Intrinsic Angular Momentum of Classical Electromagnetic Field

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Abstract

We study the question that does classical electromagnetic field have intrinsic angular momentum, and we find a surprising result that it is not possible to answer this question. We also go through how the quantity $\varepsilon_0\mathbf{E}\times\mathbf{A}$ can be used as the intrinsic angular momentum density of classical electromagnetic field.

We study a system that has been defined with a Lagrangian

$$\begin{aligned} L(A,\partial_t A, x,\dot{x}) \;&=\; \int\limits_{\mathbb{R}^3} \Big(-\frac{1}{4\mu_0}\big(\partial_\mu A_\nu(\mathbf{x}) - \partial_\nu A_\mu(\mathbf{x})\big)\big(\partial^\mu A^\nu(\mathbf{x}) - \partial^\nu A^\mu(\mathbf{x})\big) \\ &\qquad -\; A_\mu(\mathbf{x})\sum_{k=1}^K \rho_k^\mu(\mathbf{x})\Big) d^3x \;-\; \sum_{k=1}^K m_k c^2\sqrt{1-\frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}} \\ &= \int\limits_{\mathbb{R}^3} \frac{1}{2\mu_0}\Big(\Big\|\frac{1}{c}\partial_t\mathbf{A}(\mathbf{x}) \;+\; \nabla_{\mathbf{x}}A^0(\mathbf{x})\Big\|^2 \;-\; \big\|\nabla_{\mathbf{x}}\times\mathbf{A}(\mathbf{x})\big\|^2\Big) d^3x \\ &\qquad -\sum_{k=1}^K q_k\big(cA^0(\mathbf{x}_k) - \mathbf{A}(\mathbf{x}_k)\cdot\dot{\mathbf{x}}_k\big) \\ &\qquad -\sum_{k=1}^K m_k c^2\sqrt{1-\frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}}, \end{aligned}$$

where $\mu_0 \approx 4\pi\cdot 10^{-7}\mathrm{kg}\cdot\mathrm{m}/\mathrm{C}^2$ is the vacuum magnetic permeability, $c = 299792458\mathrm{m/s}$ is the speed of light, $A^\mu:\mathbb{R}^3\to\mathbb{R}$ and $\partial_t A^\mu:\mathbb{R}^3\to\mathbb{R}$ are fields for all $\mu\in\{0,1,2,3\}$, $x,\dot{x}\in\mathbb{R}^{3K}$ are coordinates, and $K\in\{1,2,3,\ldots\}$ is some constant that describes the number of particles. We use the notation $\mathbf{A}=(A^1,A^2,A^3)$. The coordinates of the vector x are arranged so that it is $x=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_K)$, and then for all $k\in\{1,2,\ldots K\}$ there is a relation $\mathbf{x}_k\in\mathbb{R}^3$, and $\mathbf{x}_k=(x_k^1,x_k^2,x_k^3)$, where $x_k^i\in\mathbb{R}$ for all $i\in\{1,2,3\}$. The

[1]In 19.11.2022 I uploaded an article with the same name to viXra, but now in 2024 I no longer believe that the logic and the conclusions in that article would be entirely correct, so that article should be ignored. This 16.1.2024 version of the article that has different logic and conclusions should be seen as replacing the old version.

vector $\dot{x}$ is arranged similarly. This means that we have K particles in 3 dimensions. The particles have masses $m_1, m_2, \ldots, m_K \in \mathbb{R}$. The quantity ρ_k^μ is a current density associated to a particle k. We have substituted

$$\rho_k^\mu(\mathbf{x}) \;=\; q_k \delta(\mathbf{x} - \mathbf{x}_k)(c, \dot{\mathbf{x}}_k),$$

which means that we assume the particles to be point-like. Here $q_k \in \mathbb{R}$ is the charge of the particle k. This model has some mathematical inconsistencies that are related to the paradoxes of classical point-like particles, but we can use this model anyway. If one does calculations with this model formally, one gets similar results as one would have gotten with more elaborate models that avoid the paradoxes.

The functional and partial derivatives of this Lagrangian are

$$\frac{\delta L}{\delta A^0(\mathbf{x})} \;=\; -\frac{1}{\mu_0} \nabla_{\mathbf{x}} \cdot \Big(\frac{1}{c}\partial_t \mathbf{A}(\mathbf{x}) \;+\; \nabla_{\mathbf{x}} A^0(\mathbf{x})\Big) \;-\; \sum_{k=1}^K q_k c \delta(\mathbf{x} - \mathbf{x}_k),$$

$$\frac{\delta L}{\delta A^i(\mathbf{x})} \;=\; -\frac{1}{\mu_0} \big(\partial_i \big(\nabla_{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x})\big) \;+\; \nabla_{\mathbf{x}}^2 A_i(\mathbf{x})\big) \;+\; \sum_{k=1}^K q_k \dot{x}_k^i \delta(\mathbf{x} - \mathbf{x}_k)$$

for $i \in \{1, 2, 3\}$,

$$\frac{\delta L}{\delta(\partial_t A^0(\mathbf{x}))} \;=\; 0,$$

$$\frac{\delta L}{\delta(\partial_t A^i(\mathbf{x}))} \;=\; \frac{1}{\mu_0 c}\Big(-\frac{1}{c}\partial_t A_i(\mathbf{x}) \;+\; \partial_i A^0(\mathbf{x})\Big),$$

$$\frac{\partial L}{\partial x_k^i} \;=\; q_k\big(-c \partial_i A^0(\mathbf{x}_k) \;+\; \partial_i \mathbf{A}(\mathbf{x}_k) \cdot \dot{\mathbf{x}}_k\big)$$

and

$$\frac{\partial L}{\partial \dot{x}_k^i} \;=\; -q_k A_i(\mathbf{x}_k) \;+\; \frac{m_k \dot{x}_k^i}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}}}.$$

The Euler-Lagrange equations of this system are

$$D_t \frac{\delta L(A(t,\bullet), \partial_t A(t,\bullet), x(t), \dot{x}(t))}{\delta(\partial_t A^0(\mathbf{x}))} \;=\; \frac{\delta L(A(t,\bullet), \partial_t A(t,\bullet), x(t), \dot{x}(t))}{\delta A^0(\mathbf{x})}$$

$$\iff \quad 0 \;=\; -\frac{1}{\mu_0} \nabla_{\mathbf{x}} \cdot \Big(\frac{1}{c}\partial_t \mathbf{A}(t,\mathbf{x}) \;+\; \nabla_{\mathbf{x}} A^0(t,\mathbf{x})\Big) \;-\; \sum_{k=1}^K q_k c \delta(\mathbf{x} - \mathbf{x}_k(t)),$$

$$D_t \frac{\delta L(A(t,\bullet), \partial_t A(t,\bullet), x(t), \dot{x}(t))}{\delta(\partial_t A^i(\mathbf{x}))} \;=\; \frac{\delta L(A(t,\bullet), \partial_t A(t,\bullet), x(t), \dot{x}(t))}{\delta A^i(\mathbf{x})}$$

$$\iff \quad \frac{1}{\mu_0 c}\partial_t\Big(-\frac{1}{c}\partial_t A_i(t,\mathbf{x}) \;+\; \partial_i A^0(t,\mathbf{x})\Big)$$

$$= \; -\frac{1}{\mu_0}\Big(\partial_i \big(\nabla_{\mathbf{x}} \cdot \mathbf{A}(t,\mathbf{x})\big) \;+\; \nabla_{\mathbf{x}}^2 A_i(t,\mathbf{x})\Big) \;+\; \sum_{k=1}^K q_k \dot{x}_k^i(t) \delta(\mathbf{x} - \mathbf{x}_k(t))$$

and

$$D_t \frac{\partial L(A(t,\bullet), \partial_t A(t,\bullet), x(t), \dot{x}(t))}{\partial \dot{x}_k^i} \;=\; \frac{\partial L(A(t,\bullet), \partial_t A(t,\bullet), x(t), \dot{x}(t))}{\partial x_k^i}$$

$$\iff \quad -q_k\big(\partial_t A_i(t,\mathbf{x}_k(t)) \;+\; \dot{\mathbf{x}}_k(t)\cdot\nabla_{\mathbf{x}} A_i(t,\mathbf{x}_k(t))\big) \;+\; D_t \frac{m_k \dot{x}_k^i(t)}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}$$

$$= \; q_k\big(- c\partial_i A^0(t,\mathbf{x}_k(t)) \;+\; \partial_i \mathbf{A}(t,\mathbf{x}_k(t))\cdot\dot{\mathbf{x}}_k(t)\big).$$

The Euler-Lagrange equations related to $\mathbf{A}$ can be rearranged into a form

$$- \partial_t\big(\partial_t \mathbf{A}(t,\mathbf{x}) \;+\; c\nabla_{\mathbf{x}} A^0(t,\mathbf{x})\big)$$

$$= \; c^2\nabla_{\mathbf{x}}\times\big(\nabla_{\mathbf{x}}\times\mathbf{A}(t,\mathbf{x})\big) \;-\; \sum_{k=1}^{K}\mu_0 c^2 q_k \dot{\mathbf{x}}_k(t)\delta(\mathbf{x}-\mathbf{x}_k(t)).$$

The Euler-Lagrange equations related to $\mathbf{x}_k$ can be rearranged into a form

$$D_t \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}$$

$$= \; q_k\Big(- \partial_t \mathbf{A}(t,\mathbf{x}_k(t)) - c\nabla_{\mathbf{x}} A^0(t,\mathbf{x}_k(t)) \;+\; \dot{\mathbf{x}}_k(t)\times\big(\nabla_{\mathbf{x}}\times\mathbf{A}(t,\mathbf{x}_k(t))\big)\Big).$$

The vacuum permittivity is related to the vacuum permeability and the speed of light according to the formula $\varepsilon_0 = \frac{1}{\mu_0 c^2}$. In this model the electric and magnetic fields can be considered to have been defined with formulas

$$\mathbf{E}(t,\mathbf{x}) \;=\; -\partial_t \mathbf{A}(t,\mathbf{x}) \;-\; c\nabla_{\mathbf{x}} A^0(t,\mathbf{x}) \quad \text{and}$$
$$\mathbf{B}(t,\mathbf{x}) \;=\; \nabla_{\mathbf{x}}\times\mathbf{A}(t,\mathbf{x}).$$

Using these quantities the Euler-Lagrange equations related to A^0 and $\mathbf{A}$ can be written as

$$\nabla_{\mathbf{x}}\cdot\mathbf{E}(t,\mathbf{x}) \;=\; \frac{1}{\varepsilon_0}\sum_{k=1}^{K} q_k \delta(\mathbf{x}-\mathbf{x}_k(t)) \qquad \text{and}$$

$$\partial_t \mathbf{E}(t,\mathbf{x}) \;=\; \frac{1}{\varepsilon_0\mu_0}\nabla_{\mathbf{x}}\times\mathbf{B}(t,\mathbf{x}) \;-\; \frac{1}{\varepsilon_0}\sum_{k=1}^{K} q_k \dot{\mathbf{x}}_k(t)\delta(\mathbf{x}-\mathbf{x}_k(t)),$$

that are two of the Maxwell's equations. The other two Maxwell's equations that are

$$\nabla_{\mathbf{x}}\cdot\mathbf{B}(t,\mathbf{x}) \;=\; 0 \qquad \text{and} \qquad \nabla_{\mathbf{x}}\times\mathbf{E}(t,\mathbf{x}) \;=\; -\partial_t \mathbf{B}(t,\mathbf{x})$$

are true even without the Euler-Lagrange equations. If we denote

$$\mathbf{F}_k(t) \;=\; D_t \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}},$$

the Euler-Lagrange equations related to $\mathbf{x}_k$ can be written as

$$\mathbf{F}_k(t) \;=\; q_k\big(\mathbf{E}(t,\mathbf{x}_k(t)) \;+\; \dot{\mathbf{x}}_k(t)\times\mathbf{B}(t,\mathbf{x}_k(t))\big)$$

that is the Lorentz force formula.

We see that our Lagrangian appears to be interesting, because it implies the basics of electromagnetism with point-like particles. The major problem with this model is that in the Lorentz force formula we assume that the fields would be well defined at locations $(t,\mathbf{x}_k(t))$, but eventually this assumption turns out to be false, which maybe means that this is all nonsense. We can try to study these formulas anyway.

The energy of this system is

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3}\frac{\delta L}{\delta(\partial_t A^\mu(\mathbf{x}))}\partial_t A^\mu(\mathbf{x})d^3x \;+\; \sum_{k=1}^{K}\frac{\partial L}{\partial\dot{x}_k^i}\dot{x}_k^i \;-\; L\\
&= \int\limits_{\mathbb{R}^3}\Big(\frac{\varepsilon_0}{2}\|\mathbf{E}(\mathbf{x})\|^2 \;+\; \frac{1}{2\mu_0}\|\mathbf{B}(\mathbf{x})\|^2 \;+\; \frac{1}{\mu_0 c}\mathbf{E}(\mathbf{x})\cdot\nabla_{\mathbf{x}}A^0(\mathbf{x})\Big)d^3x\\
&\quad+\;\sum_{k=1}^{K}\Bigg(q_k c A^0(\mathbf{x}_k) \;+\; \frac{m_k c^2}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}}}\Bigg).
\end{aligned}$$

It turns out that this formula can be simplified. If we apply integration by parts to the term $\frac{1}{\mu_0 c}\mathbf{E}(\mathbf{x})\cdot\nabla_{\mathbf{x}}A^0(\mathbf{x})$, and then use one of the Euler-Lagrange equations of this system, this term cancels out, and we are left with an energy formula

$$\frac{1}{2}\int\limits_{\mathbb{R}^3}\Big(\varepsilon_0\|\mathbf{E}(\mathbf{x})\|^2 \;+\; \frac{1}{\mu_0}\|\mathbf{B}(\mathbf{x})\|^2\Big)d^3x \;+\; \sum_{k=1}^{K}\frac{m_k c^2}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}}}.$$

At this point we face some difficulties in deciding that which formulas would best describe the energy of this system. If somebody asks that what is the system's energy $E(t)$, it looks like that we can reasonably start by answering that the energy is

$$E(t) \;=\; \int\limits_{\mathbb{R}^3}\mathcal{E}(t,\mathbf{x})d^3x \;+\; \sum_{k=1}^{K}E_k(t),$$

where $\mathcal{E}(t,\mathbf{x})$ is the energy density of the field, and where $E_1(t),E_2(t),\ldots,$ $E_K(t)$ are the energies of the particles. However, there are two different options to what the energy density and the energies of the particles can be. The first option is that we define the energy density to be

$$\mathcal{E}_{\rm a}(t,\mathbf{x}) \;=\; \frac{1}{2}\Big(\varepsilon_0\big\|\mathbf{E}(t,\mathbf{x})\big\|^2 \;+\; \frac{1}{\mu_0}\big\|\mathbf{B}(t,\mathbf{x})\big\|^2\Big) \;+\; \frac{1}{\mu_0 c}\mathbf{E}(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}A^0(t,\mathbf{x}),$$

and the particle energies to be

$$E_{\mathrm{a},k}(t) \;=\; q_k c A^0(t, \mathbf{x}_k(t)) \;+\; \frac{m_k c^2}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}.$$

The second option is that we define the energy density to be

$$\mathcal{E}_{\mathrm{b}}(t, \mathbf{x}) \;=\; \frac{1}{2}\Big(\varepsilon_0 \big\|\mathbf{E}(t, \mathbf{x})\big\|^2 \;+\; \frac{1}{\mu_0}\big\|\mathbf{B}(t, \mathbf{x})\big\|^2\Big),$$

and the particle energies to be

$$E_{\mathrm{b},k}(t) \;=\; \frac{m_k c^2}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}.$$

When we have two options like this, you might be interested to know if we could somehow decide which one of these options is the right one. It turns out that making a such decision is extremely difficult. An obvious observation is that the second option appears to use formulas that look simpler, which could be a reason to favor them, but this doesn't mean that we should judge the first option as somehow incorrect. We should recognize the fact that both of these options still use the same equations of motion, which means that they produce the same predictions for empirical tests. It will not be possible to device an experiment that would tell which one of the options is the right one. One argument in favor of the second option is that the energy density and the particle energies should not depend directly on the vector potential A^μ, because it does not have a unique time evolution due to the gauge issue. We can criticize this argument by noting that since there does not exist an experimental setup that could be used to measure the pointwise values of the energy density or the individual true particle energies, then maybe there is nothing wrong with them directly depending on A^μ. One argument in favor of the first option is that since its formulas came directly from the generic expression for energy $\pi \cdot \dot{x} - L$ without any extra modifications, it is simple in that sense. This reminds us of the problem with Occam's razor that different people can have different opinions about what is simple and what is not. Another argument in favor of the first option is that actually the second option doesn't give correct energies to all states $(A, \partial_t A, x, \dot{x})$, but only to those states that satisfy the one Euler-Lagrange equation, which makes that energy formula slightly suspicious. The used Euler-Lagrange equation did not have a second order time derivative in it, so it imposes a constraint on the vector $(A, \partial_t A, x, \dot{x})$. We can criticize this argument by noting that maybe the energies don't matter for states $(A, \partial_t A, x, \dot{x})$ that don't satisfy the Euler-Lagrange equation. Smart people hopefully recognize that these are all philosophical arguments, and it is maybe impossible to determine which one them would be correct purely in light of some mathematical facts. It is a little strange that we cannot even know the true energies of the particles, but this is how it seems to be.

Obviously these two options should not be mixed. For example, if one uses the energy density $\mathcal{E}_{\mathrm{b}}(t,\mathbf{x})$ together with the particle energies $E_{\mathrm{a},k}(t)$, then one will get incorrect results.

Let's have a look at the momentum of this system. In order to apply Noether's theorem, we must fix some vector $\mathbf{u} \in \mathbb{R}^3$, and then define a translation transformation T_α, where $\alpha \in \mathbb{R}$ is some parameter, according to formulas

$$\big(T_\alpha(A^\mu)\big)(\mathbf{x}) \;=\; A^\mu(\mathbf{x}-\alpha\mathbf{u}) \qquad \text{and} \qquad T_\alpha \mathbf{x}_k \;=\; \mathbf{x}_k + \alpha\mathbf{u}.$$

The derivative of the transformed vector potential with respect to α at the location $\alpha = 0$ is

$$\big(D_\alpha T_0(A^\mu)\big)(\mathbf{x}) \;=\; -\mathbf{u}\cdot\nabla_{\mathbf{x}} A^\mu(\mathbf{x}).$$

According to Noether's theorem the momentum of the system in the direction $\mathbf{u}$ is the quantity

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3} \big(D_\alpha T_0(A)\big)^\mu(\mathbf{x}) \frac{\delta L}{\delta(\partial_t A^\mu(\mathbf{x}))} d^3x \;+\; \sum_{k=1}^{K} (D_\alpha T_0 x_k^i) \frac{\partial L}{\partial \dot{x}_k^i} \\
&= \; -u^j \frac{1}{\mu_0 c} \int\limits_{\mathbb{R}^3} \big(\partial_j A^i(\mathbf{x})\big)\Big(-\frac{1}{c}\partial_t A_i(\mathbf{x}) \;+\; \partial_i A^0(\mathbf{x})\Big) d^3x \\
&\quad + \sum_{k=1}^{K}\sum_{i=1}^{3} u^i \Big(-q_k A_i(\mathbf{x}_k) \;+\; \frac{m_k \dot{x}_k^i}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}}}\Big).
\end{aligned}$$

We can remove $\mathbf{u}$ and conclude that the momentum vector is

$$\begin{aligned}
\mathbf{P}(t) \;&=\; -\varepsilon_0 \int\limits_{\mathbb{R}^3} \big(\nabla_{\mathbf{x}} A_i(t,\mathbf{x})\big) E^i(t,\mathbf{x}) d^3x \\
&\qquad + \sum_{k=1}^{K} \Big(q_k \mathbf{A}(t,\mathbf{x}_k(t)) \;+\; \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}\Big).
\end{aligned}$$

Equation

$$\big(\mathbf{E}(t,\mathbf{x})\times\mathbf{B}(t,\mathbf{x})\big)^j \;=\; \mathbf{E}(t,\mathbf{x})\cdot\partial_j \mathbf{A}(t,\mathbf{x}) \;-\; \mathbf{E}(t,\mathbf{x})\cdot\nabla_{\mathbf{x}} A^j(t,\mathbf{x})$$

is true for all $j \in \{1,2,3\}$. This means that we can alternatively write the momentum as

$$\mathbf{P}(t) \;=\; \varepsilon_0 \int\limits_{\mathbb{R}^3} \big(\mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) \;+\; (\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}})\mathbf{A}(t,\mathbf{x})\big) d^3x$$
$$+ \sum_{k=1}^{K} \Big(q_k \mathbf{A}(t,\mathbf{x}_k(t)) \;+\; \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}\Big).$$

We see that now we can simplify this similarly as the energy. If we apply integration by parts to the term $(\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}})\mathbf{A}(t,\mathbf{x})$, and then use the one Euler-Lagrange equation of this system, this term cancels out, and we are left with a momentum formula

$$\mathbf{P}(t) \;=\; \varepsilon_0 \int\limits_{\mathbb{R}^3} \mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) d^3x \;+\; \sum_{k=1}^{K} \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}.$$

Now we face a similar problem as what we faced with the energy. If somebody asks that what is the system's momentum $\mathbf{P}(t)$, it looks like that we can reasonably start by answering that the momentum is

$$\mathbf{P}(t) \;=\; \int\limits_{\mathbb{R}^3} \boldsymbol{\mathcal{P}}(t,\mathbf{x}) d^3x \;+\; \sum_{k=1}^{K} \mathbf{p}_k(t),$$

where $\boldsymbol{\mathcal{P}}(t,\mathbf{x})$ is the momentum density of the field, and where $\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots, \mathbf{p}_K(t)$ are the momenta of the particles. However, there are two different options to what the momentum density and the momenta of the particles can be. The first option is that we define the momentum density to be

$$\boldsymbol{\mathcal{P}}_{\mathrm{a}}(t,\mathbf{x}) \;=\; \varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) \;+\; \varepsilon_0 \mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}} \mathbf{A}(t,\mathbf{x}),$$

and the particle momenta to be

$$\mathbf{p}_{\mathrm{a},k}(t) \;=\; q_k \mathbf{A}(t,\mathbf{x}_k(t)) \;+\; \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}.$$

The second option is that we define the momentum density to be

$$\boldsymbol{\mathcal{P}}_{\mathrm{b}}(t,\mathbf{x}) \;=\; \varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}),$$

and the particle momenta to be

$$\mathbf{p}_{\mathrm{b},k}(t) \;=\; \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}.$$

We face the same philosophical arguments as with the energy. It is a little strange that we cannot know the true momenta of the particles, but this is how it seems to be.

Let's have a look at the angular momentum of this system. In order to apply Noether's theorem, we must fix some vector $\mathbf{u} \in \mathbb{R}^3$, and then define a rotation transformation R_α according to formulas

$$\begin{aligned} \big(R_\alpha(A^0)\big)(\mathbf{x}) &= A^0(e^{-\alpha \mathbf{u}\times}\mathbf{x}), \\ \big(R_\alpha(\mathbf{A})\big)(\mathbf{x}) &= e^{\alpha \mathbf{u}\times}\mathbf{A}(e^{-\alpha \mathbf{u}\times}\mathbf{x}) \quad \text{and} \\ R_\alpha \mathbf{x}_k &= e^{\alpha \mathbf{u}\times}\mathbf{x}_k. \end{aligned}$$

The derivatives of the transformed vector potential with respect to α at the location $\alpha = 0$ are

$$\begin{aligned} \big(D_\alpha R_0(A^0)\big)(\mathbf{x}) &= -\mathbf{u}\cdot(\mathbf{x}\times\nabla_{\mathbf{x}})A^0(\mathbf{x}) \quad \text{and} \\ \big(D_\alpha R_0(\mathbf{A})\big)(\mathbf{x}) &= \mathbf{u}\times\mathbf{A}(\mathbf{x}) \;-\; \big(\mathbf{u}\cdot(\mathbf{x}\times\nabla_{\mathbf{x}})\big)\mathbf{A}(\mathbf{x}). \end{aligned}$$

According to Noether's theorem the angular momentum of the system in the direction $\mathbf{u}$ is the quantity

$$\begin{aligned} &\int_{\mathbb{R}^3} \big(D_\alpha R_0(A)\big)^\mu(\mathbf{x})\frac{\delta L}{\delta(\partial_t A^\mu(\mathbf{x}))}d^3x \;+\; \sum_{k=1}^{K}(D_\alpha R_0 x_k^i)\frac{\partial L}{\partial \dot{x}_k^i} \\ &= \mathbf{u}\cdot\Bigg(\varepsilon_0\int_{\mathbb{R}^3}\Big(\mathbf{E}(\mathbf{x})\times\mathbf{A}(\mathbf{x}) \;-\; \big((\mathbf{x}\times\nabla_{\mathbf{x}})A_i(\mathbf{x})\big)E^i(\mathbf{x})\Big)d^3x \\ &\qquad + \sum_{k=1}^{K}\mathbf{x}_k\times\Big(q_k\mathbf{A}(\mathbf{x}_k) \;+\; \frac{m_k\dot{\mathbf{x}}_k}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k\|^2}{c^2}}}\Big)\Bigg) \end{aligned}$$

We can remove $\mathbf{u}$ and conclude that the angular momentum vector is

$$\begin{aligned} \mathbf{L}(t) &= \varepsilon_0\int_{\mathbb{R}^3}\Big(\mathbf{E}(t,\mathbf{x})\times\mathbf{A}(t,\mathbf{x}) \;-\; \big((\mathbf{x}\times\nabla_{\mathbf{x}})A_i(t,\mathbf{x})\big)E^i(t,\mathbf{x})\Big)d^3x \\ &\quad + \sum_{k=1}^{K}\mathbf{x}_k(t)\times\Big(q_k\mathbf{A}(t,\mathbf{x}_k(t)) \;+\; \frac{m_k\dot{\mathbf{x}}_k(t)}{\sqrt{1-\frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}\Big). \end{aligned}$$

Equation

$$\begin{aligned} -\big((\mathbf{x}\times\nabla_{\mathbf{x}})A_i(t,\mathbf{x})\big)E^i(t,\mathbf{x}) &= \mathbf{x}\times\big(\mathbf{E}(t,\mathbf{x})\times\mathbf{B}(t,\mathbf{x})\big) \\ &\quad + \mathbf{x}\times\big(\mathbf{E}(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}\big)\mathbf{A}(t,\mathbf{x}) \end{aligned}$$

is true. This is not obvious at a glance, but if one studies the right side carefully, many terms cancel, and what remains is the same as that on the left side. This means that we can alternatively write the angular momentum as

$$\mathbf{L}(t) \;=\; \varepsilon_0 \int\limits_{\mathbb{R}^3} \Big(\mathbf{E}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x}) + \mathbf{x} \times \big(\mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) \big) \;+\; \mathbf{x} \times \big(\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}} \big) \mathbf{A}(t,\mathbf{x}) \Big) d^3x + \sum_{k=1}^{K} \mathbf{x}_k(t) \times \Big(q_k \mathbf{A}(t,\mathbf{x}_k(t)) \;+\; \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}} \Big).$$

We see that now we can simplify this similarly as the energy and the momentum. If we apply integration by parts to the term $\mathbf{x} \times (\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}}) \mathbf{A}(t,\mathbf{x})$, and then use the one Euler-Lagrange equation of this system, some cancelling happens, and we are left with an angular momentum formula

$$\mathbf{L}(t) \;=\; \varepsilon_0 \int\limits_{\mathbb{R}^3} \mathbf{x} \times \big(\mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) \big) d^3x \;+\; \sum_{k=1}^{K} \mathbf{x}_k(t) \times \frac{m_k \dot{\mathbf{x}}_k(t)}{\sqrt{1 - \frac{\|\dot{\mathbf{x}}_k(t)\|^2}{c^2}}}.$$

Now we face a similar problem as what we faced with the energy and the momentum. If somebody asks that what is the system's angular momentum $\mathbf{L}(t)$, we can give two different answers. The first option is that we say that the angular momentum is

$$\mathbf{L}(t) \;=\; \int\limits_{\mathbb{R}^3} \big(\boldsymbol{\mathcal{S}}(t,\mathbf{x}) \;+\; \mathbf{x} \times \boldsymbol{\mathcal{P}}_{\mathrm{a}}(t,\mathbf{x}) \big) d^3x \;+\; \sum_{k=1}^{K} \mathbf{x}_k(t) \times \mathbf{p}_{\mathrm{a},k}(t),$$

where

$$\boldsymbol{\mathcal{S}}(t,\mathbf{x}) \;=\; \varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x})$$

is the intrinsic angular momentum density of electromagnetic field, $\boldsymbol{\mathcal{P}}_{\mathrm{a}}(t,\mathbf{x})$ is the momentum density of the field defined earlier above, and $\mathbf{p}_{\mathrm{a},k}(t)$ are the momenta of the particles defined earlier above.

The second option is that we say that the angular momentum is

$$\mathbf{L}(t) \;=\; \int\limits_{\mathbb{R}^3} \mathbf{x} \times \boldsymbol{\mathcal{P}}_{\mathrm{b}}(t,\mathbf{x}) d^3x \;+\; \sum_{k=1}^{K} \mathbf{x}_k(t) \times \mathbf{p}_{\mathrm{b},k}(t),$$

where $\boldsymbol{\mathcal{P}}_{\mathrm{b}}(t,\mathbf{x})$ is the momentum density of the field defined earlier above, and $\mathbf{p}_{\mathrm{b},k}(t)$ are the momenta of the particles defined earlier above. In this second option there is no intrinsic angular momentum density.

If we assume that our philosophy and logic has been valid above, we can continue using the same philosophy and logic here, and conclude that we cannot know which one of these two formulas for angular momentum

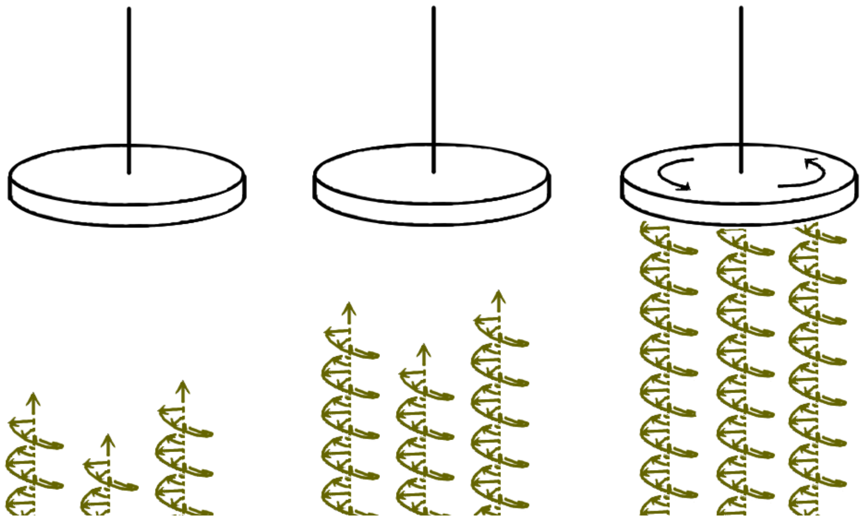

Figure 1: A simplified representation of the R. A. Beth's experiment from 1936. Circularly polarized light hits a solid object, and the solid object turns a little, apparently revealing that the circularly polarized light carries intrinsic angular momentum.

would be more correct than the other. **We obtain a crazy result that actually we cannot know whether the intrinsic angular momentum of classical electromagnetic field exists or not.**

Again, the found two options should not be mixed. For example, if one uses the intrinsic angular momentum density $\boldsymbol{\mathcal{S}}(t,\mathbf{x})$ together with the momentum density $\boldsymbol{\mathcal{P}}_{\mathrm{b}}(t,\mathbf{x})$ and the particle momenta $\mathbf{p}_{\mathrm{b},k}(t)$, then one will get incorrect results.

The existence of the quantity $\varepsilon_0\mathbf{E}\times\mathbf{A}$ is known among the mainstream physics community, but there seems to be confusion about how it should be interpreted. Many people seem to believe that this term would somehow be related to quantum theory, but no clear justification for this belief can be found.

Let's take a look at how the angular momentum density $\boldsymbol{\mathcal{S}}(t,\mathbf{x})$ seems to work in a light of a simple example. Let's define an electromagnetic field by formulas

$$\mathbf{E}(t,\mathbf{x}) \;=\; E_0\begin{pmatrix}\cos\big(k(ct-x^3)\big)\\ 0\\ 0\end{pmatrix}$$

and

$$\mathbf{B}(t,\mathbf{x}) \;=\; \frac{E_0}{c}\begin{pmatrix} 0 \\ \cos\big(k(ct-x^3)\big) \\ 0 \end{pmatrix},$$

where $E_0 \in \mathbb{R}$ and $k \in \mathbb{R}$ are some constants. These formulas describe a linearly polarized plane wave that travels in the direction of z-axis. A natural choice for a vector potential that generates this electromagnetic field is

$$A^0(t,\mathbf{x}) \;=\; 0 \qquad \text{and} \qquad \mathbf{A}(t,\mathbf{x}) \;=\; \frac{E_0}{kc}\begin{pmatrix} -\sin\big(k(ct-x^3)\big) \\ 0 \\ 0 \end{pmatrix}.$$

Now

$$\boldsymbol{\mathcal{S}}(t,\mathbf{x}) \;=\; \varepsilon_0\mathbf{E}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x}) \;=\; \mathbf{0}$$

so according to our formula there is no intrinsic angular momentum present. With this choice of $\mathbf{A}(t,\mathbf{x})$ also

$$(\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}})\mathbf{A}(t,\mathbf{x}) \;=\; \mathbf{0}$$

is true, which is nice. What happens if we instead define an electromagnetic field by formulas

$$\mathbf{E}(t,\mathbf{x}) \;=\; E_0\begin{pmatrix} \cos\big(k(ct-x^3)\big) \\ \sin\big(k(ct-x^3)\big) \\ 0 \end{pmatrix}$$

and

$$\mathbf{B}(t,\mathbf{x}) \;=\; \frac{E_0}{c}\begin{pmatrix} -\sin\big(k(ct-x^3)\big) \\ \cos\big(k(ct-x^3)\big) \\ 0 \end{pmatrix}?$$

These formulas describe a circularly polarized plane wave that travels in the direction of z-axis. A natural choice for a vector potential that generates this electromagnetic field is

$$A^0(t,\mathbf{x}) \;=\; 0 \qquad \text{and} \qquad \mathbf{A}(t,\mathbf{x}) \;=\; \frac{E_0}{kc}\begin{pmatrix} -\sin\big(k(ct-x^3)\big) \\ \cos\big(k(ct-x^3)\big) \\ 0 \end{pmatrix}.$$

Now

$$\boldsymbol{\mathcal{S}}(t,\mathbf{x}) \;=\; \varepsilon_0\mathbf{E}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x}) \;=\; \frac{\varepsilon_0 E_0^2}{kc}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so according to our formula there is a non-trivial intrinsic angular momentum present. With this choice of $\mathbf{A}(t,\mathbf{x})$ also

$$(\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}})\mathbf{A}(t,\mathbf{x}) \;=\; \mathbf{0}$$

is true, which is nice. We see that the expression $\varepsilon_0 \mathbf{E} \times \mathbf{A}$ assigns intrinsic angular momentum correctly to circularly polarized light and not to linearly polarized light, which is in agreement with the Beth's experimental result. It is unlikely that we got a result like this by pure chance, so there seems to be some truth to the expression $\varepsilon_0 \mathbf{E} \times \mathbf{A}$.

This example produces a paradox: If it is true that the two angular momentum formulas, of which one has the intrinsic angular momentum term, and the other one does not, are both equivalent, then how is it possible that one of them seems to explain the Beth's experimental result, and the other one does not? The solution to this paradox is to notice that the derivation of the angular momentum formula that doesn't have the intrinsic angular momentum term required integration by parts, and integration by parts doesn't work with plane waves. So even though we cannot tell whether the intrinsic angular momentum exists in reality or not, we can tell that if you want to use plane waves because you believe that they are a good tool to approximate reality, then with those you should use the angular momentum formula with the intrinsic angular momentum term.