THE COLLATZ CONJECTURE interpreted with graph theory and the properties of the digital root of numbers: an analytical approach.

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Abstract

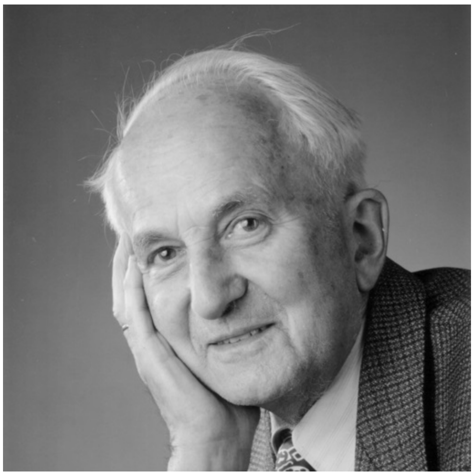
In this paper I present a study of the Collatz problem and its conjecture using graph theory and the properties of the digital root of natural numbers.

Keywords: Collatz conjecture, 3n+1 problem, number sequence, sequence of numbers, digital root, graph theory.

Löthar Collatz *(1910-1990) was a German mathematician who did research mainly in numerical analysis, with important studies and publications. He studied in Germany between 1928 and 1935 and as was customary he studied at various universities: Munich, Göttingen and Berlin.*

The Collatz-Wielandt formula for positive matrices in the Perron-Frobenius theorem is named in his honor. The 1957 article, written in conjunction with Ulrich Snogwitz, who was killed in the Bombing of Darmstadt during World War II, created the field of Spectral Graph Theory.

From 1952 until his retirement in 1978, Collatz worked at the University of Hamburg, where he founded the Institute for Applied Mathematics in 1953. After his retirement as professor emeritus, he was active in conferences.



For his great work, he was awarded many awards during his life, among which are:

- *Elected to the German Academy of Natural Sciences Leopoldina, the Academy of Sciences from the Institute of Bologna and the Academy of Modena in Italy.*

- *Honorary member of the Hamburg Mathematical Society.*

• Honorary degrees, awarded by the University of São Paulo, the Technical University of Vienna, the University of Dundee (Scotland), Brunel University (England), the University of Hannover and the Dresden Polytechnic University.

Lothar Collatz died unexpectedly of a heart attack in Varna (Bulgaria) on September 26, 1990, while attending a congress.

Influenced by his teachers Landau and Schur, he tried to represent arithmetic functions by graphs. To do this he joined n → m when f (n) = m and tried to find simple functions that would give rise to cycles. For it one should have numbers such that f(n)<n and others with f(m)>m. It was in this way that he defined the function that today we call de Collatz.

Definitions

***The Collatz conjecture** is an unsolved mathematical problem that states the following: for any natural number, can always be reduced to 1 by following a series of steps defined by operations math. The steps are defined as follows: if the number is even, divide by 2; if it is odd, multiply by 3 and add 1. The conjecture suggests that regardless of the starting number, we will arrive at eventually to number 1 after a finite number of steps, entering an infinite loop with the numbers 4→2→1→4.*

***The digital root** It is a mathematical concept that consists of adding the digits of a number and then returning to add the digits of the result, and so on, until you get a single digit. This last number is digital root of the original number. For example, the number 123 has a digital root of 6 (1 + 2 + 3 = 6). For him number 12345, its digital root is 6 (1 + 2 + 3 + 4 + 5 = 15, and 1 + 5 = 6).*
In number theory, the digital root is used to describe the structure of numbers and to investigate His properties. The classification of numbers based on their digital roots is done by separating the numbers in different groups depending on the sum of their digits.
Simplification: The digital root reduces a number to a single digit, which makes it easier to compare and analyze with other numbers.
Classification: The digital root allows to classify the numbers in groups according to their properties, which facilitates the identification of patterns and relationships between them. Modularity: The digital root is closely related to the modulo 9 calculus, which allows to investigate the structure of numbers and find relationships and mathematical properties using theory techniques of numbers.

***Graph theory** is a branch of mathematics that studies the relationship between objects and connections. that exist between them. Objects are represented as vertices (nodes) and connections are represented as edges (links) that connect them.*
Graph theory is used in a wide variety of fields, including computer science, engineering, physics, biology, sociology, economics and many other disciplines.
Some of the most important problems that have been solved with graph theory are the following: The maximum matching problem. The traveling salesman problem. The maximum flow problem. The graph coloring problem. The problem of the maximum clique.
These are just a few examples of the problems that have been solved with graph theory. In general, This theory has been applied in a wide variety of fields, including computer science, engineering, physics, biology and economics, among others.
In more precise terms, a graph is an abstract structure consisting of a set of vertices. (nodes) and a set of edges (links) that connect some or all of the vertices. Graphs can be use to represent a wide variety of real-life situations, such as social networks, maps of roads, electrical circuits, transportation networks, relationships between data sets, and much more.

Introduction

Collatz function . *By iteratively applying the Collatz function C to any number, we will arrive at number 1.*

$$C(n) \begin{cases} n/2, \text{ if } n \text{ is even} \\ 3n+1, \text{ if } n \text{ is odd} \end{cases}$$

An example of a sequence beginning with the number 212:

212, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1. If the sequence is continued, 1 iterates to 4 and this to 2 and 1, going into an infinite loop with these numbers.

The same sequence, in which the numbers have been replaced by the value of their digital roots: 5, 7, 8, 7, 8, 4, 2, 1, 5, 7, 8, 4, 2, 1.

Diagram of the possible development of any sequence generated by the function of the Collatz conjecture. The natural numbers are distributed in the vertices according to their digital root and the edges indicate the trajectory that follow other numbers in their iterations. (Figure 1)

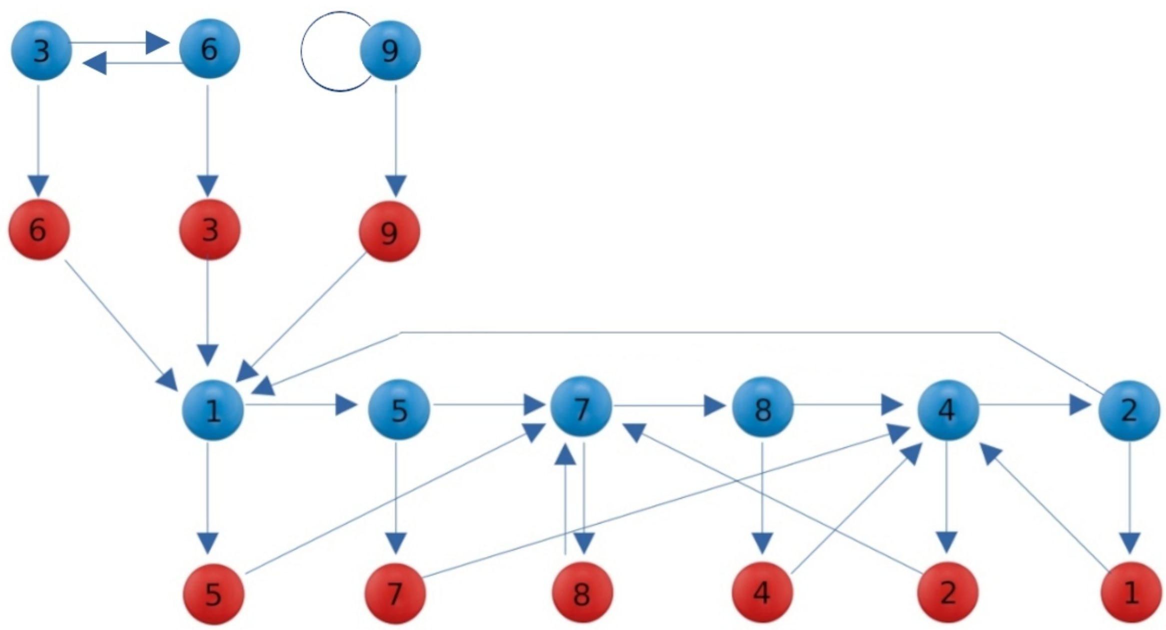


Figura 1

The natural numbers divided into 9 groups of even numbers (blue vertices) and 9 groups of numbers odd (red vertices), arranged by the value of their digital root indicated in the vertices.
The edges indicate the vertex to which the number obtained from the iteration of the previous number belongs.
Vertex numbers 3, 6, 9, 6, 3, 9, after iterating to vertex numbers 1 will no longer come out in the sequence, so we'll focus on another diagram that excludes these numbers. (Figure 2)

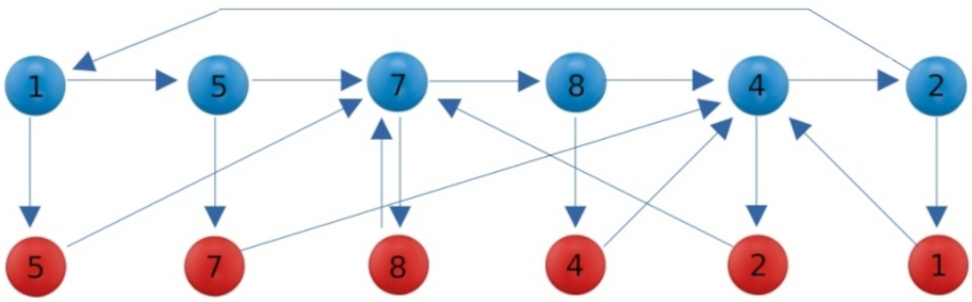


Figura 2

The diagram presents a cyclic structure that starts and ends at vertex 1, although it could be any another vertex (1, 5, 7, 8, 4, 2). The 3D diagram can be represented on the external surface in a way cylindrical.

Vertex numbers (5, 7, 8, 4, 2, 1) iterate over vertex numbers (7 and 4), which with a degree of entry 4 each, appear in the sequences more frequently than the other numbers.

In the iterations several cycles are formed: (7, 8, 7), (7, 8, 4, 2, 7), (4, 2, 1, 4), (4, 2, 7, 8, 4, 4), among others possible, but the main cycle that leads the sequences to number 1 is the one with the blue vertices of the even numbers. (Figure 3)

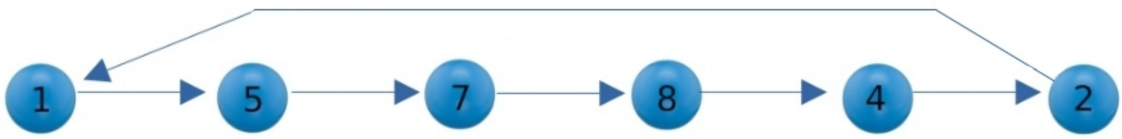


Figura 3

It is logical to ensure that certain even numbers go through this cycle one or more times, for example the number 4096, which loops through it twice.

Other numbers, with the inevitable iterations with the odd numbers of the red vertices, cause disorder and no apparent pattern in the sequences, but the cycle is the same, although the sequence later in your journey.

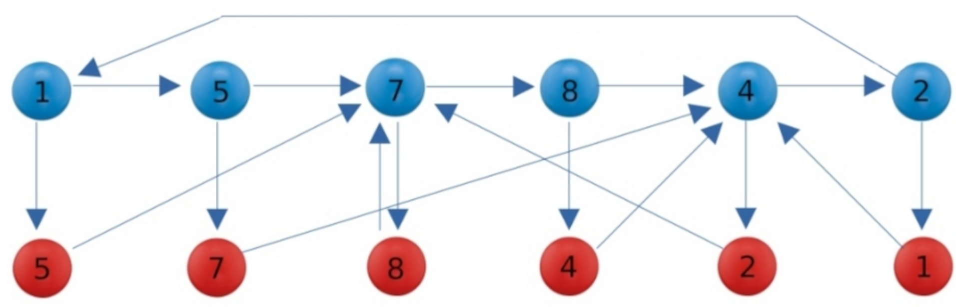


Figura 2

Again figure 2, to better visualize the following ***statements:***

1 . *All even numbers can iterate to odd numbers and these only to even numbers of the vertices 7 and 4, so all the numbers, at some point in the sequence, reach these numbers.*

2 . *The iterations of vertex numbers 7 and 4 are: 7→8→4→2.*

Vertex numbers 7 can go to vertex numbers 8, which return to vertex 7.
Vertex numbers 8 can go to vertex numbers 4, which go to vertex 4.
Vertex numbers 4 can go to vertex numbers 2, which go to vertex 7.

3 . *All numbers arrive at vertex numbers 2.*

4 . *Vertex numbers 2 will iterate over vertex numbers 1 and vertex numbers 1.*
When iterating to vertex numbers 1, the sequence can either end by reaching number 1 or iterate to a number from vertex 4, which will return to vertex 2.
When they iterate to vertex numbers 1, the sequence enters a new loop, at the end of which it will arrive again to vertex numbers 2 and the cycle will be repeated or the sequence will end.

5 . *At each iteration of an even number, it is halved, making the sequence decreasing at that point, so the more even-numbered vertices there are with more than one edge incoming, the more times the sequence will go through them and the more decreasing it will be. It is seen from the diagram that said vertices are 7 and 4, each with four incoming edges, while the other vertices only they have an incoming edge.*

Cycles have an important influence on the development of a sequence, and it is necessary to highlight what following:

cycles . *1 . The main cycle is the one of the vertices (1, 5, 7, 8, 4, 2, 1).*

2 . The vertices (7, 8, 7), with a group of even numbers and a group of odd numbers, makes the sequence is increasing and will be more increasing the longer the sequence stays in it.

3 . The vertices (7, 8, 4, 2, 7), three groups of even numbers and only one group of odd numbers, makes the sequence decreasing and will become more decreasing the longer the sequence lasts in it.

4 . The vertices (4, 2, 1, 4), two groups of even numbers and only one group of odd numbers, make that the sequence is decreasing and it will be more decreasing the longer the sequence remains in it.

There are other cycles, but these are the most important and the cycle of the vertices (7, 8, 7) is decisive in the number of steps of the sequences, since it makes them increase, in the measure of the time that it takes time to get out of it.

Below is an example of a sequence starting with the number 511, with the root values digital under each number.

511	1534	767	2302	1151	3454	1727	5182	2591	7774	3887	11662	5831	17494	8747	26242
7	4	2	7	8	7	8	7	8	7	8	7	8	7	8	7
13121	39364	19682	9841	29524	14762	7381	22144	11072	5536	2768	1384	692	346	173	520
8	7	8	4	4	2	1	4	2	1	5	7	8	4	2	7
260	130	65	196	98	49	148	74	37	112	56	28	14	7	22	11
8	4	2	7	8	4	4	2	1	4	2	1	5	7	4	2
34	17	52	26	13	40	20	10	5	16	8	4	2	1		
7	8	7	8	4	4	2	1	5	7	8	4	2	1		

The numbers highlighted in blue are the most important sections of the sequence, when the numbers They iterate in cycles.

With the number 2302 the sequence enters the cycle (7, 8, 7) and exits with the number 39364 (multiplicity 7) and in In this section, the growth of the sequence has been very important.

The sequence enters the loop (4, 2, 1, 4) with the number 29524, iterating through it until it reaches the number 22144. The same happens with the number 148 that goes up to the number 56. In these cycles, the sequence is always decreasing.

The number 1384 enters the cycle (7, 8, 4, 2, 7) and leaves with the number 98, causing an important decrease to the sequence.

Conclusion

The iteration of the sequence numbers occurs in a well-defined pattern, in which the value of its digital root strictly follows the cyclic order presented in this writing. All numbers reach others even numbers of digital root 2 and after an unknown number of iterations and cycles, finally They will reach number 1.