LAGRANGIAN APPROACH TO DERIVING THE GRAVITY EQUATIONS FOR A 3D-BRANE UNIVERSE.

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Abstract. Recently some arguments were suggested saying that our universe should be considered as a three-dimensional brane equipped with a Riemannian metric depending on the cosmological time. These arguments are based on the concept of temporal coexistence applied to events in the four-dimensional spacetime. Relying on the three-dimensional brane presentation of our universe, the four-dimensional Einstein's gravity equations were rewritten as three-dimensional time-evolution equations for the three-dimensional metric presenting the gravitational field in the framework of this three-dimensional brane model. In the present paper some of these time-evolution equations for the three-dimensional metric are rederived using the Lagrangian approach.

1. Introduction.

Subdivision of the four-dimensional spacetime into the three-dimensional space and one-dimensional time is not new in cosmology. One can see it in the Friedmann-Robertson-Walker metric[1] (see Section 1.1.3 in [1]):

$$ds^2=(dx^0)^2-R(x^0)^2\left(\frac{dr^2}{1-K\,r^2}+r^2\,d\theta^2+r^2\,\sin^2\theta\,d\phi^2\right). \tag{1.1}$$

The metric (1.1) belongs to the class of block-diagonal metrics with the direct and inverse metric tensors of the form

$$G_{ij}=\begin{Vmatrix}1&0&0&0\\0&-g_{11}&-g_{12}&-g_{13}\\0&-g_{21}&-g_{22}&-g_{23}\\0&-g_{31}&-g_{32}&-g_{33}\end{Vmatrix},\qquad G^{ij}=\begin{Vmatrix}1&0&0&0\\0&-g^{11}&-g^{12}&-g^{13}\\0&-g^{21}&-g^{22}&-g^{23}\\0&-g^{31}&-g^{32}&-g^{33}\end{Vmatrix}. \tag{1.2}$$

Local presentation of the form (1.2) in some local coordinates is applicable to any pseudo-Euclidean metric of general relativity and cosmology (see § 97 of Chapter XI in [2]). The 3D-brane universe theory from [3] prescribes the form (1.2) to the four-dimensional metric **G** globally (see also [4] and [5]). The quantities g_{ij} and g^{ij} in [3] are understood as the components of a time-dependent three-dimensional Euclidean

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[1] Here $x^0=c\,t$, where c is the speed of light and t is the cosmological time (see [6]). The other three coordinates are $x^1=r$, $x^2=\theta$, $x^3=\phi$. There are three options for the constant K: $K=1$, $K=0$, and $K=-1$ corresponding to the closed spherical universe, to the open flat universe, and to the open hyperbolic universe respectively.

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metric $\mathbf{g}$. The time derivative of the metric $\mathbf{g}$ with respect to the cosmological time t (see [6]) determines a separate tensor field $\mathbf{b}$ according to the formula

$$\frac{\dot{g}_{ij}}{2\,c}=b_{ij}. \tag{1.3}$$

Both g_{ij} and b_{ij} obey the following set of differential equations:

$$\begin{gathered}\frac{\dot{b}_{ij}}{c}-\sum_{k=1}^{3}\frac{\dot{b}^k_k}{c}\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\\ -\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{8\,\pi\,\gamma}{c^4}\,T_{ij},\end{gathered} \tag{1.4}$$

$$\sum_{k=1}^{3}\nabla_k\,b^k_j-\sum_{k=1}^{3}\nabla_j\,b^k_k=\frac{8\,\pi\,\gamma}{c^4}\,T_{0j}, \tag{1.5}$$

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\frac{R}{2}-\Lambda=\frac{8\,\pi\,\gamma}{c^4}\,T_{00}. \tag{1.6}$$

The equations (1.4), (1.5), (1.6) were derived in [3] from Einstein's gravity equations

$$r_{ij}-\frac{r}{2}\,G_{ij}-\Lambda\,G_{ij}=\frac{8\,\pi\,\gamma}{c^4}\,T_{ij}, \tag{1.7}$$

where γ is Newton's gravitational constant (see [7]):

$$\gamma\approx\ 6.674\cdot 10^{-8}\ cm^3\,\cdot\,g^{-1}\,\cdot\,s^{-2}$$

and c is the speed of light. As a result of redefining other standard units in 2019, the speed of light c now is defined as an exact physical constant (see [8]):

$$c=2.99792458\cdot 10^{10}\ cm\,\cdot\,s^{-1}.$$

The constant Λ in (1.7) is the cosmological constant. It is associated with the dark energy (see [9]). Its value is quite uncertain, but it is very close to zero (see [10]):

$$\Lambda\approx\ 10^{-56}\ cm^{-2}.$$

The term T_{ij} in the right hand side of (1.7) stands for the matter including the dark matter (see [11]) and the regular matter. It represents the components of a symmetric tensor $\mathbf{T}$ which is called the energy-momentum tensor (see [12]). The components of $\mathbf{T}$ are presented in the right hand sides of the equations (1.4), (1.5), and (1.6) as well.

The term r_{ij} in (1.7) corresponds to the components of the four-dimensional Ricci tensor $\mathbf{r}$ and r is the four-dimensional scalar curvature (see § 8 in Chapter IV of [13]). Similarly, R_{ij} in (1.4) are the components of the three-dimensional Ricci tensor $\mathbf{R}$, while R in (1.4) and in (1.6) is the three-dimensional scalar curvature.

The main goal of the present paper is to derive the equations (1.4) without using (1.7) within the three-dimensional Lagrangian approach.

2. Three-dimensional action integrals.

The four-dimensional action integral of the gravitational field is well-known (see § 2 in Chapter V of [14]). It is written as

$$S_{\text{gr}} = -\frac{c^3}{16\pi\gamma}\int(r+2\,\Lambda)\,\sqrt{-\det G}\;d^4x. \tag{2.1}$$

The total action integral should include the matter (regular and dark matter):

$$S = S_{\text{gr}} + S_{\text{mat}}. \tag{2.2}$$

We write the action integral S_{mat} in (2.2) as follows:

$$S_{\text{mat}} = \int\mathcal{L}_{\text{mat}}\,\sqrt{-\det G}\;d^4x. \tag{2.3}$$

In order to rewrite the integrals (2.1) and (2.3) in the three-dimensional formalism we apply the formulas (1.2). As a result we get

$$S_{\text{gr}} = -\frac{c^3}{16\pi\gamma}\iint(r+2\,\Lambda)\,\sqrt{\det g}\;d^3x\,dx^0, \tag{2.4}$$

$$S_{\text{mat}} = \iint\mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x\,dx^0. \tag{2.5}$$

As it was shown in [3], the four-dimensional scalar curvature r in (2.4) is expressed in a three-dimensional form as follows:

$$r = -2\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0} - R - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\,b^q_k - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\,b^q_q\,. \tag{2.6}$$

Here R is the three-dimensional scalar curvature associated with the metric $\mathbf{g}$. The formulas (2.4) and (2.5) complemented with (2.6) are sufficient in order to proceed with deriving the equations (1.4).

3. Choosing dynamic variables.

Action integrals are typically used for deriving differential equations through the stationary-action principle (see [15]). The main issue in applying this principle is the proper choice of dynamic variables. In the case of the four-dimensional action integrals (2.1) and (2.3) the dynamic variables are the components of the metric $\mathbf{G}$. Not those in (1.2), but the whole set of components of the symmetric 4×4 matrix in arbitrary coordinates where it is not blockwise diagonal.

In the case of the three-dimensional action integrals (2.4) and (2.5) we restrict ourselves to the components of the 3×3 symmetric matrix

$$g^{ij} = \begin{Vmatrix} g^{11} & g^{12} & g^{13}\\ g^{21} & g^{22} & g^{23}\\ g^{31} & g^{32} & g^{33}\end{Vmatrix}. \tag{3.1}$$

The matrix (3.1) is obtained as a part of the matrix G^{ij} from (1.2) in some special coordinates x^0, x^1, x^2, x^3. Their choice is described in [3]. These coordinates are not considered as dynamic variables within our three-dimensional approach.

4. Varying the metric.

According to the he stationary-action principle we need to organize variations of the dynamic variables and then study the changes of the action integrals under these variations. In the case of the metric (3.1) we write

$$\hat{g}^{ij} = g^{ij}(x^0, x^1, x^2, x^3) + \varepsilon\, h^{ij}(x^0, x^1, x^2, x^3), \tag{4.1}$$

where $\varepsilon \to 0$ is a small parameter. From (4.1) we derive

$$\sqrt{\det \hat{g}} = \sqrt{\det g}\left(1 - \varepsilon \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{g_{ij}\, h^{ij}}{2}\right) + \ldots\,. \tag{4.2}$$

Here in (4.2) and in what follows below through dots we denote higher order terms with respect to the small parameter ε. The formula (4.2) is similar to the formula (2.15) in § 2 of Chapter V in [14].

Substituting (2.6) into the action integral (2.4) we get four integrals

$$S_{\mathrm{gr}} = S_1 + S_2 + S_3 + S_4, \tag{4.3}$$

where

$$S_1 = \frac{c^3}{16\pi\gamma} \iint 2 \sum_{k=1}^{3} \frac{\partial b^k_k}{\partial x^0} \sqrt{\det g}\; d^3x\, dx^0, \tag{4.4}$$

$$S_2 = \frac{c^3}{16\pi\gamma} \iint (R - 2\,\Lambda) \sqrt{\det g}\; d^3x\, dx^0, \tag{4.5}$$

$$S_3 = \frac{c^3}{16\pi\gamma} \iint \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k \sqrt{\det g}\; d^3x\, dx^0, \tag{4.6}$$

$$S_4 = \frac{c^3}{16\pi\gamma} \iint \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q \sqrt{\det g}\; d^3x\, dx^0. \tag{4.7}$$

Applying the metric variation (4.1) to the integral (4.5), we get

$$\begin{aligned} \hat{S}_2 = S_2 + \frac{\varepsilon\, c^3}{16\pi\gamma} \iint \sum_{i=1}^{3}\sum_{j=1}^{3} \left(R_{ij} - \frac{R}{2}\, g_{ij} + \Lambda\, g_{ij}\right) \cdot \\ \cdot\; h^{ij} \sqrt{\det g}\; d^3x\, dx^0 + \ldots\,. \end{aligned} \tag{4.8}$$

The arguments and calculations supporting the formula (4.8) are the same as in deriving the equation (1.7) in § 2 of Chapter V in [14].

Before proceeding to the integrals (4.4), (4.6), and (4.7) let's recall that the matrix with the components $\hat{g}_{ij}$ is inverse to the matrix with the components $\hat{g}^{ij}$

in (4.1). Therefore from the formula (4.1) we derive a formula analogous to the formula (2.7) in § 2 of Chapter V in [14]:

$$\hat{g}_{ij}=g_{ij}-\varepsilon\,h_{ij}+\ldots=g_{ij}-\varepsilon\sum_{p=1}^{3}\sum_{q=1}^{3}g_{ip}\,h^{pq}\,g_{qj}+\ldots. \tag{4.9}$$

The formula (1.3) means

$$b_{ij}=\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}, \qquad\qquad \hat{b}_{ij}=\frac{1}{2}\,\frac{\partial\hat{g}_{ij}}{\partial x^0}. \tag{4.10}$$

Applying (4.10) to (4.9), we derive

$$\hat{b}_{ij}=b_{ij}-\varepsilon\sum_{p=1}^{3}\sum_{q=1}^{3}\Bigl(b_{ip}\,h^{pq}\,g_{qj}+g_{ip}\,h^{pq}\,b_{qj}+g_{ip}\,\frac{1}{2}\,\frac{\partial h^{pq}}{\partial x^0}\,g_{qj}\Bigr)+\ldots. \tag{4.11}$$

In (4.4) and in (4.7) we see the trace of the tensor field $\mathbf{b}$. Therefore we need to calculate the variation of this trace:

$$\sum_{k=1}^{3}\hat{b}^k_k=\sum_{q=1}^{3}\hat{b}^q_q=\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{b}_{ij}\,\hat{g}^{ij}. \tag{4.12}$$

Applying (4.1) and (4.11) to (4.12), we derive

$$\sum_{k=1}^{3}\hat{b}^k_k=\sum_{k=1}^{3}b^k_k-\varepsilon\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(b_{ij}\,h^{ij}+g_{ij}\,\frac{1}{2}\,\frac{\partial h^{ij}}{\partial x^0}\Bigr)+\ldots. \tag{4.13}$$

Differentiating (4.12) in x^0, we obtain the following formula:

$$\sum_{k=1}^{3}\frac{\partial\hat{b}^k_k}{\partial x^0}=\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}-\varepsilon\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\partial b_{ij}}{\partial x^0}\,h^{ij}+2\,b_{ij}\,\frac{\partial h^{ij}}{\partial x^0}+g_{ij}\,\frac{1}{2}\,\frac{\partial^2 h^{ij}}{\partial {x^0}^2}\Bigr)+\ldots. \tag{4.14}$$

The variation for the square of the trace (4.12) in (4.7) is also easily calculated:

$$\begin{gathered}\sum_{k=1}^{3}\sum_{q=1}^{3}\hat{b}^k_k\,\hat{b}^q_q=\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q-\\-\varepsilon\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(2\,b^k_k\,b_{ij}\,h^{ij}+b^k_k\,g_{ij}\,\frac{\partial h^{ij}}{\partial x^0}\Bigr)+\ldots.\end{gathered} \tag{4.15}$$

In (4.6) we see the trace of the square of the tensor field $\mathbf{b}$. Therefore we need to calculate the variation of this trace. Let's start with

$$\hat{b}^k_q=\sum_{i=1}^{3}\hat{b}_{iq}\,\hat{g}^{ik}. \tag{4.16}$$

From (4.1), (4.11), and (4.16) we derive

$$\hat{b}^k_q = b^k_q - \varepsilon \sum_{p=1}^{3} \sum_{s=1}^{3} b^k_p \, h^{ps} \, g_{sq} - \varepsilon \sum_{s=1}^{3} \frac{1}{2} \, \frac{\partial h^{ks}}{\partial x^0} \, g_{sq} + \ldots . \tag{4.17}$$

Then from (4.17) we derive the required variation of the trace for the square of $\mathbf{b}$:

$$\begin{gathered} \sum_{k=1}^{3} \sum_{q=1}^{3} \hat{b}^k_q \, \hat{b}^q_k = \sum_{k=1}^{3} \sum_{q=1}^{3} b^k_q \, b^q_k \, - \\ - \, \varepsilon \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} 2 \, b^k_i \, h^{ij} \, b_{jk} - \varepsilon \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial h^{ij}}{\partial x^0} \, b_{ij} + \ldots . \end{gathered} \tag{4.18}$$

Each of the four integrals (4.4), (4.5), (4.6), and (4.7) comprises the term $\sqrt{\det g}$. Its variation is given by the formula (4.2). Now we apply (4.18) to (4.6) taking into account (4.2). As a result we derive the following expression:

$$\begin{gathered} \hat{S}_3 = S_3 - \frac{\varepsilon \, c^3}{16\pi\gamma} \iint \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial h^{ij}}{\partial x^0} \, b_{ij} \, \sqrt{\det g} \; d^3x \, dx^0 \, - \\ - \, \frac{\varepsilon \, c^3}{16\pi\gamma} \iint \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} 2 \, b^k_i \, h^{ij} \, b_{jk} \, \sqrt{\det g} \; d^3x \, dx^0 \, - \\ - \, \frac{\varepsilon \, c^3}{16\pi\gamma} \iint \sum_{k=1}^{3} \sum_{q=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} \, b^k_q \, b^q_k \, g_{ij} \, h^{ij} \, \sqrt{\det g} \; d^3x \, dx^0 + \ldots . \end{gathered} \tag{4.19}$$

As it is usual in calculus of variations, the function h^{ij} introduced in (4.1) is assumed to be a function with compact support (see [16]). Therefore we can apply integration by parts in order to transform the first integral in (4.19). This yields

$$\begin{gathered} \hat{S}_3 = S_3 + \frac{\varepsilon \, c^3}{16\pi\gamma} \iint \sum_{i=1}^{3} \sum_{j=1}^{3} h^{ij} \, \frac{\partial (b_{ij} \, \sqrt{\det g}\,)}{\partial x^0} \; d^3x \, dx^0 \, - \\ - \, \frac{\varepsilon \, c^3}{16\pi\gamma} \iint \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} 2 \, b^k_i \, h^{ij} \, b_{jk} \, \sqrt{\det g} \; d^3x \, dx^0 \, - \\ - \, \frac{\varepsilon \, c^3}{16\pi\gamma} \iint \sum_{k=1}^{3} \sum_{q=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} \, b^k_q \, b^q_k \, g_{ij} \, h^{ij} \, \sqrt{\det g} \; d^3x \, dx^0 + \ldots . \end{gathered} \tag{4.20}$$

The further transformation of (4.20) is based on the formula

$$\frac{\partial (\sqrt{\det g}\,)}{\partial x^0} = \frac{1}{2} \sum_{k=1}^{3} \sum_{q=1}^{3} g^{kq} \, \frac{\partial g_{kq}}{\partial x^0} \, \sqrt{\det g} \, . \tag{4.21}$$

The formula (4.21) itself is derived from Jacobi's formula for differentiating determinants (see [17]). The same Jacobi's formula is used in deriving (4.2). As for the formula (4.2), applying (4.21) to it, we get

$$\begin{aligned}\hat{S}_3=S_3+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\partial b_{ij}}{\partial x^0}+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\frac{\partial g_{kq}}{\partial x^0}\,b_{ij}\,-\\-\sum_{k=1}^{3}2\,b^k_i\,b_{jk}-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g_{ij}\Bigr)\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.\end{aligned}\tag{4.22}$$

The last step in transforming the formula (4.22) is applying the first formula (4.10) to it. Then the partial derivatives of g_{kq} are expressed through b_{kq} and ultimately we get the trace of the tensor field $\mathbf{b}$ in the resulting formula:

$$\begin{aligned}\hat{S}_3=S_3+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\partial b_{ij}}{\partial x^0}+\sum_{k=1}^{3}b^k_k\,b_{ij}\,-\\-\sum_{k=1}^{3}2\,b^k_i\,b_{jk}-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g_{ij}\Bigr)\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.\end{aligned}\tag{4.23}$$

Having done with S_3, we can proceed to the fourth integral (4.7) in (4.3). In this case we apply the formulas (4.15) and (4.2). As a result we get

$$\begin{aligned}\hat{S}_4=S_4-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}2\,b^k_k\,b_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0\,-\\-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}b^k_k\,g_{ij}\,\frac{\partial h^{ij}}{\partial x^0}\,\sqrt{\det g}\;d^3x\,dx^0\,-\\-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.\end{aligned}\tag{4.24}$$

In order to transform the second integral in (4.24) we use integration by parts:

$$\begin{aligned}\hat{S}_4=S_4-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}2\,b^k_k\,b_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0\,+\\+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}h^{ij}\,\frac{\partial(b^k_k\,g_{ij}\sqrt{\det g}\,)}{\partial x^0}\,d^3x\,dx^0\,-\\-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.\end{aligned}\tag{4.25}$$

Applying the first formula (4.10) to (4.25), we can reduce (4.25) to

$$\begin{aligned}\hat{S}_4=S_4+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}h^{ij}\,g_{ij}\,\frac{\partial(b^k_k\sqrt{\det g}\,)}{\partial x^0}\,d^3x\,dx^0\,-\\-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.\end{aligned}\tag{4.26}$$

The next step is to differentiate the product $b^k_k\sqrt{\det g}$ in (4.26) and to apply (4.21). As a result we obtain the following formula:

$$\begin{aligned}
&\hat{S}_4=S_4+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{i=1}\sum^3_{j=1}h^{ij}\,g_{ij}\,\frac{\partial b^k_k}{\partial x^0}\,\sqrt{\det g}\;d^3x\,dx^0+\\
&+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{i=1}\sum^3_{j=1}\sum^3_{p=1}\sum^3_{q=1}\frac{1}{2}\,b^k_k\,h^{ij}\,g_{ij}\,g^{pq}\,\frac{\partial g_{pq}}{\partial x^0}\,\sqrt{\det g}\;d^3x\,dx^0-\\
&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{q=1}\sum^3_{i=1}\sum^3_{j=1}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.
\end{aligned}\tag{4.27}$$

And finally, if we apply the first formula (4.10) to (4.27), then we get

$$\begin{aligned}
&\hat{S}_4=S_4+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{i=1}\sum^3_{j=1}h^{ij}\,g_{ij}\,\frac{\partial b^k_k}{\partial x^0}\,\sqrt{\det g}\;d^3x\,dx^0+\\
&+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{q=1}\sum^3_{i=1}\sum^3_{j=1}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.
\end{aligned}\tag{4.28}$$

The formula (4.14) is prepared for applying to the first integral (4.4) in (4.3). It is derived by differentiating (4.13). However, it turns out that applying the formula (4.13) itself to (4.4) leads to easier calculations:

$$\begin{aligned}
&\hat{S}_1=S_1-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{i=1}\sum^3_{j=1}2\,\frac{\partial}{\partial x^0}\Bigl(b_{ij}\,h^{ij}+\\
&+g_{ij}\,\frac{1}{2}\,\frac{\partial h^{ij}}{\partial x^0}\Bigr)\sqrt{\det g}\;d^3x\,dx^0-\\
&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{i=1}\sum^3_{j=1}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.
\end{aligned}\tag{4.29}$$

Integrating by parts in the first term of (4.29), we derive

$$\begin{aligned}
&\hat{S}_1=S_1+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{i=1}\sum^3_{j=1}2\,\Bigl(b_{ij}\,h^{ij}+\\
&+g_{ij}\,\frac{1}{2}\,\frac{\partial h^{ij}}{\partial x^0}\Bigr)\,\frac{\partial(\sqrt{\det g}\,)}{\partial x^0}\;d^3x\,dx^0-\\
&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{k=1}\sum^3_{i=1}\sum^3_{j=1}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.
\end{aligned}\tag{4.30}$$

Now we apply (4.21) to (4.30). As a result we get

$$\hat{S}_1=S_1+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum^3_{i=1}\sum^3_{j=1}\sum^3_{k=1}\sum^3_{q=1}\Bigl(b_{ij}\,h^{ij}+$$

$$+\,g_{ij}\,\frac{1}{2}\,\frac{\partial h^{ij}}{\partial x^0}\biggr)\,g^{kq}\,\frac{\partial g_{kq}}{\partial x^0}\,\sqrt{\det g}\,d^3x\,dx^0\,-$$
$$-\,\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x\,dx^0+\ldots.$$

Then we apply the first formula (4.10) in order to transform the above formula:

$$\begin{aligned}\hat{S}_1=S_1&+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}2\,b_{ij}\,h^{ij}\,b^k_k\,\sqrt{\det g}\,d^3x\,dx^0\,+\\&+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}g_{ij}\,\frac{\partial h^{ij}}{\partial x^0}\,b^k_k\,\sqrt{\det g}\,d^3x\,dx^0\,-\\&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x\,dx^0+\ldots.\end{aligned}\tag{4.31}$$

Integration by parts in the second term of (4.31) yields

$$\begin{aligned}\hat{S}_1=S_1&+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}2\,b_{ij}\,h^{ij}\,b^k_k\,\sqrt{\det g}\,d^3x\,dx^0\,-\\&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}h^{ij}\,\frac{\partial(g_{ij}\,b^k_k\,\sqrt{\det g}\,)}{\partial x^0}\,d^3x\,dx^0\,-\\&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x\,dx^0+\ldots.\end{aligned}\tag{4.32}$$

Note that the second term in (4.32) is similar to the second term of (4.25). Therefore further steps are similar to (4.26), (4.27), and (4.28):

$$\begin{aligned}\hat{S}_1=S_1&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}h^{ij}\,g_{ij}\,\frac{\partial(b^k_k\,\sqrt{\det g}\,)}{\partial x^0}\,d^3x\,dx^0\,-\\&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x\,dx^0+\ldots.\end{aligned}\tag{4.33}$$

Differentiating the product $b^k_k\,\sqrt{\det g}$ in the first term of the formula (4.33) and applying (4.21) along with the first formula (4.10), we obtain

$$\begin{aligned}\hat{S}_1=S_1&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,g_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x\,dx^0\,-\\&-\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}2\,\frac{\partial b^k_k}{\partial x^0}\,g_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x\,dx^0+\ldots.\end{aligned}\tag{4.34}$$

Now we can gather the variations of all of the terms from (4.3) into one formula. As a result we get the variation of the action integral (2.4) of the gravitational field.

For this variation from (4.34), (4.8), (4.23), and (4.28) we derive the formula

$$\begin{aligned}\hat{S}_{\mathrm{gr}}=S_{\mathrm{gr}}+\frac{\varepsilon\,c^3}{16\pi\gamma}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\biggl(\frac{\partial b_{ij}}{\partial x^0}-\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}-\sum_{k=1}^{3}2\,b^k_i\,b_{jk}\,-\\-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g_{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij}+\sum_{k=1}^{3}b^k_k\,b_{ij}\,+\\+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}\biggr)\,h^{ij}\,\sqrt{\det g}\;d^3x\,dx^0+\ldots.\end{aligned}\tag{4.35}$$

Unlike the case of gravity, the action integral for the matter is not given explicitly in this paper. It is given implicitly by the formula (2.5). Therefore its variation is also written only in an implicit form:

$$\hat{S}_{\mathrm{mat}}=S_{\mathrm{mat}}+\varepsilon\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta g_{ij}}\,h_{ij}\,\sqrt{\det g}\;d^3x\,dx^0.\tag{4.36}$$

Despite being implicit, the formula (4.36) complemented with the explicit formula (4.35) is sufficient for deriving the differential equations for the metric $\mathbf{g}$ through the stationary-action principle (see [15]).

Theorem 4.1. *The stationary-action principle applied to the action integral* (2.2) *leads to the following differential equations:*

$$\begin{aligned}\frac{\partial b_{ij}}{\partial x^0}-\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}-\sum_{k=1}^{3}2\,b^k_i\,b_{jk}-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k\,-\\-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=-\frac{16\,\pi\,\gamma}{c^3}\,\frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta g_{ij}}.\end{aligned}\tag{4.37}$$

Theorem 4.1 is immediate from the formulas (4.35) and (4.36). The equations (4.37) can be written in terms of the time variable t, where $x^0=c\,t$:

$$\begin{aligned}\frac{\dot{b}_{ij}}{c}-\sum_{k=1}^{3}\frac{\dot{b}^k_k}{c}\,g_{ij}-\sum_{k=1}^{3}2\,b^k_i\,b_{jk}-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k\,-\\-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{16\,\pi\,\gamma}{c^3}\,\frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta g_{ij}}.\end{aligned}\tag{4.38}$$

The equations (4.38) can be compared with (1.4). Comparing left hand sides of (4.38) and (1.4), we find one term with $2\,b^k_i\,b_{jk}$ in (4.38) which is different from the corresponding term in (1.4). However, using symmetry of b_{ij} and g^{ij} we can transform this term as follows:

$$\sum_{k=1}^{3}2\,b^k_i\,b_{jk}=\sum_{k=1}^{3}\sum_{q=1}^{3}2\,b_{iq}\,g^{qk}\,b_{jk}=\sum_{k=1}^{3}(b_{kj}\,b^k_i+b_{ki}\,b^k_j).\tag{4.39}$$

Due to (4.39) the left hand sides of (4.38) and (1.4) do actually coincide. Comparing the right hand sides of these formulas, we derive the following formula for the components of the energy-momentum tensor:

$$T_{ij} = -2\,c\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}. \tag{4.40}$$

Note that the formula (4.40) applies only to spacial components of the energy-momentum tensor, i. e. to those where $1 \leqslant i, j \leqslant 3$. The components presented in the right hand sides of the equations (1.5) and (1.6) are not covered by this formula.

5. Concluding remarks.

Einstein's gravity equations (1.7) are written in terms 4×4 matrices r_{ij}, G_{ij}, and T_{ij}, representing three tensor fields — the Ricci tensor $\mathbf{r}$, the metric tensor $\mathbf{G}$, and the energy-momentum tensor $\mathbf{T}$, and in terms of one scalar field r representing the scalar curvature. Due to symmetry of the matrices r_{ij}, G_{ij}, and T_{ij} they constitute a set of 10 differential equations. Within the paradigm of the 3D-brane universe they are subdivided into three subsets (1.4), (1.5), and (1.6) comprising 6 equations, 3 equations and 1 equation respectively. Applying Lagrangian approach with action integrals (2.4) and (2.5), in the present paper we have shown that 6 equations of the first subset (1.4) can be derived within the framework of the 3D-brane universe paradigm (see Theorem 4.1 and the equations (4.38)). The other 4 equations (1.5) and (1.6) cannot be derived due to the lack of dynamic variables. Therefore they should be omitted from the 3D-brane universe theory.

Reducing the number of equations from 10 to 6, one can expect a larger set of solutions. The expected new solutions could correspond to new physical phenomena which are absent in the standard four-dimensional formalism or to new features in the phenomena which are already known.

6. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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