The Rational Planck-Scale Number Line; An Absolute Unitless Real Number Line Does Not Exist

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Abstract

The purpose for a number line is inextricably linked with measurement. The concept of "measurement" must include a unit length to have a notion of measure with the assigned unit. For example, a right triangle with sides $1: 1: \sqrt{2}$ has a unit length of 1 for a side. But this can be recast with the hypotenuse assigned a unit length so that the other sides are said to have irrational measure. However, not all scales result in irrational numbers. The only rational number line is the Planck-length scale number line. Here, every possible length is countably measurable. Any unit measurement scale which seeks a number with a precision below the Planck length is unmeasurable and meaningless. For example, irrational number measures with an infinite decimal expansion are meaningless below the Planck-length precision. Accordingly, an absolute unitless real number line does not exist. Therefore, the "cardinality of the real numbers" and the Continuum Hypothesis are meaningless.

I. Background

There is a very interesting and short (5:32) YouTube video, called "Why Irrational Numbers Don't Make Sense"[1] by Shirley from Learning0to1. For background purposes and later discussion, we mention some observations made in the video here:

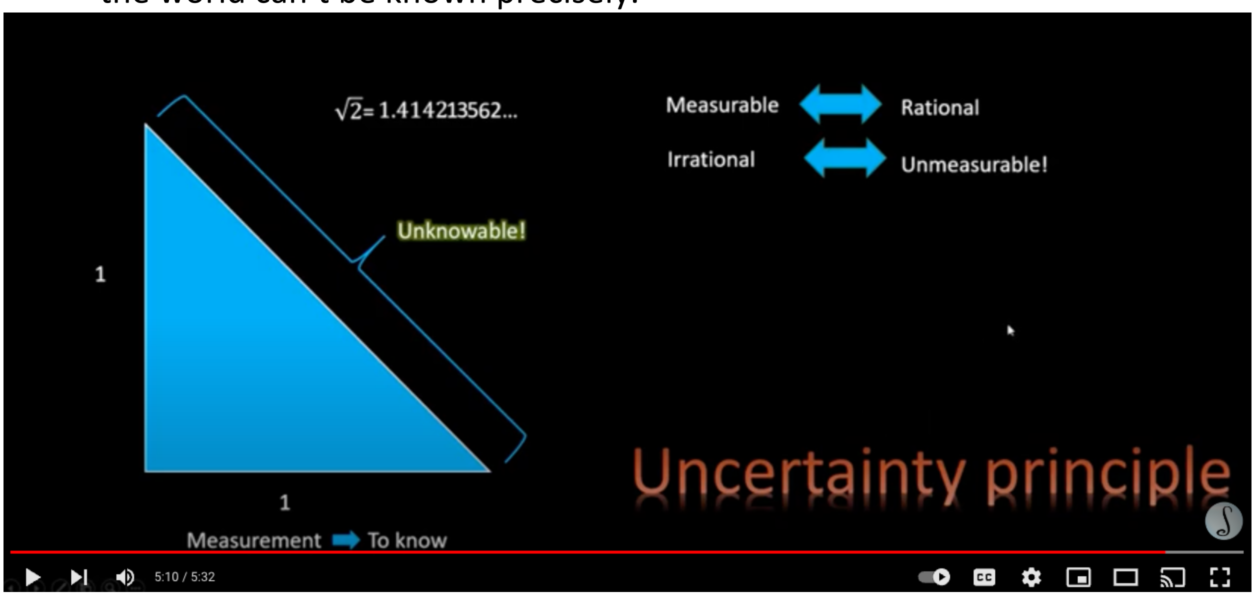
- Measurements are made using some standard unit of measure (*i.e.*, for 1 assigned unit).
- Some measurements can be made using ratios of a countable number of these units, and these are rational measurements.
- Some measurements cannot be made using ratios of a countable number of these units, and these are irrational.

The video concludes with the following text, starting at 4:02. The subsequent screen shot is taken at 5:10.

Being measurable is related to being a rational number. So, if it's irrational, then it means it can't be measured. So, if its irrational, then it means it's unmeasurable - because how can it be measurable if you can't even quote how many units it occupies. It has nothing to do with how you do the measurement. Hence eventually, irrational numbers mean something unmeasurable. But measurement is our critical tool to know the world. So, if the length is unmeasurable, that means it's unknowable.

A length of square root of 2 can be approximated with greater and greater accuracy. But you never know exactly what it is. Each decimal place is unpredictable, until you actually compute it. But, how come we see the length exists in front of us, yet there is no way of knowing.

It reminds me of the uncertainty principle in quantum mechanics - which says that the essential property of an electron, it's position and momentum (or simply velocity) can't be known precisely. It's pretty interesting that both math reasoning and physics seem to suggest that the world can't be known precisely.



II. Introduction

The YouTube video discussed in the previous section captures some very important observations about numbers, measurement, and what is knowable. In this paper, we expand upon those observations.

In the following sections we show that, 1) measurements require a finite-length unit scale, 2) the purpose/basis for having a number line is inextricably linked with measurement, 3) the Planck-length scale is the smallest possible measurement scale and the only rational measurement scale, 4) no measurement scale can have a meaningful precision below the Planck-length - since this is unmeasurable, 5) an absolute unitless real number line does not exist, 6) the cardinality of the real numbers and the continuum hypothesis are meaningless.

III. Measurements and Finite Length Unit Scales

The YouTube video shows that comparisons of lengths, such as those of a right triangle, are done using comparisons (or ratios) of sides, where a measurement unit is adopted. Alternatively, the length of a side can be made with the measurement unit itself.

Measurements made using numbers and the number line must adopt a unit scale, such as meter, inch, etc. Numbers and measurements are only relatable to each other by this unit scale.

Key Point 1: Numbers and measurement lengths are only relatable by the use of a unit length for a measurement scale. In other words, measurements must be made using units, and these units determine what can be measured.

Drawing any line on a piece of paper and designating 0 and 1 on a linear scale assigns it an implied unit length against which things can be measured; it is a "ruler" with its own assigned unit measure.

Key Point 2: The purpose/basis for having a number line is inextricably linked with measurement.

Consider a right triangle with sides 1 meter (m) and hypotenuse $\sqrt{2}$ m. Suppose a different measurement scale is adopted where this length ($\sqrt{2}$ m) is assigned a unit called "sqrt2", so that the hypotenuse has a length of 1 sqrt2. In this measurement scale, it is the side lengths of the right triangle that are irrational; they do not have a countable number of units of measure relative to 1 "sqrt2". In fact, in the scale of sqrt2, all rational lengths measured in meters, besides 0 m, are irrational (and some irrational numbers are now rational). It might seem that a specific length may always be considered to be either rational or irrational depending on the unit scale used.

However, math and science have consilience when it comes to measurement and number lines. Science shows that the smallest possible length is the Planck length, which is very small. We will consider a Planck-length number line to be a linear number line with the Planck-length as a unit. Every possible measurement length is measurable by a whole number of these units.

Key Point 3: The Planck-length unit scale provides for the only rational number line.

The Planck length scale has the ultimate precision possible in the universe. If a different unit scale, say one using the meter, is used in an attempt to specify an irrational number for a measurement, then this measurement eventually becomes rational below the Planck length scale. In other words, a decimal precision of meters becomes meaningless when it has a precision below the Planck length.

Key Point 4: Lengths can only have a precision down to the Planck length. Ultimately, all lengths have only rational measure.

In the appendix, we provide a different method to show that “irrational” numbers do not exist.

Since mathematics that involve the number line and geometry require the assignment of a measurement scale, then:

Key Point 5: An absolute unitless real number line does not exist.

The use of the Planck-Scale number line automatically resolves some paradoxes, such as Zeno’s well-known paradox of the tortoise and the hare. The Planck-length scale shows that the tortoise and hare can only traverse a whole number of Planck-lengths per unit time. Eventually, the faster moving tortoise will overtake the hare since only discrete measures are involved. So, there is no need to discuss a continuum of infinitely divisible lengths. Many other paradoxes of the infinite (including the continuum hypothesis – which presume that the size of the real numbers is larger than that of the natural numbers) disappear along with the irrationals.

Although calculus assumes that a continuous real number line exists, the concept of integrating over the number line is still valid. The discrete Planck-length unit is so small that it is practically infinitesimal (although it is really finite). The number line “appears” to be continuous in human-scale measurement and any measurement below the Planck-length is unknowable anyway. Ultimately, things can only be known to a certain precision in the universe – and it makes no sense to pretend otherwise.

V. Conclusion

The abstract concept of the counting numbers are very different from the number line and geometry. In this paper, we show that the use of the number line and measurement requires the use of a finite-length unit scale. The Planck-length scale is the smallest possible unit scale and is the only rational scale. Any other measurement scale which attempts to define a precision below the Planck-length becomes meaningless. We conclude that the absolute unitless real number line used in pure mathematics does not exist.

VI. Appendix

Here we provide another way to show that the “real” numbers can only consist of rational numbers, and that every irrational is not-a-number (NAN) if it is considered to have an infinite precision.

An integer is finite and is represented in decimal notation by a finite number of digits (with all leading 0s removed). An important observation is that any infinite string of digits, which is not finite after removing all leading 0s, is indistinguishable from any other such string in terms of size.

So, a string of digits which are all 1s:
…11111111111111….
is indistinguishable in terms of size from a string of digits which are all 9’s:
…99999999999999…

Such infinite strings of digits do not represent a number and will be called Not-A-Number (NAN). Now we show that every irrational is NAN since it is equivalent to an infinite string of digits divided by another infinite string of digits.

An irrational number such as pi (*i.e.*, π) cannot be expressed as the ratio of integers, but is supposedly represented by an infinite string of non-repeating digits: 3.14159265…

The following sequence will eventually be more accurate than any Cauchy sequence[2]:

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \quad 3.141592, \quad 3.1415926, \quad 3.14159265,$$

Or

$$\frac{3}{1}, \quad \frac{31}{10}, \quad \frac{314}{100}, \quad \frac{3141}{1000}, \quad \frac{31415}{10000}, \quad \frac{314159}{100000}, \quad \frac{3141592}{1000000}, \quad \frac{31415926}{10000000}, \quad \frac{314159265}{100000000},$$

In the limit as the sequence goes to infinity, the numerator and denominator become an infinite sequence of digits, which is meaningless. We cannot even say if the numerator or denominator is larger than the other.

Accordingly, irrationals are NAN and the "real" numbers contain only rational numbers.

We can also show that the size of a hypothetically complete infinite is NAN. Suppose that there was a size of a complete "infinite". This means that we could take an infinite string of digits, which is not finite after removing all leading 0s, and assign a size to it. But we have already shown that this is NAN.

The concept of a complete "infinity" has been discussed by Norman J. Wildberger:[3]

> The idea of `infinity' as an unattainable ideal that can only be approached by an endless sequence of better and better finite approximations is both humble and ancient, and one I would strongly advocate to those wishing to understand

mathematics more deeply. This is the position that Archimedes, Newton, Euler and Gauss would have taken, and it is a view that ought to be seriously reconsidered.

> Why is any of this important? The real numbers are where Cantor's hierarchies of infinities begins, and much of modern set theory rests, so this is an issue with widespread consequences, even within algebra and combinatorics…

He also refers to what Gauss said about a complete infinity, prior to Cantor's time: [4]

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction.

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[1] "Why Irrational Numbers Don't Make Sense." *YouTube*, YouTube, 18 June 2022, https://www.youtube.com/watch?v=dwfdq8cCL9w

[2] More precisely, it will eventually be less than any epsilon used in the formal definition of a limit.

[3] Njwildberger tangential thoughts, "Difficulties with Real Numbers.", Jan. 20 2013, https://njwildberger.com/2012/12/02/difficulties-with-real-numbers/. Accessed Nov. 27, 2022

[4] Njwildberter "Set Theory: Should You Believe?", "https://web.maths.unsw.edu.au/~norman/views2.htm

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