The Intuitive Geometric Significance of Rotation Matrices of Spin $\frac{1}{2}$ and Spin 1, and Why Half Angles in SU(2)

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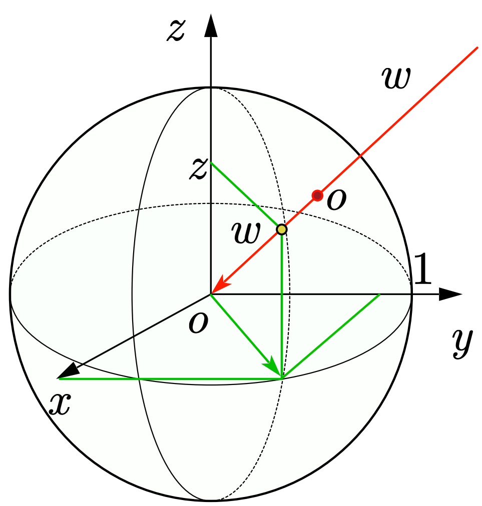
1 Intuitive Geometric Significance of the Rotation Transformation Matrces of Spin $\frac{1}{2}$,Why Half Angles in SU(2)

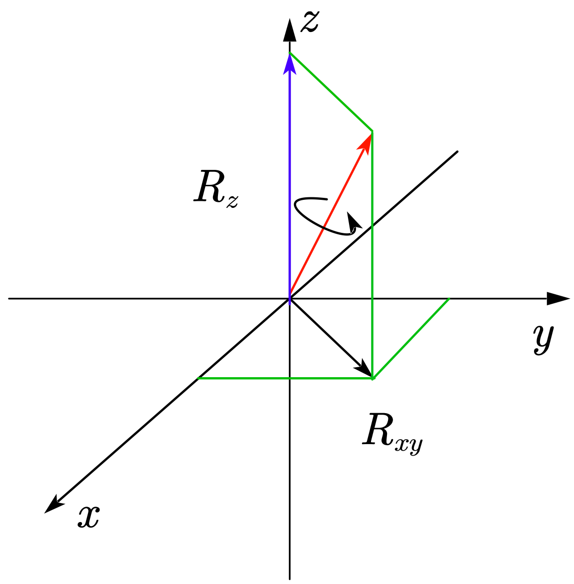
The transformation of any point on a four-dimensional unit sphere is bounded by the formular $x^2 + y^2 + z^2 + w^2 = 1$.



$$x^2 + y^2 + z^2 + w^2 = 1$$

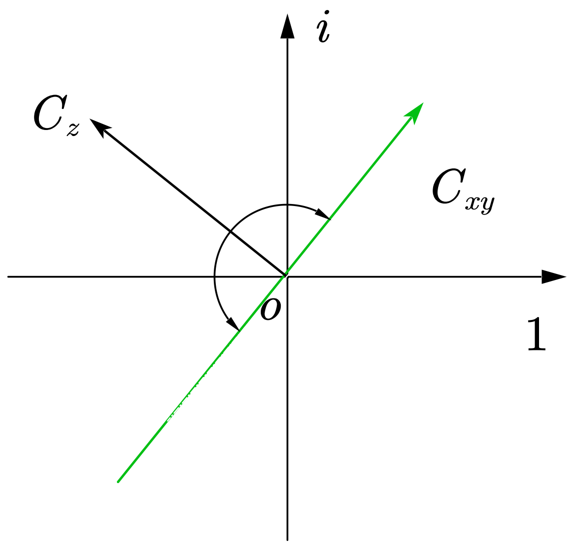
As shown in the figure above, a unit four-dimensional sphere can be represented by unit three-dimensional balls. The three-dimensional sphere is at the origin of the w-axis, the center of the three-dimensional sphere is located at one unit distance away from origin along w-axix and at origin on the x, y, and z axes respectively. The transformation of points on the four-dimensional unit sphere corresponds to the transformation of points in the three-dimensional unit ball(surface included).

It is also possible to project the potential vector of any point on a four-dimensional sphere onto two complex planes to produce two complex numbers, then map these two complex numbers onto one complex plane. The sum of the squares of the modulo lengths of these two complex numbers is a constant. Changing the value of this constant corresponds to the three-dimensional Euler rotation.



As shown in the figure above, the three-dimensional Euler rotation is not just a three-dimensional vector rotation, but in nature it represents a rigid body rotation. The rotation of axis vector R of a rigid body in 3D space can be captured by two angle parameters, while the rigid body can also rotate on its axis R, which produce another angle parameter. Usually, the sequence of Euler rotation is as follows assuming there is a fixed follow-up Cartesian coordinate system xyz on the rigid body:the rigid body firstly rotates around its z-axis by α angle, then around its y-axis by β angle, and finally rotates around its x-axis by γ angle. Thus the rigid body is able to reach any specified orientation state from its initial orientation state by rotation.

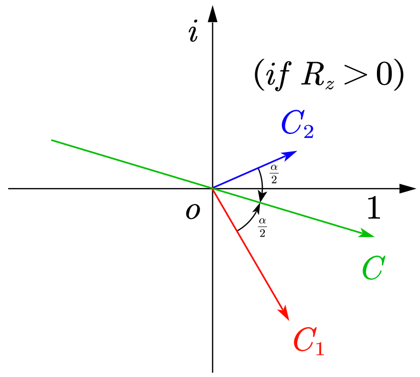
First, project the axis unit vector of a three-dimensional rigid body onto the z-axis and the xy plane to get R_z and R_{xy}. Note that R_z can be positive or negative, while R_{xy} can only be positive. Draw them on a complex plane, denoted by C_z and C_{xy} respectively:

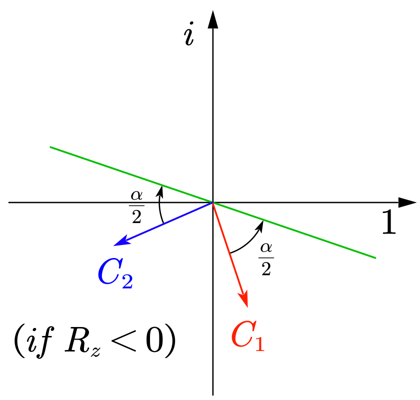


As shown in the figure above, the modulus of C_z is equal to that of R_z, and the modulus of C_{xy} is equal to that of R_{xy}. First let's find out the relationship between R_z and R_{xy} in three-dimensional space, and assume that C_z and C_{xy} is orthogonal. If R_z is positive,it points to the left side of C_{xy}, and if R_z is negative, it is points to the right side of C_{xy}.

Now, lets remove the assumption that C_z and C_{xy} must be orthogonal,but keep the rule–"If R_z is positive it points to the left side of C_{xy}, if R_z is negative it points to right side of C_{xy} side" unchanged. Otherwise, it will be impossible to disginguish whether R_z is positive or negative

Now, lets denote C_{xy} by C_1, and C_z by C_2.

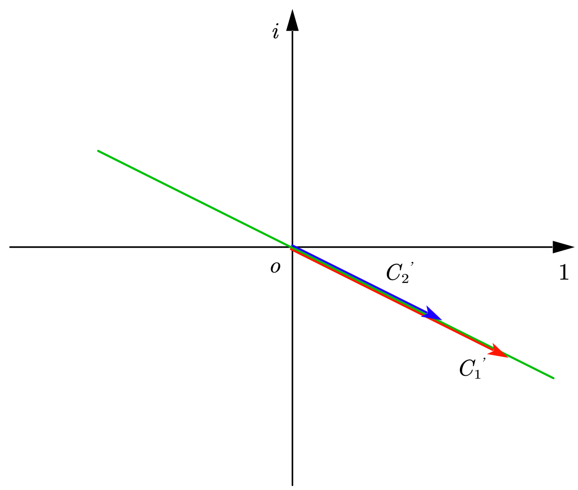


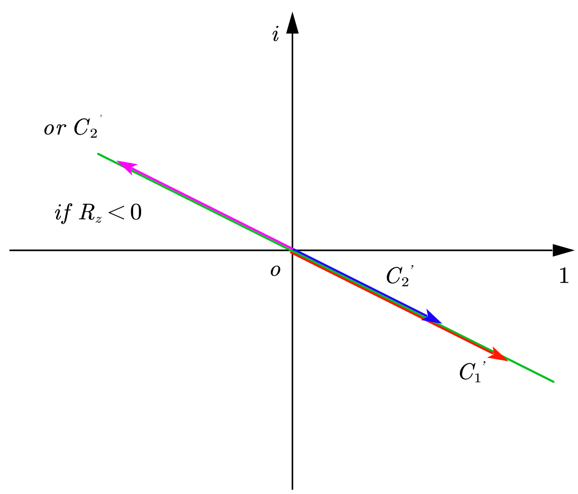


As shown in the figure above, let's draw an auxiliary vector C on the complex plane with the arrow pointing downword of the semi-complex plane.That is, its phase angle is between -π and 0. Next, draw C_1 on the right side of C, and C_2 on the left side of C under mirror-symmetrically (corresponding to R_z being positive). The angle between C_2 and C is $\frac{\alpha}{2}$, because the angle is constrainted to change between 0 and π, which corresponds to the rotation of the three-dimensional Euler rotation around the z-axis——ranging from 0 to 2π. If the R_z corresponding to C_2 is negative, then C_2 points to the same side of C_1 (ie, the right side of C). At this time, C_2 and C_1 are mirror-symmetrical about the vertical line passing through the origin of C.

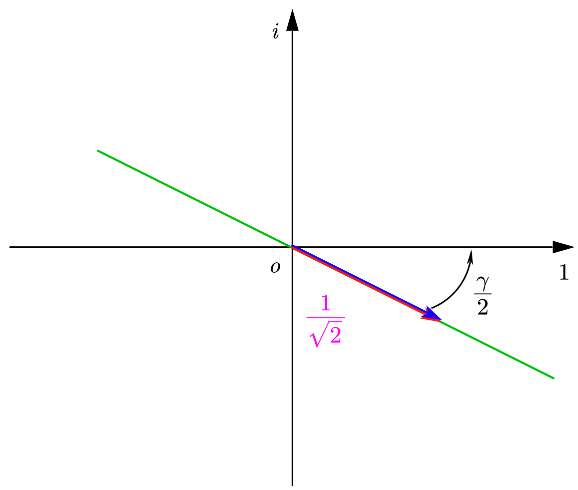
The following matrix acts on C_1 and C_2 at the same time while keeping the modulo of C_1 and C_2 unchanged, and rotate them line up as C. This rotation corresponds to the rotation of Euler rotation around the z-axis in three-dimensional space, which means just half-angle is required instead of full-angle:

$$\begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{1}$$





After C_1 and C_2 line up as C, now change the lengths of C_1 and C_2 but keep the square sum of their modulo unchanged in order for them to reach to the same direction and the same modulo $\frac{1}{\sqrt{2}}$:



This is equivalent to keep the phase between C_1' and C_2', but the former modulo changes from positive to $\frac{1}{\sqrt{2}}$, and the latter change from positive or negative modulo to $\frac{1}{\sqrt{2}}$, the sum of the squares of the two modulo remains at 1. This is equivalent to the former being the value of sine of an angle between 0 and π (always positive), the latter being that of cosine (positive or negative), and the changing range of the angle (set to $\frac{\beta}{2}$) is π (otherwise no way to guarantee that the range is still between 0 and π after the transformation), which just corresponds to the range of change of β is 2π, that is, the Euler rotation of a rigid body in three-dimensional space around the y-axis.

$$\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} e^{-i\frac{\gamma}{2}} \tag{2}$$

$$\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} R_{xy} \\ R_z \end{pmatrix} e^{-i\frac{\gamma}{2}} \tag{3}$$

Now, the two vectors are completely coincident. Their phases are the same as that of C– $e^{-\frac{i\gamma}{2}}$, and the modulo of both are $\frac{1}{\sqrt{2}}$. Lastly, they rotate $\frac{\gamma}{2}$ angle counterclockwise and reach the real axis simultaneously:

$$\begin{pmatrix} \cos\frac{\gamma}{2} & i\sin\frac{\gamma}{2} \\ i\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4}$$

The changing range of angle $-\frac{\gamma}{2}$ is between $-\pi \sim 0$, because we agreed from the beginning that the complex number C is in the lower half complex plane. So the changing range of the angle γ is $0 \sim 2\pi$, which just corresponds to the

Euler angle of rotation around x-axis in three-dimensional space. If we use the upper half complex plane for C at the beginning and perform similar operation with the range of angles used is $0 \sim \pi$, which also completely correspond to all Euler rotations in three-dimensional space. So, this is a 2-to-1 correspondence.

The three transformation matrices used above are common representation matrices of SU(2). Their exponential forms are:

$$\begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} = e^{i\sigma_z\frac{\alpha}{2}}, \tag{5}$$

$$\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} = e^{i\sigma_y\frac{\beta}{2}}, \tag{6}$$

$$\begin{pmatrix} \cos\frac{\gamma}{2} & i\sin\frac{\gamma}{2} \\ i\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} = e^{i\sigma_x\frac{\gamma}{2}}. \tag{7}$$

The exponent parts are three Pauli matrices :

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

The following are three generators of commonly used representation matrices of SU(2):

$$i\sigma_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{9}$$

This is the infinitesimal form of the change of C_1 and C_2 mentioned above:

The first matrix represents infinitesimal changes of C_1 in the direction of i in the complex plane, and infinitesimal changes of C_2 in the direction of -i. (The accumulation of this infinitesimal change is the rotation in a finite angle, that is, the reverse rotation of C_1 and C_2).

The second matrix represents infinitesimal changes of C_1' in the direction of the real axis, which is proportional to the modulus of C_2'; and infinitesimal changes of C_2' in the negative direction of the real axis, which is proportional to the modulus of C_1'. This means they tangle to adapt to each other.

The third matrix represents infinitesimal changes of both C_1'' and C_2'' in the direction of i, which is proportional to the length of each other. This means they rotate counterclockwise simultaneously.

For more information about the intuitive geometry meaning of Pauli matrices including their commutative form,please refer to my article: https://vixra.org/abs/1710.0198

2 The Intuitive Geometric Meaning of Spin 1 Under Commonly Used Rotation Matrices

The intuitive geometric meaning of spin $\frac{1}{2}$ is explained clearily. Why do we use half-angle to correspond to the full angle in three-dimensional space is also explained, please refers to https://vixra.org/abs/1810.0324.

Now let's study the usual rotation matrices of spin 1 using similar reasoning method. When a particle with spin 1 passing through the instrument along the y-axis of a Stern-Gerlach instrument, there will be possibilities of deflection–+1, 0, and -1–in the z-axis of the instrument(the up and down direction of the instrument). The three possibilities show that the particle may have spin angular momentum +1, 0, or -1 in the z-direction of the instrument. The probability amplitudes of these three possibilities are represented by three complex numbers C_1, C_0, C_2, and the sum of square of the moduloes is 1.

Expressing them in a column vector, i.e, the wavefunction of angular momentum of spin 1:

$$\begin{pmatrix} C_1 \\ C_0 \\ C_2 \end{pmatrix} \tag{10}$$

It is time to predict how the wavefunction of spin—i.e. the above column vector—change with the rotation of the instrument by an angle α around its z-axis(what are the three probabilities of the three deflections of the particle along the z-axis if the instrument rotated)? This corresponds to a transformation matrix–the rotation matrix D_z–acting on the column vector. Similarly,how to predict the result of the change in the above column vector if the instrument rotates around its y-axis by an angle of β, or by an angle of γ around its x-axis with the rotation matrix D_y or D_x acting on the column vector respectively. The spin rotation matrices D_z, D_y and D_x in these three directions are:

$$\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix} \tag{11}$$

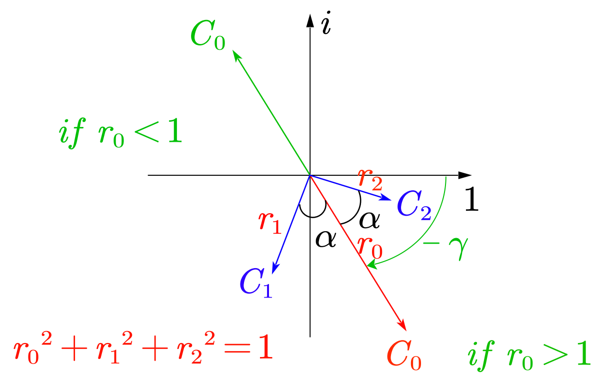
$$\begin{pmatrix} \frac{1}{2}(1+\cos(\beta)) & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}(1-\cos(\beta)) \\ -\frac{\sin(\beta)}{\sqrt{2}} & \cos(\beta) & \frac{\sin(\beta)}{\sqrt{2}} \\ \frac{1}{2}(1-\cos(\beta)) & -\frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}(1+\cos(\beta)) \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} \frac{1}{2}(1+\cos(\gamma)) & i\frac{\sin(\gamma)}{\sqrt{2}} & \frac{1}{2}(-1+\cos(\gamma)) \\ i\frac{\sin(\gamma)}{\sqrt{2}} & \cos(\gamma) & i\frac{\sin(\gamma)}{\sqrt{2}} \\ \frac{1}{2}(-1+\cos(\gamma)) & i\frac{\sin(\gamma)}{\sqrt{2}} & \frac{1}{2}(1+\cos(\gamma)) \end{pmatrix} \tag{13}$$

What are the intuitive geometric meaning of these three matrices? How do we quickly get them by intuitive geometric reasoning without rot memory or looking them up from books?

From the link at the beginning of this article, we know that the rotation matrix of spin $\frac{1}{2}$, i.e. the SU(2), is three transformations which keep the sum of the squares of the modulo unchanged of two vectors on a complex plane.The three transformations correspond to the Euler rotation SO(3) in 3D space, and it is a 2-to-1 correspondence. Similarly, we can achieve a 1-to-1 correspondence by using three transformations on the complex plane which keep the sum of the squares of the modulus constant to correspond to the Euler rotation in the 3D space.

First, we put three ordered vectors on the complex plane(three ordered complex numbers, or one three-dimensional complex vector as shown below), namely the three ordered complex numbers C_1, C_0 and C_2 are actually three possible values of 1,0 and -1 of the projection of angular momentum of spin 1 on the z axis, which corresponds to the probability amplitudes C_+, C_0, C_- bounded by the sum of their squares of modulo is 1:



$$\left\{ \begin{array}{l} c_1 = r_1 e^{-i(\nu+\alpha)} \\ c_0 = r_0 e^{-i\nu} \\ c_2 = r_2 e^{-i(\nu-\alpha)} \end{array} \right. \tag{14}$$

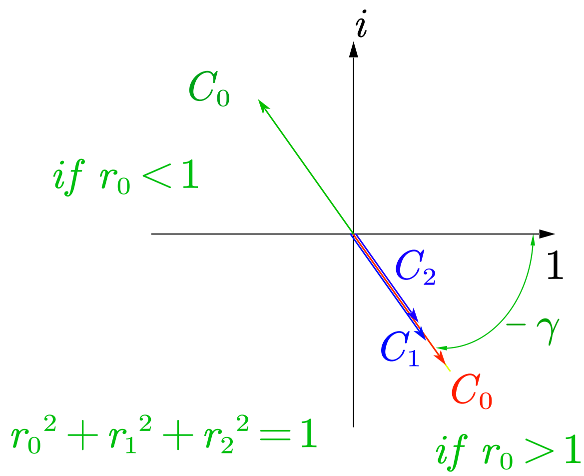
Transformation keeping the sum of the squares of r_0, r_1, and r_2 as 1 corresponds to the Euler rotation of the three-dimensional space.

In addition to the exponents θ in each $re^{i\theta}$, we also need to agree on the sign of three rs: r_1 and r_2 are always positive(including 0), and r_θ can be positive or negative. Picture the setting this way: The projection of a point on the three-dimensional unit sphere on the z-axis is the component r_z which is r_0 here, can be positive or negative; the projection onto the xy plane is the component $r_x y$ which is always positive, and $r_x y$ can be decomposed by r_1 and r_2 which are also positive.

The effect of the first matrix D_z:

$$\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} r_1 e^{-i(\gamma+\alpha)} \\ r_0 e^{-i\gamma} \\ r_2 e^{-i(\gamma-\alpha)} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_0 \\ r_2 \end{pmatrix} e^{-i\nu}. \tag{15}$$

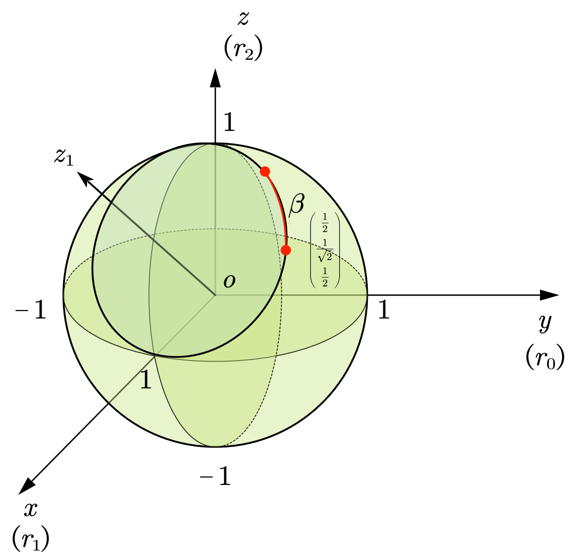
Keeping the modulus of these three complex numbers the same while C_1 and C_2 rotate in the opposite direction and line up on the same line C_0. The rotation amplitude, that is the α angle, ranges between $0 \sim 2\pi$, which completely corresponds to the three-dimensional rigid body Euler rotation angle α around the z-axis):



The above figure shows only the phase of C_1 and C_2 changes but none of their lengths .

Next, we are going to adjust only the lengths of r_1, r_0 and r_2, while keeping the sum of the squares of the modulus constant at 1. r_1, r_0, and r_2 are three real numbers, keeping the sum of the squares at 1 is equivalent to the rotation of that position vector on a unit sphere.

We make an auxiliary three-dimensional coordinate system and an unit sphere as shown in the figure below:



The equation of this unit sphere is:

$$x^2 + y^2 + z^2 = 1, \tag{16}$$

It intersects with a plane whose equation is $z = -x + 1$, and the intersecting line is a small circle on the unit sphere:

$$y^2 = -2x^2 + 2x. \tag{17}$$

The points $(1, 0, 0)$ and $(0, 0, 1)$ of the (x, y, z) coordinate system the small circle passes through are the maximum points on the x-axis and z-axis. The extreme value of the small circle on the y-axis is: $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$, at $(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$.

r_1 and r_2 are always positive, which correspond to the coordinates of the point of the small circle on the x-axis and z-axis are also always positive. r_0 can be positive or negative, and the coordinates of the point on the y-axis corresponding to the small circle can be positive or negative.

Now, the point (r_1, r_0, r_2) on the small circle rotates an angle of β clockwise along the small circle (around the z' axis in the figure, that is, the bisector of the angle between the x-axis and the z-axis) clockwise reaching the point $(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$ on the small circle. This point is exactly the maximum point along the y-axis of the small circle. Obviously, the possible rotation range of β is θ to $\frac{\pi}{2}$, which completely corresponds to the β angle that the three-dimensional Euler rotation around the y-axis. Now we need to formalize a rotation matrix around the z' axis by a beta angle which transforms the point (r_1, r_0, r_2) to the point $(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$.

We first rotate the z-axis in the figure around the y-axis by $\frac{\pi}{4}$ to the z' axis. Next, let the point on the small circle rotate $-\beta$ angle around the current z-axis, and then rotate the z-axis around the y-axis by $-\frac{\pi}{4}$ back to the original position of the z-axis.

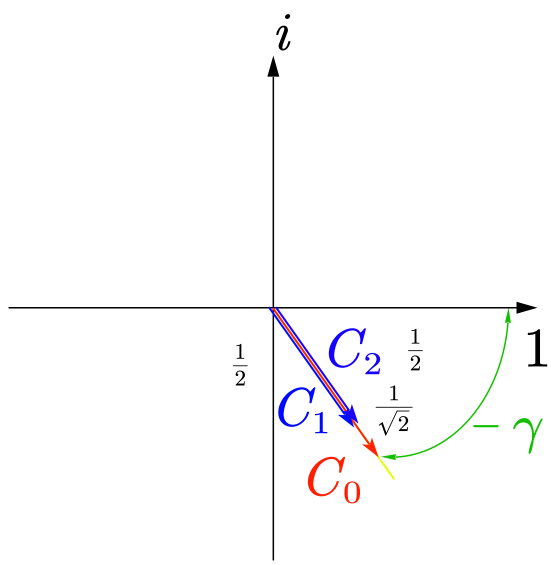
$$\begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & 0 & -\sin\left(-\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ \sin\left(-\frac{\pi}{4}\right) & 0 & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} \cdot$$

$$\begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & 0 & -\sin\left(\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{\pi}{4}\right) & 0 & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} \cos(\beta)+1 & \sqrt{2}\sin(\beta) & 1-\cos(\beta) \\ -\sqrt{2}\sin(\beta) & 2\cos(\beta) & \sqrt{2}\sin(\beta) \\ 1-\cos(\beta) & -\sqrt{2}\sin(\beta) & \cos(\beta)+1 \end{pmatrix}.$$

This is exactly the rotation matrix D_y of spin 1 around y. Simply put, the seemingly complicated matrix D_y is actually completely equivalent to the rotation matrix around the bisector of the x and z axes in the xyz Cartesian coordinate system in three-dimensional space.

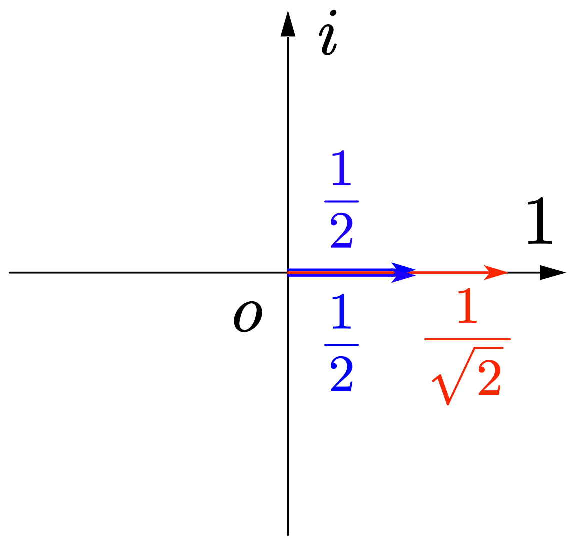
Back to the complex plane, the three complex numbers now becomes:



$$\begin{cases} C_1'' = \frac{1}{2}e^{-i\nu} \\ C_0'' = \frac{1}{\sqrt{2}}e^{-i\nu} \\ C_2'' = \frac{1}{2}e^{-i\gamma} \end{cases} \tag{18}$$

Finally, keep their respective modulus unchanged while rotating them by an angle of γ so that they all reach to the real axis. The following matrix play this role, which is exactly the geometric effect of the rotation matrix D_x of spin 1:

$$\left(\begin{array}{ccc} \frac{1}{2}(1+\cos(\gamma)) & i\frac{\sin(\gamma)}{\sqrt{2}} & \frac{1}{2}(-1+\cos(\gamma)) \\ i\frac{\sin(\gamma)}{\sqrt{2}} & \cos(\gamma) & i\frac{\sin(\gamma)}{\sqrt{2}} \\ \frac{1}{2}(-1+\cos(\gamma)) & i\frac{\sin(\gamma)}{\sqrt{2}} & \frac{1}{2}(1+\cos(\gamma)) \end{array}\right) \cdot \left(\begin{array}{c} \frac{1}{2}e^{-i\gamma} \\ \frac{1}{\sqrt{2}}e^{-i\gamma} \\ \frac{1}{2}e^{-i\gamma} \end{array}\right) = \left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{array}\right). \tag{19}$$



Postscript: Of course, the most common method of using three vectors (that is, three complex numbers) to correspond to the three-dimensional Euler rotation on a complex plane, is to line up the three complex numbers on one same straight line on that complex plane, with the same phase while the sign of the modulus of x, y and z be positive or negative.This setting corresponds to the three components of a vector in three-dimensional space. Then keep one of them unchanged while changing the length of the other two complex numbers (the length mentioned here can be positive or negative), and keep the sum of squares constant. It is a pure real transformation this way. By setting the relative parameters, the three commonly rotation matrices in three-dimensional space apply here.