Formulae yielding Pi

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Abstract
We give some integrals for Pi

Introduction

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) = 3.1415926535\ldots$$

In this note we give some integrals for Pi.

Integrals

Entry 1.

$$\pi = \int_0^1 \frac{2}{1-x^2}\left(\left(\frac{1-x}{2}\right)^{\frac{1}{4}} - \left(\frac{1-x}{2}\right)^{\frac{3}{4}} + \left(\frac{1+x}{2}\right)^{\frac{1}{4}} - \left(\frac{1+x}{2}\right)^{\frac{3}{4}}\right) dx$$

Entry 2.

$$\pi = 2\int_0^\infty \left(\left(\frac{1-\tanh x}{2}\right)^{\frac{1}{4}} - \left(\frac{1-\tanh x}{2}\right)^{\frac{3}{4}} + \left(\frac{1+\tanh x}{2}\right)^{\frac{1}{4}} - \left(\frac{1+\tanh x}{2}\right)^{\frac{3}{4}}\right) dx$$

Entry 3.

$$\pi = \int_0^\infty \frac{2}{\sinh x}\left(\left(\frac{1-\operatorname{sech} x}{2}\right)^{\frac{1}{4}} - \left(\frac{1-\operatorname{sech} x}{2}\right)^{\frac{3}{4}} + \left(\frac{1+\operatorname{sech} x}{2}\right)^{\frac{1}{4}} - \left(\frac{1+\operatorname{sech} x}{2}\right)^{\frac{3}{4}}\right) dx$$

Entry 4.

$$\pi = \int_0^{\frac{\pi}{2}} \frac{2}{\sin x}\left(\left(\frac{1-\cos x}{2}\right)^{\frac{1}{4}} - \left(\frac{1-\cos x}{2}\right)^{\frac{3}{4}} + \left(\frac{1+\cos x}{2}\right)^{\frac{1}{4}} - \left(\frac{1+\cos x}{2}\right)^{\frac{3}{4}}\right) dx$$

Entry 5.

$$\pi = \int_0^{\frac{\pi}{2}} \frac{2}{\cos x}\left(\left(\frac{1-\sin x}{2}\right)^{\frac{1}{4}} - \left(\frac{1-\sin x}{2}\right)^{\frac{3}{4}} + \left(\frac{1+\sin x}{2}\right)^{\frac{1}{4}} - \left(\frac{1+\sin x}{2}\right)^{\frac{3}{4}}\right) dx$$

Entry 6.

$$\pi = \int_0^{\frac{\pi}{2}} \frac{2}{\sin x}\left(\left(\sin\left(\frac{x}{2}\right)\right)^{\frac{1}{2}} - \left(\sin\left(\frac{x}{2}\right)\right)^{\frac{3}{2}} + \left(\cos\left(\frac{x}{2}\right)\right)^{\frac{1}{2}} - \left(\cos\left(\frac{x}{2}\right)\right)^{\frac{3}{2}}\right) \mathrm{d}x$$

Entry 7.

$$\pi = \int_0^{\infty} \frac{2}{\sinh(2x)}\left((\mathrm{e}^{-x}\sinh x)^{\frac{1}{4}} - (\mathrm{e}^{-x}\sinh x)^{\frac{3}{4}} + (\mathrm{e}^{-x}\cosh x)^{\frac{1}{4}} - (\mathrm{e}^{-x}\cosh x)^{\frac{3}{4}}\right) \mathrm{d}x$$

Entry 8.

$$\pi = \int_0^{\infty} \frac{2}{\sinh(2x)}\left((1+\coth x)^{-\frac{1}{4}} - (1+\coth x)^{-\frac{3}{4}} + (1+\tanh x)^{-\frac{1}{4}} - (1+\tanh x)^{-\frac{3}{4}}\right) \mathrm{d}x$$

Entry 9.

$$\pi = \int_0^{\frac{\pi}{2}} \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{\sin\left(\frac{x}{2}\right)}\left(1+\sin\left(\frac{x}{2}\right)\right)} + \frac{\sin\left(\frac{x}{2}\right)}{\sqrt{\cos\left(\frac{x}{2}\right)}\left(1+\cos\left(\frac{x}{2}\right)\right)}\right) \mathrm{d}x$$

Endnote

Entry 10.

$$\int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{\sin\left(\frac{x}{2}\right)}\left(1+\sin\left(\frac{x}{2}\right)\right)} \mathrm{d}x = 4\tan^{-1}\left(\frac{1}{\sqrt[4]{2}}\right)$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{x}{2}\right)}{\sqrt{\cos\left(\frac{x}{2}\right)}\left(1+\cos\left(\frac{x}{2}\right)\right)} \mathrm{d}x = 4\tan^{-1}\left(\frac{\sqrt[4]{2}-1}{\sqrt[4]{2}+1}\right)$$

References

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[2] Apelblat, A. : Tables of Integrals and Series. Verlag Harri Deutsch, 1996.
[3] Arndt, J., and Haenel, C. : π unleashed. Springer-Verlag, 2001.