On the reconstruction of the rotation from stresses with respect to rotated coordinate axes

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July 5, 2022

Abstract

In this letter a theorem is stated on the reconstruction of the rotation from stresses with respect to rotated coordinate axes. In most literature a coordinate axis rotation is defined by an angle. Motivated by practical applications, we define the rotation by a unit vector expressed in Cartesian coordinates. An example and an application from the analysis of extreme stresses clarify the theoretical result and demonstrate the practical potential of the theorem.

A stress state with respect to the axes x and y, is defined by the normal stresses σ_x, σ_y and the shear stress $\tau_{x,y}$ see Figure 1. We consider stress states for which it holds that $\sigma_x \neq \sigma_y \ \vee \ \tau_{x,y} \neq 0$.

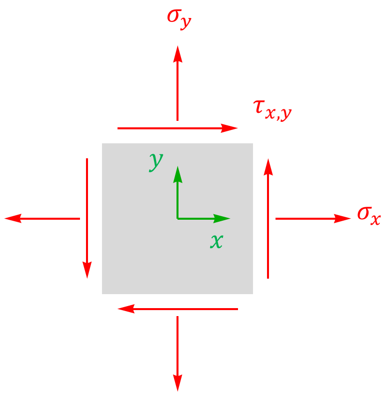

Figure 1: A stress state with respect to the axes x and y.

The direction of the $\tilde{x}$-axis as a rotation of the x-axis over a given angle φ is defined by the unit vector $\underline{e}_{\tilde{x}}$, see Figure 2.

$$\underline{e}_{\tilde{x}} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}, \ \varphi \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right]$$

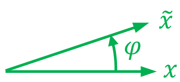

Figure 2: The $\tilde{x}$-axis as a rotation of the x-axis over a given angle φ.

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The stress state with respect to the axes $\tilde{x}$ and $\tilde{y}$, is defined by the normal stresses $\sigma_{\tilde{x}}, \sigma_{\tilde{y}}$ and the shear stress $\tau_{\tilde{x},\tilde{y}}$, see Figure 3.

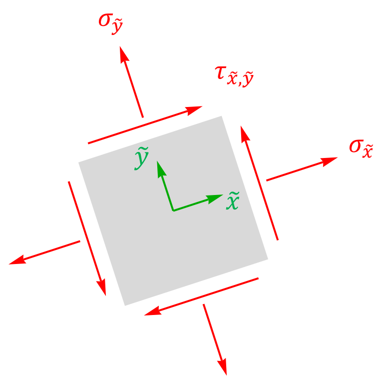

Figure 3: The stress state with respect to the axes $\tilde{x}$ and $\tilde{y}$.

The stress state with respect to the axes x and y and the stress state with respect to the axes $\tilde{x}$ and $\tilde{y}$, are for a given axis rotation related by Mohr's formulas:

$$(\sigma_{\tilde{x}} + \sigma_{\tilde{y}})/2 = (\sigma_x + \sigma_y)/2$$

$$\begin{bmatrix} (\sigma_{\tilde{x}} - \sigma_{\tilde{y}})/2 \\ -\tau_{\tilde{x},\tilde{y}} \end{bmatrix} = \begin{bmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{bmatrix} \cdot \begin{bmatrix} (\sigma_x - \sigma_y)/2 \\ -\tau_{x,y} \end{bmatrix}$$

1 Lemma

Define

$$\begin{bmatrix} c \\ s \end{bmatrix} := \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}$$

Then

$$\begin{bmatrix} \sigma_{\tilde{x}} \\ -\tau_{\tilde{x},\tilde{y}} \end{bmatrix} - \begin{bmatrix} \sigma_y \\ -\tau_{x,y} \end{bmatrix} = \lambda \cdot \begin{bmatrix} c \\ s \end{bmatrix}$$

with

$$\lambda := c \cdot (\sigma_x - \sigma_y) + 2 \cdot s \cdot \tau_{x,y}$$

Proof

Using Mohr's formulas:

$$\begin{aligned} \sigma_{\tilde{x}} - \sigma_y &= (1 + \cos(2\varphi)) \cdot (\sigma_x - \sigma_y)/2 + \sin(2\varphi) \cdot \tau_{x,y} \\ &= c^2 \cdot (\sigma_x - \sigma_y) + 2 \cdot c \cdot s \cdot \tau_{x,y} \\ &= (c \cdot (\sigma_x - \sigma_y) + 2 \cdot s \cdot \tau_{x,y}) \cdot c \\ &= \lambda \cdot c \end{aligned}$$

$$\begin{aligned} \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} &= \sin(2\varphi) \cdot (\sigma_x - \sigma_y)/2 + (1 - \cos(2\varphi)) \cdot \tau_{x,y} \\ &= c \cdot s \cdot (\sigma_x - \sigma_y) + 2 \cdot s^2 \cdot \tau_{x,y} \\ &= (c \cdot (\sigma_x - \sigma_y) + 2 \cdot s \cdot \tau_{x,y}) \cdot s \\ &= \lambda \cdot s \end{aligned}$$

□

The following theorem provides the reconstruction of the rotation from stresses with respect to the rotated axes.

2 Theorem

(1) Let $\sigma_{\tilde{x}} \neq \sigma_y \lor \tau_{\tilde{x},\tilde{y}} \neq \tau_{x,y}$, then

$$\underline{e}_{\tilde{x}} = \operatorname{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

(2) Let $\sigma_{\tilde{x}} = \sigma_y \land \tau_{\tilde{x},\tilde{y}} = \tau_{x,y}$, then

for $\tau_{x,y} \neq 0$

$$\underline{e}_{\tilde{x}} = \operatorname{sgn}(\tau_{x,y}) \cdot \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} / \left\| \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} \right\|$$

and for $\tau_{x,y} = 0$

$$\underline{e}_{\tilde{x}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Proof

(1) From $\lambda \neq 0$ and the lemma it follows

$$\operatorname{sgn}(\sigma_{\tilde{x}} - \sigma_y) = \operatorname{sgn}(\lambda) = \frac{|\lambda|}{\lambda},$$

$$\underline{e}_{\tilde{x}} = \frac{|\lambda|}{\lambda} \cdot \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} / \left\| \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} \right\| = \operatorname{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

(2) From $\lambda = 0$ it follows

for $\tau_{x,y} \neq 0$

$$\operatorname{sgn}\left(\frac{2 \cdot \tau_{x,y}}{c}\right) = \operatorname{sgn}(\tau_{x,y}) = \frac{|\tau_{x,y}|}{\tau_{x,y}},$$

$$\underline{e}_{\tilde{x}} = \left|\frac{2 \cdot \tau_{x,y}}{c}\right| \cdot \frac{c}{2 \cdot \tau_{x,y}} \cdot \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} / \left\| \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} \right\| = \operatorname{sgn}(\tau_{x,y}) \cdot \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} / \left\| \begin{bmatrix} 2 \cdot \tau_{x,y} \\ \sigma_y - \sigma_x \end{bmatrix} \right\|$$

and for $\tau_{x,y} = 0 \; (\sigma_x \neq \sigma_y)$

$$\lambda = c \cdot (\sigma_x - \sigma_y) + 2 \cdot s \cdot \tau_{x,y} = c \cdot (\sigma_x - \sigma_y) + 2 \cdot s \cdot 0$$

$$0 = c \cdot (\sigma_x - \sigma_y) = 0 \Leftrightarrow c = 0$$

$$\mathrm{c} = \cos(\varphi) = 0 \land \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right] \Rightarrow \varphi = \frac{\pi}{2} \Rightarrow s = \sin(\varphi) = 1$$

$$\underline{e}_{\tilde{x}} = \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

□

3 Example

Given:

A stress state with respect to the axes x and y, see Figure 4.

$$\sigma_x = +70\ \mathrm{N/mm^2}$$
$$\sigma_y = -10\ \mathrm{N/mm^2}$$
$$\tau_{x,y} = -30\ \mathrm{N/mm^2}$$

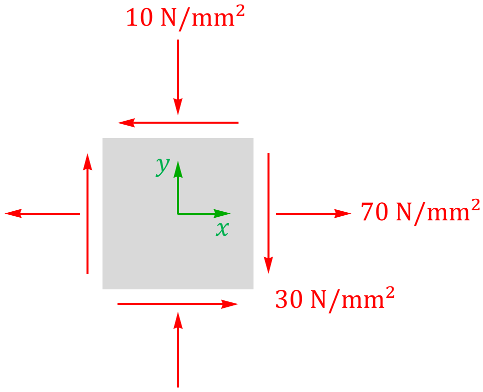

Figure 4: A given stress state with respect to the axes x and y.

A stress state with respect to the axes $\tilde{x}$ and $\tilde{y}$, see Figure 5.

$$\sigma_{\tilde{x}} = +78\ \mathrm{N/mm^2}$$
$$\sigma_{\tilde{y}} = -18\ \mathrm{N/mm^2}$$
$$\tau_{\tilde{x},\tilde{y}} = +14\ \mathrm{N/mm^2}$$

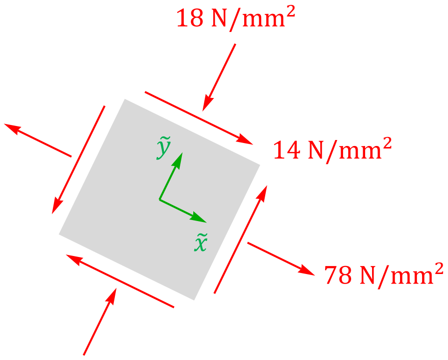

Figure 5: The stress state with respect to the axes $\tilde{x}$ and $\tilde{y}$.

Using the theorem for the reconstruction of the rotation, see Figure 6:

$$\underline{e}_{\tilde{x}} = \mathrm{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

$$= \mathrm{sgn}(78 - -10) \cdot \begin{bmatrix} 78 - -10 \\ -30 - 14 \end{bmatrix} / \left\| \begin{bmatrix} 78 - -10 \\ -30 - 14 \end{bmatrix} \right\| = \mathrm{sgn}(88) \cdot \begin{bmatrix} +88 \\ -44 \end{bmatrix} / \left\| \begin{bmatrix} +88 \\ -44 \end{bmatrix} \right\|$$

$$= 1 \cdot \begin{bmatrix} +2 \\ -1 \end{bmatrix} / \left\| \begin{bmatrix} +2 \\ -1 \end{bmatrix} \right\|$$

$$= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} +2 \\ -1 \end{bmatrix}$$

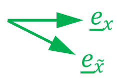

Figure 6: The unit vector $\underline{e}_x$ and the reconstructed unit vector $\underline{e}_{\tilde{x}}$.

Extreme stress states:

$$\sigma_{min} \leq \sigma_{\tilde{x}} \leq \sigma_{max}$$

$$\sigma_{min} \leq \sigma_{\tilde{y}} \leq \sigma_{max}$$

$$\tau_{min} \leq \tau_{\tilde{x},\tilde{y}} \leq \tau_{max}$$

Let

$$\sigma_m := \frac{\sigma_x + \sigma_y}{2}, \qquad R := \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{x,y}\right)^2}$$

Then

$$\sigma_{min} = \sigma_m - R \qquad \sigma_{max} = \sigma_m + R$$

$$\tau_{min} = -R \qquad \tau_{max} = +R$$

For the extreme normal stresses the following properties hold:

$$\sigma_{\tilde{x}} = \sigma_{max} \quad \Leftrightarrow \quad \sigma_{\tilde{y}} = \sigma_{min}$$

$$\sigma_{\tilde{x}} = \sigma_{min} \quad \Leftrightarrow \quad \sigma_{\tilde{y}} = \sigma_{max}$$

$$\sigma_{\tilde{x}} = \sigma_{max} \quad \Rightarrow \quad \tau_{\tilde{x},\tilde{y}} = 0$$

$$\sigma_{\tilde{x}} = \sigma_{min} \quad \Rightarrow \quad \tau_{\tilde{x},\tilde{y}} = 0$$

For the extreme shear stresses the following properties hold:

$$\tau_{\tilde{x},\tilde{y}} = \tau_{max} \quad \Rightarrow \quad \sigma_{\tilde{x}} = \sigma_{\tilde{y}}$$

$$\tau_{\tilde{x},\tilde{y}} = \tau_{min} \quad \Rightarrow \quad \sigma_{\tilde{x}} = \sigma_{\tilde{y}}$$

4 Application from the analysis of extreme stresses

For the stress state displayed in Figure 4, we have

$$\sigma_m := \frac{\sigma_x+\sigma_y}{2} = \frac{(+70)+(-10)}{2} = 30 \text{ N/mm}^2$$

$$R := \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \left(\tau_{x,y}\right)^2} = \sqrt{\left(\frac{(+70)-(-10)}{2}\right)^2 + (-30)^2} = 50 \text{ N/mm}^2$$

$$\sigma_{max} = \sigma_m + R = 30 + 50 = +80 \text{ N/mm}^2$$

$$\sigma_{min} = \sigma_m - R = 30 - 50 = -20 \text{ N/mm}^2$$

$$\tau_{max} = +R = +50 \text{ N/mm}^2$$

$$\tau_{min} = -R = -50 \text{ N/mm}^2$$

Reconstruction of the unit vector $\underline{e}_{\tilde{x}}$ for the maximal normal stress $\sigma_{\tilde{x}} = \sigma_{max}$, see Figure 7.

Using the theorem for $(\sigma_{\tilde{x}}, \tau_{\tilde{x},\tilde{y}}) \coloneqq (\sigma_{max}, 0)$:

$$\underline{e}_{\tilde{x}} = \mathrm{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

$$= \mathrm{sgn}(80 - -10) \cdot \begin{bmatrix} 80 - -10 \\ -30 - 0 \end{bmatrix} / \left\| \begin{bmatrix} 80 - -10 \\ -30 - 0 \end{bmatrix} \right\| = \mathrm{sgn}(+90) \cdot \begin{bmatrix} +90 \\ -30 \end{bmatrix} / \left\| \begin{bmatrix} +90 \\ -30 \end{bmatrix} \right\|$$

$$= +1 \cdot \begin{bmatrix} +3 \\ -1 \end{bmatrix} / \left\| \begin{bmatrix} +3 \\ -1 \end{bmatrix} \right\|$$

$$= \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} +3 \\ -1 \end{bmatrix}$$

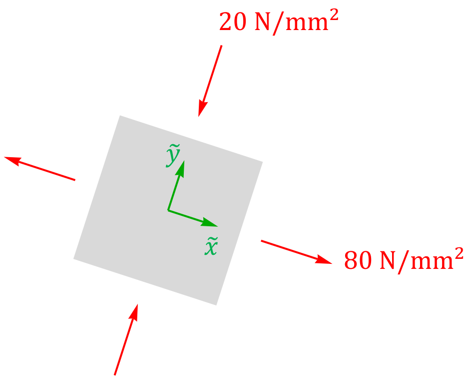

Figure 7: The stress state with $\sigma_{\tilde{x}} = \sigma_{max}$.

Reconstruction of the unit vector $\underline{e}_{\tilde{x}}$ for the minimal normal stress $\sigma_{\tilde{x}} = \sigma_{min}$, see Figure 8.

Using the theorem for $(\sigma_{\tilde{x}}, \tau_{\tilde{x},\tilde{y}}) \coloneqq (\sigma_{min}, 0)$:

$$\underline{e}_{\tilde{x}} = \mathrm{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

$$= \mathrm{sgn}(-20 - -10) \cdot \begin{bmatrix} -20 - -10 \\ -30 - 0 \end{bmatrix} / \left\| \begin{bmatrix} -20 - -10 \\ -30 - 0 \end{bmatrix} \right\| = \mathrm{sgn}(-10) \cdot \begin{bmatrix} -10 \\ -30 \end{bmatrix} / \left\| \begin{bmatrix} -10 \\ -30 \end{bmatrix} \right\|$$

$$= -1 \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix} / \left\| \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\|$$

$$= \frac{1}{\sqrt{10}} \cdot \begin{bmatrix} +1 \\ +3 \end{bmatrix}$$

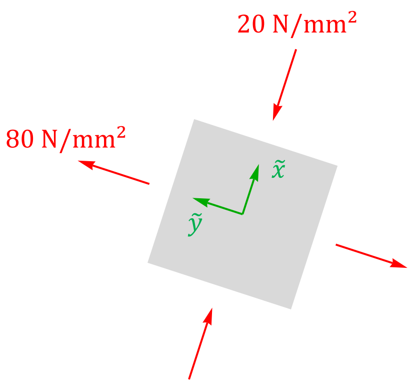

Figure 8: The stress state with $\sigma_{\tilde{x}} = \sigma_{min}$.

Reconstruction of the unit vector $\underline{e}_{\tilde{x}}$ for the maximal shear stress $\tau_{\tilde{x},\tilde{y}} = \tau_{max}$, see Figure 9.

Using the theorem for $(\sigma_{\tilde{x}}, \tau_{\tilde{x},\tilde{y}}) \coloneqq (\sigma_m, \tau_{max})$:

$$\underline{e}_{\tilde{x}} = \text{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

$$= \text{sgn}(30 - -10) \cdot \begin{bmatrix} 30 - -10 \\ -30 - +50 \end{bmatrix} / \left\| \begin{bmatrix} 30 - -10 \\ -30 - +50 \end{bmatrix} \right\| = \text{sgn}(+40) \cdot \begin{bmatrix} +40 \\ -80 \end{bmatrix} / \left\| \begin{bmatrix} +40 \\ -80 \end{bmatrix} \right\|$$

$$= +1 \cdot \begin{bmatrix} +1 \\ -2 \end{bmatrix} / \left\| \begin{bmatrix} +1 \\ -2 \end{bmatrix} \right\|$$

$$= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

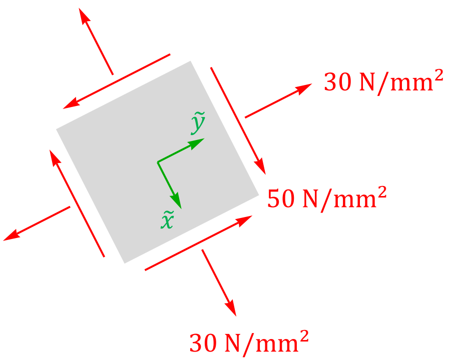

Figure 9: The stress state with $\tau_{\tilde{x},\tilde{y}} = \tau_{max}$.

Reconstruction of the unit vector $\underline{e}_{\tilde{x}}$ for the minimal shear stress $\tau_{\tilde{x},\tilde{y}} = \tau_{min}$, see Figure 10.

Using the theorem for $(\sigma_{\tilde{x}}, \tau_{\tilde{x},\tilde{y}}) \coloneqq (\sigma_m, \tau_{min})$:

$$\underline{e}_{\tilde{x}} = \text{sgn}(\sigma_{\tilde{x}} - \sigma_y) \cdot \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} / \left\| \begin{bmatrix} \sigma_{\tilde{x}} - \sigma_y \\ \tau_{x,y} - \tau_{\tilde{x},\tilde{y}} \end{bmatrix} \right\|$$

$$= \text{sgn}(30 - -10) \cdot \begin{bmatrix} 30 - -10 \\ -30 - -50 \end{bmatrix} / \left\| \begin{bmatrix} 30 - -10 \\ -30 - -50 \end{bmatrix} \right\| = \text{sgn}(+40) \cdot \begin{bmatrix} +40 \\ +20 \end{bmatrix} / \left\| \begin{bmatrix} +40 \\ +20 \end{bmatrix} \right\|$$

$$= +1 \cdot \begin{bmatrix} +2 \\ +1 \end{bmatrix} / \left\| \begin{bmatrix} +2 \\ +1 \end{bmatrix} \right\|$$

$$= \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} +2 \\ +1 \end{bmatrix}$$

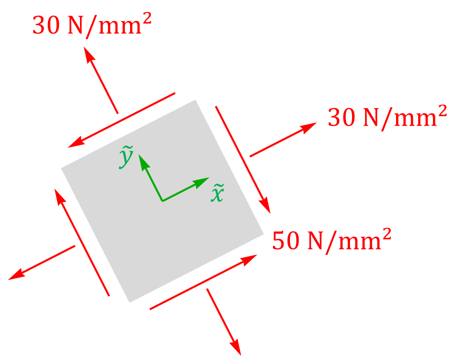

Figure 10: The stress state with $\tau_{\tilde{x},\tilde{y}} = \tau_{min}$.

5 Acknowledgement

The authors acknowledge the support of Charlotte Creusen and Ad Klein both affiliated with the Department of Engineering of Zuyd University of Applied Sciences.