P. Bungus : Sums of three cubes

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Abstract

Bungus equation and Pi

Introduction

Question: Equation

$$x^3+y^3+z^3=w^3 \quad , x,y,x,w\in\mathbb{N}=\{1,2,3,4,...\} \tag{1}$$

Solution: P. Bungus, ~ 1591 ,

$$x=3\,,\,y=4\,,\,z=5\,,\,w=6 \tag{2}$$

$$3^3+4^3+5^3=6^3 \tag{3}$$

According to Dickson (reference 2, p.550) , equation (3) had already been noted by P. Bungus in 1591.

Bungus equation and Pi

Recall that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+...$$

Entry 1. If $3^3+4^3=6^3-5^3$ we have

$$\pi=\sum_{k=0}^{\infty}2^{-4k}\binom{2k}{k}\sum_{n=0}^{k}\binom{k}{n}\frac{(17/27)^{k-n}}{2n+1}+2\sum_{k=0}^{\infty}(-1)^k\,2^{-2k}\binom{2k}{k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n(3/4)^{3k-3n}}{2n+1} \tag{4}$$

$$\pi=\sum_{k=0}^{\infty}2^{-4k}\binom{2k}{k}\left(\frac{17}{27}\right)^{k}{}_2F_1\left(\frac{1}{2},-k;\frac{3}{2};-\frac{27}{17}\right)+$$
$$2\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\left(\frac{3}{4}\right)^{3k}{}_2F_1\left(\frac{1}{2},-k;\frac{3}{2};\frac{64}{27}\right) \tag{5}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function.

Entry 2. If $3^3+5^3=6^3-4^3$ we have

$$\pi=\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\left(\frac{11}{27}\right)^{k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^{n}(27/44)^{n}}{2n+1}+$$
$$2\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^{n}(3/5)^{3k-3n}}{2n+1} \tag{6}$$

$$\pi=\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\left(\frac{11}{27}\right)^{k}{}_2F_1\left(\frac{1}{2},-k;\frac{3}{2};\frac{27}{44}\right)+$$
$$2\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\left(\frac{3}{5}\right)^{3k}{}_2F_1\left(\frac{1}{2},-k;\frac{3}{2};\frac{125}{27}\right) \tag{7}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function.

Entry 3. If $4^3+5^3=6^3-3^3$ we have

$$\pi=\sum_{k=0}^{\infty}2^{-2k}\binom{2k}{k}\left(\frac{67}{256}\right)\sum_{n=0}^{k}\binom{k}{n}\frac{(64/67)^{n}}{2n+1}+$$
$$2\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^{n}(4/5)^{3k-3n}}{2n+1} \tag{8}$$

$$\pi=\sum_{k=0}^{\infty}2^{-2k}\binom{2k}{k}\left(\frac{67}{256}\right)^{k}{}_2F_1\left(\frac{1}{2},-k;\frac{3}{2};-\frac{64}{67}\right)+$$
$$2\sum_{k=0}^{\infty}(-1)^{k}2^{-2k}\binom{2k}{k}\left(\frac{4}{5}\right)^{3k}{}_2F_1\left(\frac{1}{2},-k;\frac{3}{2};\frac{125}{64}\right) \tag{9}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function.

Entry 4. If $3^3+4^3+5^3=6^3$ we have

$$\pi=2\sum_{k=0}^{\infty}\binom{2k}{k}2^{-2k}\left(\frac{5}{6}\right)^{3k}\sum_{n=0}^{k}\binom{k}{n}\frac{1}{2n+1}\left(\frac{1}{2}\sqrt{\frac{1}{2}}\left(\frac{3}{5}\right)^{3n}+\frac{2}{3}\sqrt{\frac{2}{3}}\left(\frac{4}{5}\right)^{3n}\right) \tag{10}$$

$$\pi=2\sum_{k=0}^{\infty}\binom{2k}{k}2^{-2k}\left(\frac{2}{3}\right)^{3k}\sum_{n=0}^{k}\binom{k}{n}\frac{1}{2n+1}\left(\frac{1}{2}\sqrt{\frac{1}{2}}\left(\frac{3}{4}\right)^{3n}+\frac{5}{6}\sqrt{\frac{5}{6}}\left(\frac{5}{4}\right)^{3n}\right) \tag{11}$$

$$\pi=2\sum_{k=0}^{\infty}\binom{2k}{k}2^{-2k}\left(\frac{1}{2}\right)^{3k}\sum_{n=0}^{k}\binom{k}{n}\frac{1}{2n+1}\left(\frac{2}{3}\sqrt{\frac{2}{3}}\left(\frac{4}{3}\right)^{3n}+\frac{5}{6}\sqrt{\frac{5}{6}}\left(\frac{5}{3}\right)^{3n}\right) \tag{12}$$

Related equations

Entry 5. If $3^3+4^3+5^3=6^3$ and $i=\sqrt{-1}$ we have

$$4^3+5^3=3^3-3^4+3^5 \tag{13}$$

$$\left(\frac{1}{2}\right)^3+\left(\frac{2}{3}\right)^3+\left(\frac{5}{6}\right)^3=1 \tag{14}$$

$$\left(\frac{4}{3}\right)^3+\left(\frac{5}{3}\right)^3=7 \tag{15}$$

$$3^3+5^3=2^3\left(3^3-2^3\right) \tag{16}$$

$$\left(\frac{3}{2}\right)^3+\left(\frac{5}{2}\right)^3=3^3-2^3 \tag{17}$$

$$\frac{3^3+5^3}{3+5}=3^3-2^3 \tag{18}$$

$$\left(-3+i\sqrt{3}\right)^3+\left(-1+i\sqrt{3}\right)^3+\left(1+i\sqrt{3}\right)^3=\left(3+i\sqrt{3}\right)^3 \tag{19}$$

$$\left(-3-i\sqrt{3}\right)^3+\left(-1-i\sqrt{3}\right)^3+\left(1-i\sqrt{3}\right)^3=\left(3-i\sqrt{3}\right)^3 \tag{20}$$

$$\left(-3+i\sqrt{6}\right)^3+\left(2+i\sqrt{6}\right)^3+\left(7+i\sqrt{6}\right)^3=\left(-6+2i\sqrt{6}\right)^3 \tag{21}$$

$$\left(-3-i\sqrt{6}\right)^3+\left(2-i\sqrt{6}\right)^3+\left(7-i\sqrt{6}\right)^3=\left(-6-2i\sqrt{6}\right)^3 \tag{22}$$

$$\left(-9+i\sqrt{159}\right)^3+\left(-5+i\sqrt{159}\right)^3+20^3=24^3 \tag{23}$$

$$\left(-9-i\sqrt{159}\right)^3+\left(-5-i\sqrt{159}\right)^3+20^3=24^3 \tag{24}$$

$$\left(-3+i\sqrt{15}\right)^3+4^3+\left(-1+i\sqrt{15}\right)^3=6^3 \tag{25}$$

$$\left(-3-i\sqrt{15}\right)^3+4^3+\left(-1-i\sqrt{15}\right)^3=6^3 \tag{26}$$

$$5^3+4^3+3^3=\left(5+4-3\right)^3 \tag{27}$$

$$6^3-5^3-4^3=\left(-6+5+4\right)^3 \tag{28}$$

$$6^3-4^3-3^3=\left(6-4+3\right)^3 \tag{29}$$

$$6^3-5^3-3^3=\left(6-5+3\right)^3 \tag{30}$$

$$\left(-15+3\sqrt{30}\right)^3+\left(22-3\sqrt{30}\right)^3+\left(-13+3\sqrt{30}\right)^3=6^3 \tag{31}$$

$$\left(-15-3\sqrt{30}\right)^3+\left(22+3\sqrt{30}\right)^3+\left(-13-3\sqrt{30}\right)^3=6^3 \tag{32}$$

Endnote

Entry 6.

$$\pi=2\sin^{-1}\left(\frac{6^3+5^3-3^3}{60\sqrt{30}}\right)+2\sin^{-1}\left(\frac{6^3-5^3+3^3}{108\sqrt{2}}\right)-2\sin^{-1}\left(\frac{6^3-5^3-3^3}{30\sqrt{15}}\right) \tag{33}$$

$$\pi=2\sin^{-1}\left(\frac{6^3+5^3-4^3}{60\sqrt{30}}\right)+2\sin^{-1}\left(\frac{6^3-5^3+4^3}{96\sqrt{6}}\right)-2\sin^{-1}\left(\frac{6^3-5^3-4^3}{80\sqrt{5}}\right) \tag{34}$$

References

1. P.Bungus, 1591 , 1618 Numerorum Mysteria , Pars Altera 65.
2. Dickson, L.E., History of the Theory of Numbers, Vol II , Dover, 2005 , 550-562.
3. Mahler, K., Note on Hypothesis K of Hardy and Littlewood, Journal of the London Mathematical Society, 11(2): 136-138, 1936.
4. Mordell, L. J., On sums of three cubes, Journal of the London Mathematical Society, second series, 17(3): 139-144, 1942.