Value-Counting Up to N

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Abstract

Some interesting properties arise when value-counting the integers sequentially up to N using N digits or fingers and comparing the number of values to the prime-exact equation; with a simple method for testing primes and prime powers (particularly Mersenne and Fermat primes).

Definition

We're going to be value-counting the integers up to a fixed integer N and use that integer to see which numbers, when counted by value, are counted within a row. In other words, using our ten fingers as an example, we're counting to see which numbers up to ten, when counted sequentially by value, would completely fall within a set of ten fingers.

Value-Counting

Odd Numbers

As a child, Carl Friedrich Gauss is believed to have showed that the sum of the first N natural numbers (positive integers) is equal to $\frac{N\times(N+1)}{2}$.

For any odd number N, the sum S = $\frac{N\times(N+1)}{2}$ is divisible by N. When N is a prime or one less than a prime, S is divisible by that prime, with the exception of N = 2. For any even number N, S = $\frac{N}{2}$ (mod N).

If we count the value of the integers for the first 23 integers, we get the following table.

Table 1 – Value-Counting Up to 23

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0	**1**	**2**	2	**3**	3	3	**4**	4	4	4	**5**	5	5	5	5	**6**	6	6	6	6	6	**7**	7
1	7	7	7	7	7	**8**	8	8	8	8	8	8	8	**9**	9	9	9	9	9	9	9	9	**10**
2	10	10	10	10	10	10	10	10	10	**11**	11	11	11	11	11	11	11	11	11	11	**12**	12	12
3	12	12	12	12	12	12	12	12	12	**13**	13	13	13	13	13	13	13	13	13	13	13	13	**14**
4	14	14	14	14	14	14	14	14	14	14	14	14	14	**15**	15	15	15	15	15	15	15	15	15
5	15	15	15	15	15	**16**	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	**17**	17
6	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	**18**	18	18	18	18	18	18	18
7	18	18	18	18	18	18	18	18	18	18	**19**	19	19	19	19	19	19	19	19	19	19	19	19
8	19	19	19	19	19	19	**20**	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20
9	20	20	20	**21**	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21
10	21	**22**	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22
11	**23**	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23

The first column was started at 0 to keep with all the reminders modulo N. For each of the cells highlighted yellow, the total integers counted fit in one row.

For each of the cells highlighted orange, the total integers counted sum to the M^{th} triangular number (OEIS A000217[1]) in reverse order (i.e., sum of positive integers less than N − M). This arises from the fact that each number is one less than the number after it.

For any odd number N, the following numbers will always fit in a row:

- The first M numbers such that $\frac{M\times(M+1)}{2} < N$.
  - **Note**: 1 is always included. For N = 23, M = 6.
- p for being odd and dividing $\frac{N\times(N+1)}{2}$ or, equivalently, p dividing $\frac{(N-1)\times N}{2}$ since the last tally of p − 1 fits perfectly at the end of the last cell.
  - **Note**: There are $\frac{p+1}{2}$ rows.
- p − 1 for being one less than p and ending at the last cell.
- p − M for the complementary symmetrical nature of the numbers where M + 1 is the first integer that does not fit in a row.

The rest of the numbers require a more complicated test. However, if one were to place all the numbers in a line, then one can see that each new number starts at one more than the triangular numbers (OEIS A000124[2]) mod N.

Thus, for N = 23, the sequence (mod 23) is: 0, 1, 3, 6, 10, 15, 21, 5, 13, 22, 9, **20**, 9, 22, 13, 5, 21, 15, 10, 6, 3, 1, 0. The sequence, starting from 0, is the triangular numbers. The sequence from the end is N − n, with the last n starting at N and decreasing by the triangular numbers. Thus, the sequence is symmetric about the $\frac{N+1}{2}$ position (20 in **bold**) for all odd N.

Looking at the number of integers that fit and don't fit for an odd prime number such as 23, we get the following pattern.

Table 2 – Fits vs. Does Not Fit

Fits	1	2	3	4	5	6	8	9	11	13	16	22
Does Not Fit	'22'	21	20	19	18	17	15	14	12	10	7	'1'
Sum	23	23	23	23	23	23	23	23	23	23	23	23

In addition to the number p, and adding the numbers 1 and p − 1 in quotes for Does Not Fit since they cancel with the two that Fit, we get an estimated value e(N) of the actual count a(N) as $\left\lceil\frac{N}{2}\right\rceil + 1$, again due to the complementary symmetry of the fitting on the sums from both ends of the table when N is prime. Thus, for N = 23, we have 13 integers that fit perfectly in a row and 10 that do not. This, however, is only guaranteed to be true for odd primes p, their powers, and their power-of-two multiples. See the Output section for a summary of the first 51 integers.

Looking at half the numbers in the N = 23 sequence, we get two instances of each half of the numbers, with the exception of the middle number having only one. This is the same as looking at the starting point of all the numbers (in **bold**) from Table 1 to get the following pattern.

Table 3 – n vs. 24 – n

Place	0	1	3	6	10	15	21	5	13	22	9	20
n	1	2	3	4	5	6	7	8	9	10	11	12
24 – n	23	22	21	20	19	18	17	16	15	14	13	12
Sum	24	24	24	24	24	24	24	24	24	24	24	24

For odd numbers, this is due to the fact that the odd number fills up the last row. Thus, the starting values of all numbers pair off against each other when N is odd.

Starting backwards from N, we have some row parity relationship for (1, N) (i.e., either they have the same row parity or they don't). (2, N − 1) will have the opposite row parity relationship than (1, N), with the remaining row parity relationships alternating from the previous pair.

Table 4 – Row Parity for N = 21

k	N – k + 1	row parity
1	21	=
2	20	≠
3	19	=
4	18	≠
5	17	=
6	16	≠
7	15	=
8	14	≠
9	13	=
10	12	≠
11	11	=

Even Numbers

For 2N, every k starts at the same position as 2N − k + 1 in N columns. Due to the half row shift at the end of 2N, k and 2N − k + 1 are always on opposite sides of their pairs in N.

Miscellaneous

As a result, for any N, if no numbers start at column k, then for M∈ℕ, no numbers shall start at k + m×N for M×N, where m < M (i.e., all columns = k (mod N) in an M×N table). Similarly, for any number that starts at column k, all columns = k (mod N) shall have a starting number in an M×N table. The case for M = 2, with k + N for 2N, follows from this. The general argument follows similarly. The case for N = 2^n shows that every column has a number as its starting point starting with N = 1 and doubling.

Equation Estimate

Calculating the results for the first 1,000 integers, we get the following count signs when comparing the actual count to the equation estimate count ($\left\lceil \frac{N}{2} \right\rceil + 1$).

Table 5 – Count Sign When Comparing Actual Count to Equation Estimate Count

Sign	Prime	Non-Prime	Count	Integer Type
<	1	9	10	Powers of 2
=	167	264	431	Only one odd prime factor
>	0	559	559	All others

Proof

Theorems

Theorem 1

The actual count for an odd prime power (i.e., N = p^n, where p is a prime) is equal to $\left\lceil \frac{N}{2} \right\rceil + 1 \; or \; \frac{p^n+3}{2}$.

Note: For an odd prime power p^n, $T_k = 0 \; (mod \; p^n)$ iff k = 1 or p^n.

Proof (I): Since there are only two zeros for $T_k = 0 \; (mod \; p^n)$ and for any odd number N, the numbers 1, N − 1, and N always fit, then every row in $\left[1, \cdots, \frac{N+1}{2} - 2\right]$ must end with a number that Does Not Fit (i.e., DNF count = $\frac{N-3}{2}$). So, the actual count of N is $N - \frac{N-3}{2} = \frac{N+3}{2}$.

■

Proof (II): For 0 ≤ k ≤ N − 1, k + 1 starts at $\frac{k(k+1)}{2}$ and ends at $\frac{k(k+1)}{2} + k$ on a continuous line.

Therefore, the positions of k + 1 is the closed set $\left[\frac{k(k+1)}{2} \; (mod \; N), \cdots, \frac{k(k+1)}{2} \; (mod \; N) + k\right]$.

Thus, we need to show how many k's have $\frac{k(k+1)}{2} \; (mod \; N) + k < N$.

Let $T_k = \frac{(k-1)k}{2}$ to allow for T_1 = 0.

Note: There's a slight confusion in differentiating between the numbers [1, …, N] and the column mods [0, …, N − 1].

For any odd N, the triangular numbers $T_k = T_{N-k+1} \; (mod \; N), 1 \le k \le N$. The sequence is symmetric around $T_{\frac{N+1}{2}}$.

To prove the actual counts equation, we need only show that for N an odd prime power, k Fits iff N − k Does Not Fit, for $2 \le k \le N - 2$, since the numbers 1, N − 1, and N will always Fit.

If k Fits, then $\frac{(k-1)k}{2} (mod \; N) + k - 1 = T_k (mod \; N) + k - 1 < N$.

For N − k, where $2 \le k \le N - 2$,

$$
\begin{aligned}
\frac{(N-k-1) \times (N-k)}{2} \; (mod \; N) + N - k - 1 &= T_{N-k} \; (mod \; N) + N - k - 1 \\
&= T_{k+1} \; (mod \; N) + N - k - 1 \\
&= (T_k + k) \; (mod \; N) + N - k - 1 \\
&\ge (1 + k) + N - k - 1, since \; T_k (mod \; N) + k - 1 < N \\
&= N
\end{aligned}
$$

Since the left-hand side is greater than or equal to N, it Does Not Fit.

Note: For an odd prime power p^n, $T_k = 0 \; (mod \; p^n)$ iff k = 1 or p^n. Since $T_k \; (mod \; N) + k - 1 < N$, T_k (mod N) can be replaced with greater than or equal to 1 since k ≤ N − 2 implies k + 1 < N even without the mod and T_k is never zero for the k constraints. Thus, $(T_k + k)\ (mod\ N) \geq k + 1$ and the previous inequality holds for the constraints $2 \leq k \leq N - 2$.

Similarly, if N – k Fits, then $\frac{(N-k-1)\times(N-k)}{2}(mod\ N) + N - k - 1 = T_{N-k}(mod\ N) + N - k - 1 < N$.

$$\begin{aligned}\frac{(k-1)k}{2}\ (mod\ N) + k - 1 &= T_k\ (mod\ N) + k - 1 \\ &= T_{N-k+1}\ (mod\ N) + k - 1 \\ &= (T_{N-k} + N - k)\ (mod\ N) + k - 1 \\ &\geq (1 + N - k) + k - 1, since\ T_{N-k}\ (mod\ N) + N - k - 1 < N \\ &= N\end{aligned}$$

Since the left-hand side is at a greater than or equal to N, it Does Not Fit.

Thus, $\frac{N-3}{2} + 3 = \frac{N+3}{2} = \frac{p^n+3}{2}$ of the numbers belong to Fits and $\frac{N-3}{2} = \frac{p^n-3}{2}$ of the numbers belong to Does Not Fit.

■

Theorem 2

If N is any odd number that is not a prime power, then the actual count is greater than the estimated count.

Proof: For an odd prime power p^n, $T_k = 0\ (mod\ p^n)$ iff k = 1 or p^n.

If N is not an odd prime power, then there exist at least two other numbers (k, N – k + 1) such that $T_k = T_{N-k+1} = 0\ (mod\ N)$.

Let c be the count of numbers that satisfy $T_k = 0\ (mod\ N)$ (i.e., all numbers that start at column zero).

Note: c is even and greater than zero. c appears to be equal to twice the number of unique odd primes in N.

For any integer N, N = $\left\lfloor\frac{N}{2}\right\rfloor + \left\lceil\frac{N}{2}\right\rceil$.

The total number of rows in N is $\left\lceil\frac{N+1}{2}\right\rceil$.

Then the total number for Does Not Fit in any N is $\left\lceil\frac{N+1}{2}\right\rceil - c$ and a(N) = $N - \left(\left\lceil\frac{N+1}{2}\right\rceil - c\right) = \left\lfloor\frac{N-1}{2}\right\rfloor + c$.

Thus, $a(N) = \left\lfloor\frac{N-1}{2}\right\rfloor + c > \left\lceil\frac{N+3}{2}\right\rceil$ when N is odd and c > 2.

Note: N = p^n and c = 2 proves Theorem 1.

■

Note: Thus, one way to test if an odd number is a potential prime is to check if the actual count for Fits is equal to the estimated value. For Mersenne and Fermat numbers, this test would prove primality due to Catalan's conjecture[3].

Theorem 3

The actual count for an even number (2N) is equal to the actual count of N plus $\left\lfloor \frac{N}{2} \right\rfloor$.

Proof: From Theorem 2, we have $a(N) = \left\lfloor \frac{N-1}{2} \right\rfloor + c$.

$$
\begin{aligned}
a(2N) &= \left\lfloor \frac{2N-1}{2} \right\rfloor + c \\
&= \left\lfloor \frac{N-1}{2} \right\rfloor + \left\lfloor \frac{N}{2} \right\rfloor + c \\
&= a(N) + \left\lfloor \frac{N}{2} \right\rfloor
\end{aligned}
$$

■

Theorem 4

The equality holds for all numbers divisible by exactly one odd prime (i.e., N = $2^m \times p^n$, where p is a single prime; OEIS A336101[4]).

Proof by induction for N = $2^m \times p^n$ since Theorem 1 covers the odd perfect prime powers.

Proof: Given a(p^n) = $\left\lceil \frac{p^n}{2} \right\rceil + 1$.

Assume a($2^m \times p^n$) = $\left\lceil \frac{2^m \times p^n}{2} \right\rceil + 1$ for a fixed m ≥ 0.

Then by Theorem 3,

$$
\begin{aligned}
a(2N) &= a(N) + \left\lfloor \frac{N}{2} \right\rfloor \\
&= a(2^m \times p^n) + \left\lfloor \frac{2^m \times p^n}{2} \right\rfloor \\
&= \left\lceil \frac{2^m \times p^n}{2} \right\rceil + 1 + \left\lfloor \frac{2^m \times p^n}{2} \right\rfloor \\
&= 2^m \times p^n + 1 \\
&= \left\lceil \frac{2^{m+1} \times p^n}{2} \right\rceil + 1
\end{aligned}
$$

Note: N = $2^m \times p^n$ and c = 2 in Theorem 2 also proves this theorem.

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Theorem 5

The less than occurs when N is a power of 2 (i.e., N = 2^n). The actual count is half that power of 2 (i.e., 2^{n-1}), while the estimated count is one more than the actual count (i.e., $2^{n-1} + 1$).

Proof: Given a(1) = a(2) = 1.

Assume $a(2^n) = 2^{n-1}$ for n ≥ 1.

By Theorem 3 and induction on n, we get

$$a(2^{n+1}) = a\left(\frac{2^{n+1}}{2}\right) + \left\lfloor\frac{2^{n+1}}{4}\right\rfloor$$
$$= a(2^n) + \lfloor 2^{n-1} \rfloor$$
$$= 2^{n-1} + \lfloor 2^{n-1} \rfloor$$
$$= 2^n$$

Note: N = 2^n and c = 1 in Theorem 2 also proves this theorem.

■

Note: Thus, one way to test if an even number is a power of two is to check if the actual count for Fits is (one) less than the estimated value. 1 also passes this test.

The greater than is for the remaining types of N and appears to start with mostly a delta of 2. However, running a modified version of the Main code up to 100,000 shows the interesting pattern that emerges as listed in the following table.

Table 6 – Distribution of Deltas

Delta (d)	Count (≤ 100,000)
−1	16
0	21,172
2	46,068
6	27,999
14	4,628
30	117

Conjectures

Conjecture 1

The delta sequence is 2 less than a power of 2 (a(n) = 2^n − 2; OEIS A000918[5]).

From Theorem 2, d = c − 2. It suffices to show that c is a power of two.

Note: The count density of these deltas appears to have a right-tailed distribution. It would be interesting to see how this pattern grows.

No pattern for which numbers have the same delta is easily apparent. However, the first instance that results in a new value for each of these deltas is listed in the following table.

Table 7 – First Instance of Delta

Delta (d)	First Instance (N)	Primorial (p#)
-1	1	2#/2
0	3	3#/2
2	15	5#/2
6	105	7#/2
14	1,155	11#/2
30	15,015	13#/2

Another pattern that appears to relate to this is the count M for when $T_n = 0 \ (mod\ N), 1 \leq n \leq N$. The value of M increases for the first time and is equal to 2^m when N = $\frac{p\#}{2}$, the $(m+1)^{th}$ half-primorial (OEIS A070826[6]).

Table 8 – Count M for When T_n = 0 (mod N)

$M = 2^m$	First Instance (N)	Primorial (p#)
$1 = 2^0$	1	2#/2
$2 = 2^1$	3	3#/2
$4 = 2^2$	15	5#/2
$8 = 2^3$	105	7#/2
$16 = 2^4$	1,155	11#/2
$32 = 2^5$	15,015	13#/2

Conjecture 2

The sequence of numbers that arises from selecting the first delta difference instance from our sequence of deltas is one half of the product of the first n primes (i.e., $\frac{p\#}{2}$).

Note: Another way of generating all primes is to keep track of the primorials for the first instance of the ever-increasing deltas.

Proving these conjectures is beyond the scope of this paper.

Corollaries

Corollary 1

Let $d(p^{n+1}) = \frac{a(p^{n+1})-a(p)}{p}$, where p is an odd prime, then

$$\begin{aligned} d(p^{n+1}) &= \mathrm{p} \times d(p^n) + d(p^2) \\ &= \mathrm{p} \times d(p^n) + \frac{p-1}{2} \\ &= \frac{p^n - 1}{2} \end{aligned}$$

$\therefore a(p^{n+1}) = a(p) + p \times \frac{p^n-1}{2}$.

Proof: A corollary of Theorem 1.

■

Corollary 2

$a(p^{n+1}) = p \times a(p^n) - 3 \times \frac{p-1}{2}$, where p is an odd prime and n > 1.

Proof: A corollary of Theorem 1.

■

Corollary 3

$a(2p) = 2a(p) - 2 = p + 1$, where p is an odd prime.

Proof: A corollary of Theorem 1 and Theorem 3.

■

Corollary 4

$a(2^k p) = 2^{k-1} \times p + 1$, where p is an odd prime.

Proof: A corollary of Theorem 1 and Theorem 3.

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Corollary 5

$a(3p) = 3a(p) - 1$, where p is an odd prime greater than 3.

Proof: By the Euclidean Algorithm[7], if (p, 3) = 1, then there exists integers r, s such that pr – 3s = 1.

Let k = pr, then T_k = 0 (mod 3p).

From Theorem 2, we're assuming uniqueness of k so that c = 4 and

$$\begin{aligned} a(3p) &= \left\lfloor \frac{3p-1}{2} \right\rfloor + 4 \\ &= \frac{3p+7}{2} \\ &= 3\left(\frac{p+3}{2}\right) - 1 \\ &= 3a(p) - 1 \end{aligned}$$

■

Note: Proving that c equals twice the number of unique odd primes is key here and has been left out.

Corollary 6

$a(3p) = 2 \times (a(2p) - 1) + 1$, where p is an odd prime.

Proof: Follows from Corollary 5.

■

Corollary 7

$a(3p) = a(2p) + a(p) + 1$, where the prime p ≠ 3.

Proof: Follows from ■

Note: Proving that c equals twice the number of unique odd primes is key here and has been left out.

Corollary 6.

■

Corollary 8

$a(3 \times 2^n) = 3a(2^n) + 1 = 3 \times 2^{n-1} + 1$, for all n ≥ 1.

Proof: A corollary of Theorem 3.

■

Other Patterns

$$a(k \times p) = C + \sum_{i=1}^{k-1} C_i \times a(i \times p)$$

Table 9 – Other Actual Fit Patterns

a(k×p), k =	C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	p > k	Exception(s)
3	1	1	1														p > 3	p = 2
4	-2	-1	1	1													p > 3	
5	0	-1	-1	1	1												p > 5	
6	-2	1	-1	1	1												p > 5	p = 3, 5
7	-1	1	1	-1	-1	1	1										p > 7	p = 3
8	-4	-1	1	1	-1	-1	1	1									p > 7	
9	-7	0	0	3													p > 7	p = 5, 7
9	0	-1	1	1	1	-1	1										p > 7	p = 2, 7
9	-2	0	-1	1	1	-1	-1	1	1								p > 7	
10	-4	0	0	-1	1	1	-1	-1	1	1							p > 7	p = 5
11	-3	1	1	-1	-1	1	1	-1	-1	1	1						p > 11	
12	-4	-1	1	1	-1	-1	1	1	-1	-1	1	1					p > 11	
13	-2	0	-1	1	1	-1	-1	1	1	-1	-1	1	1				p > 13	p = 3
14	-4	0	0	-1	1	1	-1	-1	1	1	-1	-1	1	1			p > 13	p = 3, 5
15	1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1		p > 13	p = 2, 5, 7
16	-10	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	p > 13	

Finding a pattern for which this equality holds is proving to be complicated. Many of these coefficients have the (1, 1, −1, −1) pattern starting at k – 1 (e.g., an exception appears to be k = 6), but coming up with the constant C that's only dependent on k and the other coefficients has proven elusive. In general, the summation appears to hold true for a(k × p) when p > k.

Note: The equality is not necessarily a unique summation for each k.

Proof: I'm tapping out.

Observations

Observation 1

For any odd N, $\frac{N+1}{2}$ Fits for N = 8k + 1 and 8k + 3; otherwise, Does Not Fit; and $\frac{N-1}{2}$ Fits for N = 8k + 5 and 8k + 7; otherwise, Does Not Fit.

Proof: For any odd N, if k Fits, then $\frac{(k-1)k}{2} (mod\ N) + k - 1 < N$.

For N = 8k + 1, if $\frac{N+1}{2}$ Fits, then

$$\begin{aligned}\frac{\left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right)}{2}(mod\ N)+\frac{N+1}{2}-1 &= \frac{N^2-1}{8}(mod\ N)+\frac{N-1}{2}\\ &= \frac{(8k+1)^2-1}{8}(mod\ 8k+1)+\frac{8k}{2}\\ &= \frac{8k(8k+2)}{8}(mod\ 8k+1)+4\text{k}\\ &= k(8k+2)(mod\ 8k+1)+4\text{k}\\ &= k+4\text{k}\\ &= 5k\\ &< N\end{aligned}$$

For N = 8k + 3, if $\frac{N+1}{2}$ Fits, then

$$\begin{aligned}\frac{\left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right)}{2}(mod\ N)+\frac{N+1}{2}-1 &= \frac{N^2-1}{8}(mod\ N)+\frac{N-1}{2}\\ &= \frac{(8k+3)^2-1}{8}(mod\ 8k+3)+\frac{8k+2}{2}\\ &= \frac{(8k+2)(8k+4)}{8}(mod\ 8k+3)+4\text{k}+1\\ &= \big((4k+1)(2k+1)\big)(mod\ 8k+3)+4\text{k}+1\\ &= \big(k(8k+3)+(3k+1)\big)(mod\ 8k+3)+4\text{k}+1\\ &= 3k+1+4\text{k}+1\\ &= 7k+2\\ &< N\end{aligned}$$

For N = 8k + 5, if $\frac{N+1}{2}$ Does Not Fit, then

$$\frac{N^2-1}{8}(mod\ N)+\frac{N+1}{2}-1=9k+5\geq N$$

For N = 8k + 7, if $\frac{N+1}{2}$ Does Not Fit, then

$$\frac{N^2-1}{8}(mod\ N)+\frac{N+1}{2}-1=11k+9\geq N$$

For N = 8k + 1, if $\frac{N-1}{2}$ Does Not Fit, then

$$\frac{(N-3)(N-1)}{8}(mod\ N)+\frac{N-1}{2}-1=9k\geq N$$

For N = 8k + 3, if $\frac{N-1}{2}$ Does Not Fit, then

$$\frac{(N-3)(N-1)}{8}(mod\ N)+\frac{N-1}{2}-1=11k+3\geq N$$

For N = 8k + 5, if $\frac{N-1}{2}$ Fits, then

$$\frac{(N-3)(N-1)}{8}(mod\ N) + \frac{N-1}{2} - 1 = 5k + 2 < N$$

For N = 8k + 7, if $\frac{N-1}{2}$ Fits, then

$$\frac{(N-3)(N-1)}{8}(mod\ N) + \frac{N-1}{2} - 1 = 6k + 5 < N$$

■

Observation 2

For any even N, $\frac{N}{2}$ Fits for N ≠ 0 (mod 8); otherwise, Does Not Fit. In addition, $\frac{N}{2}$ starts at $\frac{(n-1)\times N}{4}$ for N = 8k + 2n.

Proof: Let M = $\frac{N}{2}$. Then N = 2(4k + n) and M = 4k + n, where n = 0, 1, 2, or 3.

If k Fits, then $\frac{(k-1)k}{2}(mod\ N) + k - 1 < N.$

For M = 4k + n, if M Fits, then let the End Position (EP) =

$$\frac{(M-1)\mathrm{M}}{2}(mod\ N) + \mathrm{M} - 1 = \frac{(4k+n-1)(4k+n)}{2}(mod\ 8k+2n) + 4k + n - 1$$

For n = 1, EP = 4k = $\frac{N}{2}$ – 1 < N = 8k + 2. Thus, $\frac{N}{2}$ starts at 0 and the previous number (4k > 0) fits in N, but the next number (4k + 1) does not.

For n = 2, EP = 6k + 2 = $\frac{3N}{4}$ – 1 < N = 8k + 4. Thus, $\frac{N}{2}$ starts at $\frac{N}{4}$ and neither the previous number (4k + 1 > 1) nor the next number (4k + 3) fit in N.

For n = 3, EP = 8k + 5 = N – 1 < N = 8k + 6. Thus, $\frac{N}{2}$ starts at $\frac{N}{2}$ and the previous number (4k + 2) and the next number (4k + 4) also fit in N.

For n = 0, EP = 10k – 1 = $\frac{5N}{4}$ – 1 ≥ N = 8k. Thus, $\frac{N}{2}$ starts at $\frac{3N}{4}$ and the previous number (4k + 3) and the next number (4k + 5) also fit in N.

■

Observation 3

Some interesting patterns arise when calculating $C(k,p) = p - 2 \times \left(a\big((k+1)\times p\big) - a(k \times p)\right)$, where k is a fixed positive integer and p is an odd prime that varies. The result is a constant for p > k.

Observation 4

For any power of 2 (i.e., N = 2^n), the number of starting columns used in N is N, while the number of starting columns unused is 0 (see Even Numbers).

For any odd prime (i.e., N = p), the number of starting columns used in N is $\frac{p+1}{2}$, while the number of starting columns unused is $\frac{p-1}{2}$. This is due to the fact that when N is prime, $T_a \neq T_b$ (mod N) for a ≠ b (mod N) and $\frac{N+1}{2} = N - \frac{N+1}{2} + 1$ (mod N) for all odd N. Thus, another and possibly faster approach to check if an odd number is prime is to count the number of starting columns used up to $\frac{N+1}{2}$ and see if it equals $\frac{N+1}{2}$.

If N is odd but not a prime and greater than one, the number of starting columns used in N is less than the number of starting columns unused.

If N is even and divisible by at most a single unique odd prime number (i.e., N = $2^n p^m$, with n ≥ 1 and m ≤ 1), then the number of starting columns used in N is $\frac{N+2^n}{2}$, while the number of starting columns unused is $\frac{N-2^n}{2}$.

If N is even and has at least two prime factors, not necessarily distinct, the number of starting columns used in N is less than the number of starting columns unused.

Table 10 – Number of Used vs. Unused Starting Columns

Used	Sign	Unused	Integer Type (N)
N	>	0	Powers of 2 (2^n)
$\frac{N+1}{2}$	>	$\frac{N-1}{2}$	Odd primes
unknown	<	unknown	Odd non-primes > 1
$\frac{N+2^n}{2}$	>	$\frac{N-2^n}{2}$	Even, with at most a single unique odd prime factor ($2^n p^m$, with n ≥ 1 and m ≤ 1); Covers the first two conditions as well; For m = 1, the Unused count is equal to $\varphi(N)$
unknown	<	unknown	All other even numbers
unknown	=	unknown	None

Conclusion

It's interesting to see how there are patterns within patterns in the integer sequence and all the different ways an integer can be proven to be prime. This method requires only simple algebra to prove whether any number is a perfect prime power or not.

A similar pattern occurs if we use the powers of two for the equation estimate to make those deltas equal since the equation estimate would strictly be a division by two.

Code (Python)

Main

```
import math
import os
import sympy

f = "counting.txt"
if os.path.exists(f"{f}"):
    os.remove(f"{f}")

f = open(f"counting.txt", "a")
# p = prime, a = actual count, s = sign, e = estimated count, d = delta, c = count increase
f.write("n\tprime\ta\ts\te\td\tc\tlist\n")

n = 1000
s = “”
c = 0

for i in range(1, n + 1):
    sum = 0
    e = int(math.ceil(i/2) + 1)

    a = 0
    l = []

    for j in range(1, i + 1):    # The end can be modified to stop at N − M and adding two to a.
        sum += j
        if (sum <= i):
            l.append(j)
            a += 1
        if (sum >= i):
            sum -= i

    if (a < e):
        s = "<"
    elif (a > e):
        s = ">"
    else:
        s = "="

    f.write("%i\t%s\t%i\t%s\t%i\t%i\t%s\t[%s]\n" % (i, sympy.isprime(i), a, s, e, a - e, ("", str(a))[a > c], str(l)[1:-1]))

    if (a > c):
        c = a

f.close()
```

Primality Testing

```
# Note:   This test normally verifies that an odd number is a potential prime since it could still be a prime power.
#         The code was designed for Mersenne numbers, but can be modified to test the primality of any odd number,
#         including Fermat numbers.  This approach can be improved for better efficiency.

import math
import sympy

n = 32
p = 1

for i in range(1, n + 1):
    # For Mersenne numbers, mp is not a prime if p is not a prime.
    p = sympy.nextprime(p)

    mp = 2**p - 1
    e = int(math.ceil(mp/2) + 1)

    sum = 0
    a = 0
    np = “”
    stop = p - 2**int(p/2 + 1) + 1
    pc = int(100*(i/n))

    for j in range(1, stop):
        sum += j
        if (sum <= mp):
            a += 1
            if (a + 2 > e):
                np = "not "
                break
        if (sum >= mp):
            sum -= mp
        print(f"%{pc} - %{int(100*j/(stop-1))}  ", end="\r")

    print(f"2 ^ %i - 1 = %i is %sa prime (%i, %i)." % (p, mp, np, a, e))
```

Output

Table 11 – Value-Counting Up to 51

n	prime	a	s	e	d	c	list
1	False	1	<	2	-1	1	[1]
2	True	1	<	2	-1		[1]
3	True	3	=	3	0	3	[1, 2, 3]
4	False	2	<	3	-1		[1, 2]
5	True	4	=	4	0	4	[1, 2, 4, 5]
6	False	4	=	4	0		[1, 2, 3, 4]
7	True	5	=	5	0	5	[1, 2, 3, 6, 7]
8	False	4	<	5	-1		[1, 2, 3, 5]
9	False	6	=	6	0	6	[1, 2, 3, 5, 8, 9]
10	False	6	=	6	0		[1, 2, 3, 4, 5, 7]
11	True	7	=	7	0	7	[1, 2, 3, 4, 6, 10, 11]
12	False	7	=	7	0		[1, 2, 3, 4, 6, 8, 9]
13	True	8	=	8	0	8	[1, 2, 3, 4, 6, 8, 12, 13]
14	False	8	=	8	0		[1, 2, 3, 4, 6, 7, 8, 10]
15	False	11	>	9	2	11	[1, 2, 3, 4, 5, 6, 7, 9, 10, 14, 15]
16	False	8	<	9	-1		[1, 2, 3, 4, 5, 7, 9, 12]
17	True	10	=	10	0		[1, 2, 3, 4, 5, 7, 9, 11, 16, 17]
18	False	10	=	10	0		[1, 2, 3, 4, 5, 7, 8, 9, 11, 14]
19	True	11	=	11	0		[1, 2, 3, 4, 5, 7, 8, 10, 13, 18, 19]
20	False	11	=	11	0		[1, 2, 3, 4, 5, 7, 8, 10, 12, 15, 16]
21	False	14	>	12	2	14	[1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 21]
22	False	12	=	12	0		[1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 14, 17]
23	True	13	=	13	0		[1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 16, 22, 23]
24	False	13	=	13	0		[1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 16, 19]
25	False	14	=	14	0		[1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 18, 24, 25]
26	False	14	=	14	0		[1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 15, 17, 21]
27	False	15	=	15	0	15	[1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 17, 20, 26, 27]
28	False	15	=	15	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 19, 23]
29	True	16	=	16	0	16	[1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 16, 18, 21, 28, 29]
30	False	18	>	16	2	18	[1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25]
31	True	17	=	17	0		[1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 15, 17, 20, 23, 30, 31]
32	False	16	<	17	-1		[1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 15, 17, 19, 22, 26]
33	False	20	>	18	2	20	[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15, 17, 19, 21, 22, 25, 32, 33]
34	False	18	=	18	0		[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19, 21, 24, 28]
35	False	21	>	19	2	21	[1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16, 18, 20, 21, 23, 27, 34, 35]
36	False	19	=	19	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 18, 20, 23, 26, 30]
37	True	20	=	20	0		[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16, 18, 20, 22, 25, 28, 36, 37]
38	False	20	=	20	0		[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16, 18, 19, 20, 22, 24, 27, 32]
39	False	23	>	21	2	23	[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 19, 21, 24, 26, 27, 30, 38, 39]
40	False	21	=	21	0		[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 16, 17, 19, 21, 23, 26, 29, 34]
41	True	22	=	22	0		[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 17, 19, 21, 23, 25, 28, 32, 40, 41]
42	False	24	>	22	2	24	[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 17, 19, 20, 21, 23, 25, 27, 28, 31, 35, 36]
43	True	23	=	23	0		[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 17, 18, 20, 22, 24, 27, 30, 34, 42, 43]
44	False	23	=	23	0		[1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 17, 18, 20, 22, 24, 26, 29, 32, 33, 37]
45	False	26	>	24	2	26	[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 20, 22, 24, 26, 29, 32, 35, 36, 44, 45]
46	False	24	=	24	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 20, 22, 23, 24, 26, 28, 31, 34, 39]
47	True	25	=	25	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 20, 21, 23, 25, 28, 30, 33, 37, 46, 47]
48	False	25	=	25	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 19, 21, 23, 25, 27, 30, 32, 33, 36, 41]
49	False	26	=	26	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 19, 21, 23, 25, 27, 29, 32, 35, 39, 48, 49]
50	False	26	=	26	0		[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 19, 21, 23, 24, 25, 27, 29, 31, 34, 38, 43]
51	False	29	>	27	2	29	[1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19, 21, 22, 24, 26, 28, 31, 33, 34, 37, 41, 50, 51]

References

[1] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000217, ***Triangular numbers***

[2] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000124, ***Central polygonal numbers***

[3] Wikipedia, The Free Encyclopedia (2022), https://en.wikipedia.org/w/index.php?title=Catalan%27s_conjecture, ***Catalan's conjecture***

[4] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A336101, ***Numbers divisible by exactly one odd prime***

[5] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000918, ***a(n) = 2^n − 2***

[6] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A070826, ***One half of product of first n primes***

[7] Wikipedia, The Free Encyclopedia (2022), https://en.wikipedia.org/wiki/Euclidean_algorithm, **Euclidean algorithm**