mixed. As one might suspect, the sets D and B can be generalized to any denominator and any base and these relationships continue to hold.

In each of these three cases there are two central questions. How many digits are needed and what are the digits. Why does 1/2 have a single digit 5 as in .5 and 1/4 have two in .25? Why does $.1\overline{6}$ have one non-repeating digit and one repeating digit in this mixed case? Why does 1/3 have a single repeating digit in $.\overline{3}$ and 1/7 have 6 repeating digits as in $.\overline{142857}$. One might surmise that the number of digits, called the period of these pure repeating decimals, increases; but, 1/11 just has 2, $.\overline{09}$, 1/13 has 6 digits, and 1/17 has 16? So it is more mysterious than one might suspect.

These questions having been answered for fractions $1/d$, decimal representations of reduced c/d are found by just multiplying the integers generated for $1/d$ by c: Thus $3/4 : 3(25) = 75$; $3/7 : 3(142857) = 428571$. The mixed case is, as we shall see, a combination of the finite and pure cases and two multiplications are necessary.

The Finite Case

The number of digits needed for the finite case is the minimum m such that $b^m(1/d)$ is an integer. So assuming we are in base 10, 1/2 with $d = 2$ is such that $b^1(1/d) = 10(1/2) = 5$, an integer. The integer is also the digit in the representation. For 1/4, $10^2(1/4) = 25$ and dividing both sides gives $1/4 = .25$. This same pattern carries over to any base.

Lemma 1. *If $D \subset B$ then there exists a minimum m such that*

$$b^m \frac{1}{d} = i,$$

where i is an integer.

Proof. Suppose $b = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$ with $x_j > 0$ for $1 \leq j \leq k$. As $D \subset B$, $d = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$ with $y_j \geq 0$ for $1 \leq j \leq k$. That is if a specific prime is not a prime factor of d it will have an exponent of 0.

Using the Archimedian property of the reals, there exists a minimum m_j such that $m_j x_j \geq y_j$ for each j. Let

$$m = \max\{m_j : 1 \leq j \leq k\}.$$

Note: as all m_j can't be zero $m > 0$. This means

$$b^m \frac{1}{d} = \frac{p_1^{mx_1} p_2^{mx_2} \cdots p_k^{mx_k}}{p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}} = p_1^{mx_1 - y_1} p_2^{mx_2 - y_2} \cdots p_k^{mx_k - y_k} \tag{1}$$

is an integer. If m' is less than m then for some j, $1 \leq j \leq k$, $m' x_j - y_j < 0$ and (1) is not an integer. $\square$

We know m exists and we can use a *while* loop to find it. But Hardy in a footnote provides a detail that can make the calculation of m more definitive. He doesn't provide a proof. We'll make it a lemma and give it a proof.

Lemma 2. *If* $b = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$ *with* $x_j > 0$ *for* $1 \leq j \leq k$ *and* $d = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$ *with* $y_j \geq 0$ *for* $1 \leq j \leq k$ *with* $D \subset B$*, then the minimum* m *such that*

$$b^m \frac{1}{d} = i,$$

an integer, is given by

$$max\{\overline{\frac{y_1}{x_1}}, \overline{\frac{y_2}{x_2}}, \ldots, \overline{\frac{y_r}{x_r}}\},$$

where the over line indicates the fractions are rounded up.

Proof. We argue by induction on the number of shared prime factors between b and d. If b and d share just one prime, p, then let $b = p^x$ and $d = p^y$, where x and y are positive integers. As

$$b^{y/x} \frac{1}{d} = \frac{(p^x)^{y/x}}{p^y} = 1,$$

this power does give the needed integer, but y/x might not be an integer. There are however just three possibilities $y < x$, $x = y$, and $y > x$. If $y < x$, then $y/x < 1$ and $n = x - y$ is a positive integer. Also

$$\frac{y}{x} + \frac{x-y}{x} = 1 \text{ and } \frac{(p^x)^{(y/x + (x-y)/x)}}{p^y} = p^{x-y}$$

and $\overline{\frac{y}{x}} = y/x + (x-y)/x = 1$.

If $y > x$, then $y = kx + r$ where k is a positive integer and r is an integer between 0 and $x-1$. The $r = 0$ case is clear. If $r \neq 0$,

$$\frac{y}{x} = k + \frac{r}{x} \text{ and } \overline{\frac{y}{x}} = k + \frac{r}{x} + \frac{x-r}{x} = k+1.$$

Now

$$\frac{(p^x)^{\overline{\frac{y}{x}}}}{p^y} = \frac{(p^x)^{k+\frac{r}{x}+\frac{x-r}{x}}}{p^y} = \frac{p^{kx+r+x-r}}{p^y} = \frac{p^{y+x-r}}{p^y} = p^{x-r},$$

an integer.

If $y = x$, then $y/x = 1$ and this case is obvious. Let $m = \overline{[\frac{y}{x}]}$.

Suppose the theorem is true for $n = k$. That is suppose b and d share k prime factors and

$$\frac{(p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k})^{m_k}}{p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}} = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

with

$$m_k = \max \left\{ \overline{\frac{y_1}{x_1}}, \overline{\frac{y_2}{x_2}}, \ldots, \overline{\frac{y_k}{x_k}} \right\} .$$

Consider

$$\frac{(p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} p_{k+1}^{x_{k+1}})^m}{p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k} p_{k+1}^{y_{k+1}}} = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \frac{p_{k+1}^{m_{k+1} x_{k+1}}}{p_{k+1}^{y_{k+1}}} = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} p_{k+1}^{s_{k+1}},$$

with $s_j \geq r_j$ and $m = \max \{m_k, m_{k+1}\}$. We have used the $k = 1$ case with

$$\frac{p_{k+1}^{m_{k+1} x_{k+1}}}{p_{k+1}^{y_{k+1}}}$$

to generate m_{k+1} for this $k+1$ prime. $\square$

Notice that we could use any large enough m value. For example,

$$10^5 \frac{1}{4} = 2^3 5^5 = 10^3 5^2 = 25000 \text{ and } \frac{25000}{10^5} = .25,$$

but this is certainly inelegant.

Theorem 1. *The Finite Case. If $D \subset B$, the number of digits in the base b representation of $1/d$ is finite. Further the representation is given by*

$$\frac{1}{d} = \frac{i}{b^m} = .(i)_{(b^m)} = .i_1 \cdots i_m, \tag{2}$$

where m is the least exponent such that $b^m \frac{1}{d}$ is an integer and this integer is given as $i_1 \cdots i_m$ in base b.

Proof. Using Lemma 1,

$$b^m \frac{1}{d} = i,$$

an integer. This implies $1/d = i/b^m$ and when i is converted to base b, this is i_b. When the decimal point is moved m places to the left we have the finite decimal of (2). $\square$

Example 1. Find the base 10 decimal representation of $1/2^3 5^4$. Using Lemma 2, the ratios of d over b's exponents for the shared factors are $\overline{3/1}$ and $\overline{4/1}$ and the maximum is 4, so $m = 4$.

$$10^4 \frac{1}{2^3 \cdot 5^4} = 2.$$

giving

$$\frac{2}{10^4} = .0002.$$

Example 2. Find the decimal representation of $1/3^3 5^4 = 1/16875$ in base 15. This will be given by

$$15^4 \frac{1}{3^3 5^4} = 3.$$

As the symbol 3 has the same meaning in base 15 as in base 10, we have $.0003_{(15)}$ as the answer. We confirm this by noting

$$\frac{3}{15^4} = \frac{1}{3^3 5^4}.$$

Example 3. What is $1/d$ in base b if $d = 3^5 = 243$ and $b = 2^2 3 = 12$? We have

$$b^5 \frac{1}{d} = 2^{10} = 1024_{10}$$

and this implies

$$\frac{1}{d} = \frac{1024_{10}}{b^5},$$

but this doesn't mean we can move the decimal in the numerator over by 5 and get the answer. We must convert the integer 1024 to base 12, then we can so move the decimal point. Using the *tower of divisions* method we arrive at $1024_{10} = 714_{12}$ and conclude the answer is $.00714_{12}$. By tower of divisions I am referring to long divisions performed in sequence where the quotient becomes the dividend and the remainders are concatenated to give the conversion. These algorithms are generally performed in high school algebra classes manually, but they are a natural for spreadsheets.

The Pure Repeating Case

The pure repeating case is best motivated by reverse engineering the result. Suppose we have a pure repeating decimal in base b, can we conclude that the reduced fraction it converges to will have a denominator, a d relatively prime to b? We can evaluate a pure repeating decimal; it is a geometric series with the first, zero exponent dropped.

Lemma 3. *A pure repeating decimal* $.\overline{x_1 x_2 \ldots x_p}$ *in base* b *converges to the fraction*

$$\frac{x_1 x_2 \ldots x_p}{b^p - 1}. \tag{3}$$

Proof. A repeating decimal is a form of the geometric series:

$$.\overline{(x)}_b = x \sum_{j=1}^{\infty} \frac{1}{b^j} = \frac{x}{b-1}. \tag{4}$$

The number of repeating digits is called the period. For periods greater than one, say p, the formula is similar:

$$.\overline{(x_1 x_2 \ldots x_p)}_b = x_1 x_2 \ldots x_p \sum_{j=1}^{\infty} \frac{1}{(b^p)^j} = \frac{x_1 x_2 \ldots x_p}{b^p - 1}. \tag{5}$$

□

We are on the right track; as b and $b-1$ are relatively prime[1], (4) shows that decimals with period one obey the rule $D \cap B = \emptyset$ implies pure repeating. In the general case, we need to show that

$$\frac{x_1 x_2 \dots x_p}{b^p - 1} = \frac{1}{d}, \tag{6}$$

for some p where $(b, d) = 1$. But (6) says that $x_1 x_2 \dots x_p \cdot d = b^p - 1$, that there exists a p such that $b^p \equiv 1$ mod d.

Viola: Fermat Euler's theorem says that if $(d, b) = 1$ then $b^{\phi(d)} \equiv 1$ mod d, where $\phi(d)$ is Euler's ϕ function, giving the number of numbers less than and relatively prime to d. Hence such a number does exist. Whether or not it is the least such exponent need not concern us yet. The digits fall out neatly.

Theorem 2. *Pure repeating decimals. If $B \cap D = \emptyset$, then there exists p such that $b^p \equiv 1$ mod(d) and $1/d$ is a pure repeating decimal of period p with digits given by*

$$i = \frac{b^p - 1}{d}. \tag{7}$$

Proof. Using Fermat-Euler's theorem, let $p = \phi(d)$, then

$$b^p \equiv 1 \text{ mod(d)}. \tag{8}$$

By definition of modularity there exists an integer i such that

$$b_p - 1 = i \cdot d \text{ or } \frac{1}{b^p - 1} = \frac{1}{di} \text{ or } \frac{i}{b^p - 1} = \frac{1}{d}.$$

Using Lemma 3, it follows that $1/d$ is a pure repeating decimal in base b of period p. $\square$

Notice that with pure repeating decimals we once again calculate an integer.

Example 4. Find 1/7 in base 10 without using the tower of division technique, but using the above Theorem.

Using Maple (or manual calculations) we determine that $10^6 \equiv 1$ mod 7. That is we iterate through $1, 2, \dots$ and find that 6 works. We know $\phi(7) = 6$, so we know we will find this least value. Then a division gives the digits:

$$\frac{10^6 - 1}{7} = 142857,$$

[1] Try some examples, attempt a proof, look up a proof if stuck.

but this means

$$142857 \sum_{j=1}^{\infty} \frac{1}{(10^6)^j} = .142857 + .000000142857 + \cdots = .\overline{142857}.$$

If b is other than base 10, the integer calculated will need to be converted to base b.

Example 5. Find $1/d$ in base b when $d = 7$ and $b = 5$. Using a spreadsheet, a calculator, or Maple we determine that $5^6 \equiv 1 \bmod 7$ and find the integer using

$$\frac{5^6 - 1}{7} = 2232.$$

Using a spreadsheet implementation of the tower of division algorithm this is converted to $32412_{(5)}$ and then upon division by 5^6 we arrive at $1/7 = .\overline{032412}_{(5)}$.

Example 6. Find $1/d$ in base $b = 10$ when $d = 11$, 13, and 17. As these denominators are all prime and as Euler's phi function evaluated at a prime is one less than the prime, we should get periods of 10, 12 and 16 for these d values. But we find $1/11 = .\overline{09}$, $1/13 = .\overline{076923}$, and $1/17 = .\overline{0588235294117647}$. The fine print is that Fermat-Euler guarantees the existence of a number, not that it is the least such number that works. We can iterate through all the values between 1 and $\phi(b)$ and we will be assured of finding the least p that works.

Like the use of inefficiently large m values for the finite and mixed cases, too large exponents, the b^ps, in the pure case will give the same number, but inelegantly. For example, $1/11 = .\overline{09}$, using period 2 and $1/11 = .\overline{0909090909}$ using period 10 – the same number. This least such value is called the order of b mod d and it is known to always be a divisor of $\phi(b)$.

The Mixed Case

Mixed decimals are logically last, as they have have finite and pure repeating parts.

Theorem 3. *Mixed decimals. If $D \cap B \neq \emptyset$ and $D \not\subset B$, then there exists a least m such that*

$$\frac{1}{d} = \frac{i + \frac{r'}{d'}}{b^m}, \tag{9}$$

where $(d, d') = 1$ and $r'/d' \in (0, 1)$.

Proof. By Lemma 1 there exists a least m for the primes shared by d and b. Using this m

$$b^m \frac{1}{d} = \frac{r}{d'} = i + \frac{r'}{d'} \tag{10}$$

and dividing by b^m gives (9). $\square$

Example 7. Find 1/6 base 10. We follow the pattern given in (10):

$$10^1 \cdot \frac{1}{6} = \frac{5}{3} = 1 + \frac{2}{3}.$$

We know using repeated divisions that $2/3 = .\overline{6}$, so we arrive at $1/6 = .1\overline{6}$, base 10.

Example 8. Find the non-repeating and repeating parts of $1/d$ in base b if $d = 2 \cdot 3^2 \cdot 5 = 90$ and $b = 2 \cdot 3^2 = 18$. Complete the conversion. We follow the pattern given in (10) and use $m = 1$ based on ratios of shared prime exponents:

$$b\frac{1}{d} = (2 \cdot 3^2)\frac{1}{2 \cdot 3^2 \cdot 5} = \frac{1}{5}.$$

Here the non-repeating part is .0. The repeating part is 1/5. We seek an exponent x such that $18^x \equiv 1 \bmod 5$; $x = 4$ works and

$$i = \frac{18^4 - 1}{5} = 20995.$$

This converted to base 18 is (3)(10)(14)(7) where the base ten numbers in parentheses are symbols base 18. Dividing this by 18^4 moves the decimal to the left for .(3)(10)(14)(7). Combining the finite and repeating parts and remembering $m = 1$, we have

$$\frac{1}{90} = \frac{0 + .(3)(10)(14)(7)}{18^1} = .0\overline{(3)(10)(14)(7)}.$$

Example 9. Find the non-repeating and repeating parts of $1/d$ in base b if $d = 2^4 \cdot 3 = 48$ and $b = 2^2 \cdot 5 = 20$. One can determine the power m using the greatest integer function on the ratios of exponents. So $m = 2$, as $\overline{4/2} = \overline{2/1} = 2$ with the only shared prime 2.

$$b^2 \frac{1}{d} = (2^2 \cdot 5)^2 \frac{1}{2^4 \cdot 3} = \frac{5^2}{3}.$$

As
$$\frac{5^2}{3} = 8 + \frac{1}{3},$$
the non-repeating part is .08. Note: the digit 8 is the same in base 10 and in the base needed here base 20. We look for x such that $20^x \equiv 1 \bmod 3$. With $x = 2$ we have $20^2 - 1 = 399$. This gives
$$\frac{20^2 - 1}{3} = 133_{10}.$$
As $133_{10} = (6)(13)$, $1/3 = .\overline{(6)(13)}$ is the repeating part. Finally, using $m = 2$ we have
$$\frac{1}{48} = \frac{8 + \frac{1}{3}}{20^2} = .08 + .00\overline{(6)(13)} = .08\overline{(6)(13)}.$$

Conclusion

Lemmas 1 and 2 are simple, obvious, and ungainly all at the same time. It is perhaps for this reason that Hardy didn't want to dig into these weeds.

References

[1] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.