Two Paradoxes in Quantum Mechanics for Two Particles on a Circle

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Abstract

Two paradoxes in quantum mechanics for two particles on a circle are presented.

1 Two particles on a circle and centre of mass coordinates

Consider two nonidentical free particles on a circle. Particle one with of moment of inertia I_1 and particle two with moment of inertia I_2. As coordinates we can use θ_1, θ_2 where θ_1 is the coordinate of particle one and θ_2 is the coordinate of particle two. We can also use Θ, θ where Θ is the coordinate of the centre of mass and θ the relative coordinate. The coordinates are related by

$$\Theta = \frac{I_1\theta_1 + I_2\theta_2}{I_1 + I_2} \qquad \theta = \theta_2 - \theta_1 \tag{1}$$

We have by (1) that

$$H \equiv -\frac{\hbar^2}{2I_1}\frac{\partial^2}{\partial\theta_1^2} - \frac{\hbar^2}{2I_2}\frac{\partial^2}{\partial\theta_2^2} = -\frac{\hbar^2}{2(I_1+I_2)}\frac{\partial^2}{\partial\Theta^2} - \frac{\hbar^2}{2(\frac{I_1I_2}{I_1+I_2})}\frac{\partial^2}{\partial\theta^2} \tag{2}$$

Let $\psi(\theta_1, \theta_2)$ be an eigenfuction of H in θ_1, θ_2 coordinates so

$$-\frac{\hbar^2}{2I_1}\frac{\partial^2\psi}{\partial\theta_1^2}(\theta_1,\theta_2) - \frac{\hbar^2}{2I_2}\frac{\partial^2\psi}{\partial\theta_2^2}(\theta_1,\theta_2) = E\psi(\theta_1,\theta_2) \tag{3}$$

Particle one at coordinate θ_1 or at coordinate $\theta_1 + 2\pi$ is at the same point on the circle. The wave function at this point has only one value hence

$$\psi(\theta_1 + 2\pi, \theta_2) = \psi(\theta_1, \theta_2) \tag{4}$$

Similarly for particle two

$$\psi(\theta_1, \theta_2 + 2\pi) = \psi(\theta_1, \theta_2) \tag{5}$$

As a result

$$E \in S_{\theta_1\theta_2} \equiv \left\{\frac{k^2\hbar^2}{2I_1} + \frac{n^2\hbar^2}{2I_2} : k, n \in \mathbb{Z}\right\} \tag{6}$$

Let $\Psi(\Theta, \theta)$ be an eigenfunction of H in Θ, θ coordinates so

$$-\frac{\hbar^2}{2(I_1+I_2)}\frac{\partial^2\Psi}{\partial\Theta^2}(\Theta,\theta) - \frac{\hbar^2}{2(\frac{I_1I_2}{I_1+I_2})}\frac{\partial^2\Psi}{\partial\theta^2}(\Theta,\theta) = E\Psi(\Theta,\theta) \tag{7}$$

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The centre of mass at coordinate Θ or at coordinate $\Theta + 2\pi$ is at the same point on the circle hence

$$\Psi(\Theta + 2\pi, \theta) = \Psi(\Theta, \theta) \tag{8}$$

Now $\Psi(\Theta, \theta) = \Psi(\Theta, \theta_2 - \theta_1)$ and particle two at coordinate θ_2 or at coordinate $\theta_2 + 2\pi$ is at the same point on the circle hence

$$\Psi(\Theta, \theta_2 + 2\pi - \theta_1) = \Psi(\Theta, \theta_2 - \theta_1) = \Psi(\Theta, \theta) \tag{9}$$

as a result

$$E \in S_{\Theta\theta} \equiv \left\{ \frac{k^2\hbar^2}{2(I_1 + I_2)} + \frac{n^2\hbar^2}{2(\frac{I_1 I_2}{I_1+I_2})} : k, n \in \mathbb{Z} \right\} \tag{10}$$

We expect $S_{\theta_1\theta_2} = S_{\Theta\theta}$ but this is not the case [1].

2 Two particles on a circle and probability densities

Consider two nonidentical particles, each on a circle, with wave function at $t = 0$

$$\psi(\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}} \sin(\theta_1 + \theta_2) \tag{11}$$

Note particle one and particle two can be on separate circles that are far apart. The probability density of particle one at $t = 0$ is then

$$\int_0^{2\pi} |\psi(\theta_1, \theta_2)|^2 d\theta_2 = \frac{1}{2\pi} \tag{12}$$

In general the wave function of particle one at $t = 0$ is

$$\psi_1(\theta_1) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{in\theta_1}}{\sqrt{2\pi}} \tag{13}$$

where $|c_n|^2$ is the probability of finding the value $n\hbar$ when angular momentum of particle one is measured.

Measuring angular momentum of particle one yields only values of $\hbar$ and $-\hbar$ with equal probability hence using (13) we have at $t = 0$ that

$$|\psi_1(\theta_1)|^2 = \frac{1}{\pi} \cos^2(\theta_1 + \phi) \tag{14}$$

for some constant ϕ. This differs from (12).

References

[1] K. De Paepe, Physics Essays, September 2008