PROOF THAT EXISTS INFINITELY MANY PRIMES OF THE FORM $n^2 + 1$

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Abstract. In this paper it is going to be proved that exists infinitely many primes that are of the form $n^2 + 1$.

1. Introduction

In this paper, it is going to be proved that there are infinitely many primes p of the form $p = n^2 + 1$ (the problem is known as 4^{th} of the four "unattackable" problems listed in the 1912 Fifth Congress of Mathematicians in Cambridge by Landau [13, 3]; the first problem, Goldbach's conjecture, has been recently disproved in [4]; the second problem, twin prime conjecture, has been recently proved in [5]). Using a very similar methodology to the one that has been recently proposed in [5, 6], it is going to be proved that exists infinitely many primes of the form $n^2 + 1$. It will become clear that, analogously, can be proved that exists infinitely many primes of the form $n^2 + a$, where $n, a \in \mathbb{N}$.

Remark. In this paper, the author sticks to the idea that number of the even numbers or number of natural numbers that are divisible by 3, and so on, can be expressed as the fraction of the natural numbers that corresponds to the density of the subset of natural numbers of interest, **which is not in accordance with adopted set theory**. So, for the time being, this paper can be considered as a thought experiment. (Working hypothesis is that it should not be allowed to make copies (clones) of the elements of the set without counting them - axiomatic idea that same elements are counted only once is not compatible with the rest of the classical math and it is that axiom (and not infinity itself) that creates a bunch of unusual phenomena that could not be found in finite cases. Applying similar ideas on finite sets it is possible to create a very unusual phenomena as well.)

Remark. In this paper any infinite series in the form $c_1 \cdot l \pm c_2$ is going to be called a thread defined by number c_1 (in literature these forms are known as arithmetic progressions – however, it seems that the term thread is probably better choice in this context). Here c_1 and c_2 are numbers that belong to the set of natural numbers (c_2 can also be zero and usually is smaller than c_1) and l represents an infinite series of consecutive natural numbers in the form $(1, 2, 3, \ldots)$.

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2. Special solutions of the quadratic equation

In this chapter, two theorems that will be used in the proof of the main thesis of this work, that there are infinitely many prime numbers of the form $n^2 + 1$, where $n \in \mathbb{N}$, are going to be proved. Some definitions and well-known theorems, will be presented, first. Let's start with the fundamental theorem of arithmetic.

Theorem 2.1 ([1, Theorem 2.10], see also [11, Theorem 5])**.** *Every integer $n > 1$ can be expressed as a product of prime numbers (with one or more factors).*

The following theorems and definitions are related to the number of solutions of a quadratic equation using Legendre symbol.

Definition 2.2. Let $(a, m) = 1$. If the congruence $x^2 \equiv a \mod m$ has a solution, then we say that a is a quadratic residue modulo m. Otherwise, we say that a is a quadratic nonresidue modulo m. [1]

Definition 2.3. Let p be an odd prime number. The Legendre's symbol $\left(\frac{a}{p}\right)$ is equal to 1 if a is a quadratic residue modulo p, it is equal to -1 if a is a quadratic nonresidue modulo p, and it is equal to 0 if $p \mid a$. [1]

The following properties of Legendre' symbol can be found in papers [1], [14], [12], [2], [11]:

(1) If $m_1 \equiv m_2 \mod p$, then $\left(\frac{m_1}{p}\right) = \left(\frac{m_2}{p}\right)$.
(2) $\left(\frac{m_1 m_2}{p}\right) = \left(\frac{m_1}{p}\right)\left(\frac{m_2}{p}\right)$.
(3) If p and q are distinct odd prime numbers, then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$.
(4) If p is odd prime number, then $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.
(5) If p is odd prime number, then $\left(\frac{1}{p}\right) = 1$.
(6) If p is odd prime number, then
$$\left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \mod 4, \\ -1, & \text{if } p \equiv 3 \mod 4. \end{cases}$$
(7) If $p \in \mathbb{P} \setminus \{2, 3\}$, then
$$\left(\frac{3}{p}\right) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \mod 12, \\ -1, & \text{if } p \equiv \pm 5 \mod 12. \end{cases}$$
(8) If $p \in \mathbb{P} \setminus \{2, 3, 5\}$, then
$$\left(\frac{5}{p}\right) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \mod 10, \\ -1, & \text{if } p \equiv \pm 3 \mod 10. \end{cases}$$

Theorem 2.4 ([1, Theorem 4.2 (Euler's criterion)])**.** *For any integer a and any odd prime number p, we have*
$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \mod p.$$

Theorem 2.5 ([7, Zadanie 53])**.** *If $p \in \mathbb{P}$, $(a, p) = 1$, $n \in \mathbb{N}$, then $x^n \equiv a \mod p$ has a solution if and only if $a^{\frac{p-1}{d}} \equiv 1 \mod p$, where $d = (n, p - 1)$.*

The following two theorems will be used later in this paper.

Theorem 2.6. *The equation* $m^2 = (4i-1)j-i \equiv -i \mod (4i-1)$ *has no solutions when* $4i-1 \in \mathbb{P}$.

Proof. It will be shown that $\left(\frac{-i}{4i-1}\right) \equiv -1 \mod (4i-1)$, where $4i-1 \in \mathbb{P}$. Then, using Theorems 2.4 and 2.5, we get that the congruence $m^2 = (4i-1)j-i \equiv -i \mod (4i-1)$ has no solutions.

a) For $i=1$ we have $\left(\frac{-1}{3}\right) \equiv -1 \mod 3$ (by property (6)).
b) For $i \in \mathbb{N} \setminus \{1\}$ we obtain

$$\left(\frac{-i}{4i-1}\right) = \left(\frac{-1}{4i-1}\right)\left(\frac{i}{4i-1}\right) = -\left(\frac{i}{4i-1}\right)$$

(by properties: (2), (6)). We will show that $\left(\frac{i}{4i-1}\right) \equiv 1 \mod (4i-1)$.

From Theorem 2.1 it follows that

$$i = 2^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_k^{n_k},$$

where $p_k \in \mathbb{P}$, $n_k \in \mathbb{N} \cup \{0\}$ for every $k \in \mathbb{N}$. Hence we have

$$\left(\frac{i}{4i-1}\right) = \left(\frac{2}{4i-1}\right)^{n_1} \cdot \left(\frac{p_2}{4i-1}\right)^{n_2} \cdot \ldots \cdot \left(\frac{p_k}{4i-1}\right)^{n_k}$$

by property (1). Note that if i is an even number ($n_1 \geqslant 1$) then

$$\left(\frac{2}{4i-1}\right) = (-1)^{i(2i-1)} = 1.$$

It will be shown that $\left(\frac{p_k}{4i-1}\right) \equiv 1 \mod (4i-1)$ for every $k \geqslant 2$, then the proof of this case will be completed. Let $k \geqslant 2$. By property (3) we know that

$$\left(\frac{p_k}{4i-1}\right)\left(\frac{4i-1}{p_k}\right) = \left(\frac{p_k}{4i-1}\right)\left(\frac{-1}{p_k}\right) = (-1)^{\frac{p_k-1}{2}\cdot(2i-1)}.$$

If $p_k = 4l-1$, where $l \in \mathbb{N}$, then

$$\left(\frac{p_k}{4i-1}\right)\left(\frac{-1}{p_k}\right) = -\left(\frac{p_k}{4i-1}\right) = -1$$

(by property (6)). Hence

$$\left(\frac{p_k}{4i-1}\right) = 1.$$

If $p_k = 4l+1$, where $l \in \mathbb{N}$, then

$$\left(\frac{p_k}{4i-1}\right)\left(\frac{-1}{p_k}\right) = \left(\frac{p_k}{4i-1}\right) = 1$$

(by property (6)). Hence

$$\left(\frac{p_k}{4i-1}\right) = 1.$$

Therefore $\left(\frac{p_k}{4i-1}\right) \equiv 1 \mod (4i-1)$. Since we have $\left(\frac{2}{4i-1}\right) \equiv 1 \mod (4i-1)$ and $\left(\frac{p_k}{4i-1}\right) \equiv 1 \mod (4i-1)$ for every $k \geqslant 2$ so it has been proved that following holds for all $i \in \mathbb{N}$:

$$\left(\frac{-i}{4i-1}\right) \equiv -1 \mod (4i-1),$$

where $4i-1 \in \mathbb{P}$.

$\square$

Theorem 2.7. *The equation* $m^2 = (4i+1)j + i \equiv i \mod (4i+1)$ *has a solution when* $4i+1 \in \mathbb{P}$.

Proof. It will be shown that $\left(\frac{i}{4i+1}\right) \equiv 1 \mod (4i+1)$, where $4i+1 \in \mathbb{P}$. Then, using Theorems 2.4 and 2.5, we get that the congruence $m^2 = (4i+1)j + i \equiv i \mod (4i+1)$ has a solution.

a) For $i = 1$ we have $\left(\frac{1}{5}\right) \equiv 1 \mod 5$ (by property (5)).

b) For $i \in \mathbb{N} \setminus \{1\}$ we will show that $\left(\frac{i}{4i+1}\right) \equiv 1 \mod (4i+1)$.

From Theorem 2.1 we obtain that

$$i = 2^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_k^{n_k},$$

where $p_k \in \mathbb{P}$, $n_k \in \mathbb{N} \cup \{0\}$ for every $k \in \mathbb{N}$. Hence we have

$$\left(\frac{i}{4i+1}\right) = \left(\frac{2}{4i+1}\right)^{n_1} \cdot \left(\frac{p_2}{4i+1}\right)^{n_2} \cdot \ldots \cdot \left(\frac{p_k}{4i+1}\right)^{n_k}$$

by property (1). Note that if i is an even number ($n_1 \geqslant 1$) then

$$\left(\frac{2}{4i+1}\right) = (-1)^{i(2i+1)} = 1.$$

It will be shown that $\left(\frac{p_k}{4i+1}\right) \equiv 1 \mod (4i+1)$ for every $k \geqslant 2$, then the proof of this case will be completed. Let $k \geqslant 2$. By properties (3) and (5) it is known that

$$\left(\frac{p_k}{4i+1}\right)\left(\frac{4i+1}{p_k}\right) = \left(\frac{p_k}{4i+1}\right) = (-1)^{(p_k-1)i}.$$

Hence

$$\left(\frac{p_k}{4i+1}\right) = 1.$$

Therefore $\left(\frac{p_k}{4i+1}\right) \equiv 1 \mod (4i+1)$ for every $k \geqslant 2$, which means that the following holds for all $i \in \mathbb{N}$:

$$\left(\frac{i}{4i+1}\right) \equiv 1 \mod (4i+1),$$

where $4i+1 \in \mathbb{P}$.

$\square$

3. Proof that exists infinitely many primes in the form n^2+1

It is well known that all odd numbers bigger than 1 can be expressed by one of the following forms $l = 4k-1$ or $l = 4k+1$, where $k \in \mathbb{N}$. It is obvious that numbers of the form n^2+1 cannot be primes if n is an odd number (the only exception is number 1 which can generate the only even prime 2, and 2 can be represented as $2 = 1^2 + 1$). So, the case of interest is when n is an even number, i.e. $n = 2m$, where $m \in \mathbb{N}$. In that case

$$n^2 + 1 = 4m^2 + 1.$$

In order to have prime numbers of that form, it is necessary to eliminate all composite numbers of the form $4m^2+1$. Since all numbers of the form $4m^2+1$

are actually odd numbers of the form $4k + 1$, it is easy to conclude that composite numbers of the form $4m^2 + 1$ can be expressed in the following form

$$4m^2 + 1 = (4i - 1)(4j - 1),$$

or in the form

$$4m^2 + 1 = (4i + 1)(4j + 1),$$

where $i, j \in \mathbb{N}$. Now, it is easy to see that the following equations hold

$$m^2 = (4i - 1)j - i, \tag{1}$$

or

$$m^2 = (4i + 1)j + i. \tag{2}$$

A few cases are going to be considered:
Case $i = 1$: $m^2 = 3j - 1$ or $m^2 = 5j + 1$.
Case $i = 2$: $m^2 = 7j - 2$ or $m^2 = 9j + 2 = 3(3j + 1) - 1$.
Case $i = 3$: $m^2 = 11j - 3$ or $m^2 = 13j + 3$.

From examples it can be concluded (and it will be further analyzed in Appendix A) that threads that are defined by odd prime numbers and that have certain quadratic residues, should be removed. For instance, it is known that there is no solution to the equation

$$m^2 \equiv -1 \mod 3,$$

so not a single thread defined by prime number 3 is going to be removed. On the other side we know that equation

$$m^2 \equiv 1 \mod 5,$$

has two solutions, so threads $5j + 1$ and $5j - 1$ are going to be removed. (It is known that quadratic residue equation has no solution or has exactly 2 solutions [9, Lemma 9.1]).

From Theorems 2.6 and 2.7 from the previous section, it can be seen that equation (1) has no solutions, and equation (2) always has a solution.

Now, a three stage process for obtaining prime numbers in the form $n^2 + 1$ is going to be presented.

STAGE 1. From the set of all natural numbers n remove all odd numbers, except number 1. As it was already mentioned, number 1 creates the only even prime 2, and 2 can be written as $2 = 1^2 + 1$. However, we are going to ignore number 2 in the analysis that follows, since it has no impact on the final conclusion.

STAGE 2. The numbers that are left after Stage 1 are even numbers $n = 2m$, $m \in \mathbb{N}$. From numbers m it is necessary remove all threads defined by odd prime numbers p that are in the form $pj - l$, where j is any natural number and l represents a specific natural number that is obtained as a solution of certain quadratic residue equation (2). We denote the amount of numbers that are left after this stage, as N_{S_2}.

STAGE 3. From the numbers m that are left after Stage 2, remove all threads defined by odd prime numbers p that are in the form $pj + l$, where j is any natural number and l represents a specific natural number that is obtained as a solution of certain quadratic residue equation (2). We denote the amount of numbers that are left after this stage as N_{S_3}.

In order to prove that exist infinitely many prime numbers of the form n^2+1, the processes in the Stage 2 and Stage 3 are going to be compared to processes generated by Sundaram sieve [8] (which is equivalent to Erathostenes sieve).

STAGE 2a. We index the even numbers. Implement Erathostenes sieve on indices of the even numbers (or equivalent Sundaram sieve). In that case, it is known that only the indices that are equal to prime numbers are going to be left. Their number is infinite. The number of those numbers that are smaller than some natural number t, is denoted as $\pi(t)$ and then the following equation holds [10] $\pi(t) \approx \frac{t}{\ln(t)}$. From [10, Corollary 1] we know that the following holds $\pi(t) \geq \frac{t}{\ln(t)}$ for $t \geq 17$. It will be explained that N_{S_2} is bigger than the number of prime numbers. From this, it is easy to understand that there will be infinitely many numbers left after the completion of Stage 2.

In order to understand why Stage 2 leaves more numbers than Stage 2a, table that follow is going to be analyzed. The table present the fraction of the numbers that are eliminated in each step of the processes defined by Stage 2 and Stage 2a.

TABLE 1. Comparison of the Stage 2a and Stage 2 – threads defined by a few smallest primes.

Step	Stage 2a	Step	Stage 2
1	Remove even numbers (except 2) amount of numbers left is 1/2.	1	Remove numbers defined by thread defined by 5 amount of numbers left is 4/5.
2	Remove numbers defined by thread defined by 3 (obtained for $i=1$) amount of numbers left is 2/3 of the numbers that are left after previous step.	2	Remove numbers defined by thread defined by 13 (because equation (2) has solution in this case) amount of numbers left is 12/13 of the numbers that are left after previous step.
3	Remove numbers defined by thread defined by 5 (obtained for $i=2$) amount of numbers left is 4/5 of the numbers that are left after previous step.	3	Remove numbers defined by thread defined by 17 (because equation (2) has solution in this case) amount of numbers left is 16/17 of the numbers that are left after previous step.
4	Remove numbers defined by thread defined by 7 (obtained for $i=3$) amount of numbers left is 6/7 of the numbers that are left after previous step.	4	Remove numbers defined by thread defined by 29 (because equation (2) has solution in this case) amount of numbers left is 28/29 of the numbers that are left after previous step.
5	Remove numbers defined by thread defined by 11 (obtained for $i=5$) amount of numbers left is 10/11 of the numbers that are left after previous step.	5	Remove numbers defined by thread defined by 37 (because equation (2) has solution in this case) amount of numbers left is 36/37 of the numbers that are left after previous step.

STAGE 3a. We index the numbers left after Stage 2 with consecutive natural numbers. Implement Erathostenes sieve (or equivalent Sundaram sieve) on indices of the numbers that are left after the implementation of the process defined by Stage 2. In that case, it is known that only the indices that are equal to the prime numbers are going to be left. It is simple to understand that their number is infinite. It will be shown that N_{S_3} is bigger than the number of numbers left by the process in the Stage 3a. That will lead to the conclusion that exists infinitely many prime numbers of the form n^2+1.

In order to understand why Stage 3 leaves more numbers than Stage 3a, table that follow is going to be analyzed. The table present the fraction of the numbers that are eliminated in each step of the processes defined by Stage 3 and Stage 3a.

Table 2. Comparison of the Stage 3a and Stage 3 – threads defined by a few smallest primes.

Step	Stage 3a	Step	Stage 3
1	Remove even numbers (except 2) amount of numbers left is 1/2.	1	Remove numbers defined by thread defined by 5 amount of numbers left is 3/4.
2	Remove numbers defined by thread defined by 3 (obtained for $i = 1$) amount of numbers left is 2/3 of the numbers that are left after previous step.	2	Remove numbers defined by thread defined by 13 (because equation (2) has solution in this case) amount of numbers left is 11/12 of the numbers that are left after previous step.
3	Remove numbers defined by thread defined by 5 (obtained for $i = 2$) amount of numbers left is 4/5 of the numbers that are left after previous step.	3	Remove numbers defined by thread defined by 17 (because equation (2) has solution in this case) amount of numbers left is 15/16 of the numbers that are left after previous step.
4	Remove numbers defined by thread defined by 7 (obtained for $i = 3$) amount of numbers left is 6/7 of the numbers that are left after previous step.	4	Remove numbers defined by thread defined by 29 (because equation (2) has solution in this case) amount of numbers left is 27/28 of the numbers that are left after previous step.
5	Remove numbers defined by thread defined by 11 (obtained for $i = 5$) amount of numbers left is 10/11 of the numbers that are left after previous step.	5	Remove numbers defined by thread defined by 37 (because equation (2) has solution in this case) amount of numbers left is 35/36 of the numbers that are left after previous step.

Values of the fractions presented in the Table 1 and Table 2 are asymptotically correct (in the finite case those values are only approximately correct - for details see [5]).

It can be noticed that it is possible that threads defined by the same number in the first and the second column of Table 2 will not remove the same percentage of numbers. The reason is obvious - consider for instance the thread defined by 5: in the first column it will remove 1/5 of the numbers left, but in the second column it will remove 1/4 of the numbers left, since the thread defined by 5 in Stage 2 has already removed one fifth of the numbers. So, only odd numbers that give residual 0, 1, 2 and 3 when they are divided by 5 are left, and there is approximately same number of numbers that give residuals 0, 1, 2 and 3, when the number is divided by 5 (the numbers of those numbers are asymptotically the same). The same way of reasoning can be applied for all other threads defined by the same prime in different columns (see [5]). From Table 1 and Table 2 can be seen that in every step, threads in the second column will leave bigger percentage of numbers than the corresponding threads in the first column. This could be easily proved using the methodology presented in [5]. From Table 1 and Table 2 it can be seen that bigger amount of numbers is left in every step of the second column than in the the first column. From that, it can be concluded that after every step, part of the numbers that is left in the second column is bigger than number of numbers left in the first column (that is also noticeable if we consider amount of numbers left after removal of all numbers generated by threads that are defined by all prime numbers smaller than some natural number). From previous analysis it can be concluded that the number of numbers m that is left after Stage 3 is infinite, since it is known that the number of numbers left after Stage 3a is infinite. Since every number m corresponds to one number n, it can safely be concluded that the number of primes of the form n^2+1 is infinite. That concludes the proof.

Without going into the details, here, we propose an approximate formula for the number of prime numbers of the form n^2+1, that are smaller than some natural number m:

$$\pi_{2,1}(m) = 0.9468 \frac{\sqrt{\frac{m}{4}}}{\sqrt[e]{\ln\left(\sqrt{\frac{m}{4}}\right)}\sqrt[e]{\ln(\sqrt{\frac{m}{4}}) - \ln\left(\sqrt[e]{\ln\left(\sqrt{\frac{m}{4}}\right)}\right)}}.$$

The following table presents the quality of approximation for several values of m.

TABLE 3. Comparison of the true (T) and approximate (A) values of the number of prime numbers of the form n^2+1 that are smaller than natural number m.

m	10^1	10^2	10^3	10^4	10^5	10^6	10^7
T	2	4	10	19	51	112	316
A	2	3	7	18	48	129	358

4. Proof that exists infinitely many primes in the form n^2+a

Using the approach proposed in Section 3, it is not difficult to be proved that exists infinite number of primes in the form n^2+a, where $n, a \in \mathbb{N}$. For every a it should be determined if the number n^2+a should be of the form $4k-1$ or $4k+1$. In the case $4k-1$ the analysis is a bit easier since the number of the form $4k-1$ is composite only in the case when one of the factors is of the form $4i+1$ and the other of the form $4j-1$. Theorems 2.6 and 2.7 do not hold for all a and they cannot be used, but their usage is not critical part of the proof, anyway. Apart from those two differences, analysis is completely analogous.

5. Appendix A

Here it is going to be proved that m in (1) and (2) is represented by threads defined by odd prime numbers. Now, the form of (1) and (2) for some values of i will be checked.

Case $i = 1$: $m = 3j - 1, \quad m = 5j + 1,$
Case $i = 2$: $m = 7j - 2, \quad m = 9j + 2 = 3(3j + 1) - 1,$
Case $i = 3$: $m = 11j - 3, \quad m = 13j + 3,$
Case $i = 4$: $m = 15j - 4 = 3(5j - 1) - 1, \quad m = 17j + 4,$
Case $i = 5$: $m = 19j - 5, \quad m = 21j + 5 = 7(3j + 1) - 2,$
Case $i = 6$: $m = 23j - 6, \quad m = 25j + 6 = 5(5j + 1) + 1,$
Case $i = 7$: $m = 27j - 7 = 3(9j - 2) - 1, \quad m = 29j + 7,$
Case $i = 8$: $m = 31j - 8, \quad m = 33j + 8 = 11(3j + 1) - 3.$

It can be seen that m is represented by the threads that are defined by odd prime numbers. From examples, it can be seen that if $(4j+1)$ or $(4j-1)$ represent a composite number, m that is represented by thread defined by that number also has a representation by the the thread defined by one of the prime factors of that composite number. That can be proved easily in the general case, by direct calculation. Here, one case is going to be analyzed. Assume that $4j + 1$ is a composite number and that $4l + 1$ is one of its prime factors. Then, the following holds

$$4j + 1 = (4l + 1)(4s + 1)$$

where $l, s \in \mathbb{N}$. That leads to

$$j = 4ls + l + s.$$

The simple calculation leads to

$$m = (4l + 1)(4s + 1)i + 4ls + l + s = (4l + 1)(4s + 1)i + s(4l + 1) + l$$

or

$$m = (4l + 1)((4s + 1)i + s) + l$$

which means

$$m = (4l + 1)f + l,$$

and that represents the already existing form of the representation of m for the factor $4l + 1$, where

$$f = (4s + 1)i + s.$$

In the same way this can be proved for the all other cases of interest.

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